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Mixture-based Adaptive Probabilistic Control

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SUMMARY

Quasi-Bayes algorithm, combined with stabilized forgetting, provides a tool for efficient recursive estimation of dynamic probabilistic mixture models. They can be interpreted either as models of closed-loop with switching modes and controllers or as a universal approximation of a wide class of non-linear control loops.

Fully probabilistic control design extended to mixture models makes basis of a powerful class of adaptive controllers based on the receding-horizon certainty equivalence strategy.

Paper summarizes the basic elements mentioned above, classifies possible types of control problems and provides solution of the key one referred to as "simultaneous" design. Results are illustrated on mixtures with components formed by normal auto-regression models with external variable (ARX).

KEY WORDS: Bayesian identification, fully probabilistic control, finite mixtures, recursive estimation.

1. INTRODUCTION

Complex technological processes running in different operational modes require control of high quality and reliability. The quality of control of these systems in the presence of large uncertainty on the controlled process can be judged according to ability: i) to deal with different operation modes; ii) to recognize properly the current situation; iii) to provide fast and stable solution of the control design problem and take appropriate actions and, iv) to learn and adapt sufficiently quickly to a new operation mode.

The multi-model framework complemented by different modelling techniques contributes to all of the mentioned aspects. The framework models a nonlinear dynamic system by a weighted sum of simpler local models each valid within the specific operation mode of the operating

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space, for a detail review see [1]. The local model representation employs a number of modelling techniques, in particular, neural networks [2], differential equations [3], regression type models [4], fuzzy systems [5], statistical mixture models [6].

The multi-modelling approach includes the following sub-problems: i) determining the number of operation modes; ii) "curse of dimensionality" whereby the computational complexity extremely increases; iii) providing the optimal blending of the local models; iv) lack of a direct relationship between the dynamics of multiple model system and the dynamics of local models.

As one of the possible ways of solution, the frequently used switching supervisory control (SSC) [7] should be mentioned. It assumes that the system belongs to a priori known sets of admissible models. Each such a set is associated with a controller possessing satisfactory closed-loop performance. Then, the switching among different controllers, ensuring required system performance, is done. It calls for the inference on performance of candidate loops and for the choice of the proper criterion [8]. The control quality can further be improved by introducing an adaptive model [9] into the set of admissible models. The control design employing such types of models is successfully applied to linear time-invariant systems. Recently, the SSC of non-linear systems with constant parameters has been developed as well [10].

The multi-model approach seems to create a natural framework for describing the complex processes and many significant results have been achieved. However, the available approaches remain to be work consuming, need a deep knowledge of system dynamics and often are application dependent with a number of restrictions on their use. The amount of available knowledge indicates that it is a proper time for attempts to solve the overall multi-model based control design in a systematic way. An attempt of this type is presented here.

The proposed approach falls into multi-model framework with the controlled system represented by a dynamic probabilistic mixture model. The probabilistic extraction of information from process data, behind the adopted approach, allows us to overcome the dependence on application and does not require detailed knowledge of system dynamics. Furthermore, the related algorithms and used techniques mostly have a long tradition [11] and already proved to be efficient and accurate.

The overall idea is as follows. Assume, that the system under control operates in different operation modes. Historical process data, fully describing these operation modes, are processed by quasi-Bayes algorithm [12] combined with stabilized forgetting technique [13] to build the system model. The obtained model reflects significant operational modes each associated with so-called *mixture component*. During the subsequent control design, the components, yielding the best system performance in the past, drive the control actions. Usually, the expected value of data dependent loss function is used for judging the performance quality. Here, Kullback-Leibler distance [14] is employed to this purpose. It measures the distance of the *probability density function (pdf)* describing closed-loop behavior to a desired closed-loop behavior, referred to as *user ideal*, also expressed by a pdf. The designed controller respects noise properties and is found *explicitly*.

Paper layout is as follows. Preparatory Section 2 contains a brief overview of the probabilistic mixture modelling complemented by estimation and prediction issues. Sections 3 and 4 form the core of the paper and describes solution of control design and its application to the normal ARX model. Sections 5 and 6 provide an illustrative example and concluding remarks, respectively.

2. ADAPTIVE ESTIMATION OF MIXTURES

This Section introduces the necessary notions and techniques to work with. More detailed description can be found in [12], [15]. In this text, $x(t)$ stands for $(x_1, \ldots, x_t)$, x^* denotes set of possible values of a variable x and $\mathring{x}$ is the number of entries in x^*. The symbol $\{f(x_{i;t})\}_{t \in t^*}$ denotes probability (density) function (p(d)f) of the ith entry of quantity x at discrete time $t \in t^* \equiv \{1, \ldots, \mathring{t}\}$, where $\mathring{t} \leq \infty$. Time index follows the semicolon in the subscript when multiple indices are used. The symbol x' means transposition of x.

2.1. Mixture model

The classic definition of probabilistic mixture and the variety of techniques of its estimation have been described in [6]. Here, the relevant essence is only presented.

Consider the process quantities (y_t, u_t, Θ, c_t) consist of the observed controlled system output y_t and directly manipulated system input u_t together with unobserved unknown time-invariant parameter $\Theta \in \Theta^*$ and a random pointer to the active mixture component $c_t \in c^* \equiv \{1, \ldots, \mathring{c}\}$, $\mathring{c} < \infty$.

The distribution that approximates (describes) the relation between quantities involved is assumed to be in the *component form* as follows

$$f(d(\mathring{t}), c(\mathring{t}), \Theta) \equiv Di_\alpha(\kappa_0) \prod_{t \in t^*} f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t)\alpha_{c_t} \prod_{c \in c^*} f(\Theta_c), \quad \text{where} \tag{1}$$

$f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t)$ is called (parameterized) *component*; it is usually uni-modal pdf of *data item* $d_t = [y_t, u_t]'$; the number of components $\mathring{c}$ is assumed to be finite and fixed;

$\phi_{c_t;t-1}$ is observable *state of* c_t*-th component*; the phase form $\phi_{c;t-1} \equiv [d'_{t-1}, \ldots, d'_{t-\partial_c}, 1]'$ of the state is considered; finite fixed orders ∂_c are assumed;

α_{c_t} is constant probability of the pointer c_t, called *component weight*; their collection forms probabilistic vector $\alpha \in \alpha^* \equiv \left\{\alpha_c \geq 0,\ \sum_{c \in c^*} \alpha_c = 1\right\}$;

Θ is *mixture parameter* parameterizing the model; it is formed of the component parameters and weights, i.e. $\Theta = \{\Theta_c, \alpha_c\}_{c \in c^*}$;

$f(\Theta_c)$ are prior pdfs of unknown $\Theta_c \in \Theta_c^*$ parameterizing individual components $c \in c^*$;

$Di_\alpha(\kappa_0) \propto \prod_{c \in c^*} \alpha_c^{\kappa_{c;0}-1}$ is prior Dirichlet pdf [16] of component weights, determined by the vector statistic κ_0 with non-negative entries $\kappa_{c;0}$; α^* is its support; the symbol $\propto$ means that right hand side has to be normalized to the unit integral to get equality.

In the closed loop, the data items $d_t \equiv (d_{1;t}, \ldots, d_{\mathring{d};t})$ are multivariate, i.e. $\mathring{d} > 1$. Using the chain rule, the individual components can be decomposed into parameterized *factors*

$$f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t) = \prod_{i=1}^{\mathring{d}} f(d_{i;t}|d_{i+1;t}, \ldots, d_{\mathring{d};t}, \phi_{c_t;t-1}, \Theta_{c_t}, c_t) \equiv \prod_{i=1}^{\mathring{d}} f(d_{i;t}|\psi_{ic_t;t}, \Theta_{ic_t}, c_t), \tag{2}$$

with *regression vectors* $\psi_{ic_t;t} = [d_{i+1;t}, \ldots, d_{\mathring{d};t}, \phi'_{c_t;t-1}]'$. The decomposition allows to model jointly scalar entries $d_{i;t}$, called *factor outputs*, and to parameterize factors individually.

Moreover, only some parameters Θ_{ic} from Θ_c may be needed to describe the i-th factor in (2). The factorized description allows to treat factor outputs of a different nature, for instance, this enables joint modelling of mixed continuous and discrete data [15]. To simplify the presentation, the model description at component level (1) with $\psi_{c;t} = [u'_t, \phi'_{c;t-1}]'$, is used below.

Components (1) belonging to *exponential family* are considered

$$f(d_t|\psi_{c;t}, \Theta_c, c) = A(\Theta_c) \exp\left[B'(\Theta_c) C(\Psi_{c;t})\right], \quad \text{where} \tag{3}$$

$A(\Theta_c) \geq 0$ is a known scalar function of unknown component parameters Θ_c;

$B(\Theta_c)$ is a known vector function of unknown component parameters Θ_c;

$C(\Psi_{c;t})$ is a known vector function of *data vector* $\Psi_{c;t} \equiv [y'_{c;t}, \psi'_{c;t}]'$.

It should be stressed that the introduced model is the mixture model. This becomes obvious when unobserved pointers to components are excluded by marginalization

$$f(d_t|d(t-1), \Theta) = f(d_t|\phi_{1;t-1}, \ldots, \phi_{\mathring{c};t-1}, \Theta) = \sum_{c_t \in c^*} \alpha_{c_t} f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t). \tag{4}$$

The expression (4) extends usual finite mixtures as it consists of dynamic components. It is, however, still restricted to have constant component weights, i.e. $f(c_t = c|d(t-1), \Theta) = \alpha_c$. The assumption is used to get feasible recursive estimation and can be weakened so that slow changes are admitted by employing stabilized forgetting [13]. This extension is often sufficient as the pf $f(c_{t+1} = c|d(t), \Theta)$ forms bounded martingale with respect the σ-algebra generated by observations $d(t)$. Thus, it converges *almost surely (a.s.)* for $t \to \infty$, [17].

2.2. Recursive mixture estimation and prediction

Adaptive control is based on recursively estimated model. The approximate recursive quasi-Bayes mixture estimation [12], that slightly extends known techniques [6], is outlined here.

The conjugate prior pdfs on mixture parameters are used. The constructed approximate estimation preserves their form, i.e. the posterior pdfs have the form

$$f(\Theta|d(t)) \quad \propto \quad \prod_{c \in c^*} \alpha_c^{\kappa_{c;t}-1} A^{\nu_{c;t}}(\Theta_c) \exp\left[B'(\Theta_c) V_{c;t}\right], \quad \text{for } t \in t^* \cup \{0\}, \text{ where} \tag{5}$$

$A(\Theta_c)$, $B(\Theta_c)$ are functions from the definition of exponential family (3);

κ_t is vector statistics determining Dirichlet pdf on α;

$\nu_{c;t},\ c \in c^*$, are scalar non-negative statistics called the *number of degrees of freedom* of the respective components;

$V_{c;t},\ c \in c^*$, are statistics describing jointly with $\nu_{c;t}$ the posterior pdfs on parameters Θ_c.

The Bayes rule applied to the model (1) and pdfs (5) after measuring new data d_{t+1} gives

$$f(\Theta, c_{t+1}|d(t+1)) \propto \prod_{c \in c^*} \alpha_c^{\kappa_{c;t}+\delta_{c_{t+1}c}-1} A^{\nu_{c;t}+\delta_{c_{t+1}c}}(\Theta_c) \exp\left[B'(\Theta_c)(V_{c;t} + \delta_{c_{t+1}c} C(\Psi_{c;t+1})\right],$$

with $\delta_{c_{t+1}c}$ being Kronecker symbol that equals one for identical indices and is zero when they differ. $C(\Psi_{c;t+1})$ is the vector function from the definition of exponential family (3).

Marginalization would be the correct exclusion of the unobserved c_{t+1} for further estimation step. It spoils, however, the assumed feasible form of the posterior pdf. This forces the approximate replacement of the unknown $\delta_{c_{t+1}c}$ by its expectation, by the probability

$$w_{c;t+1} \equiv f(c_{t+1} = c|d(t+1)) = \frac{f(d_{t+1}|d(t), c)\kappa_{c;t}}{\sum_{c \in c^*} f(d_{t+1}|d(t), c)\kappa_{c;t}},$$

which is determined by the *predictive pdf* within c-th component

$$\begin{aligned} f(d_{t+1}|d(t), c) &= \int f(d_{t+1}|d(t), \Theta_c, c) f(\Theta_c|d(t), c)\, d\Theta_c = \\ &= \frac{\int A^{\nu_{c;t}+1}(\Theta_c) \exp\left[B'(\Theta_c)(V_{c;t} + C(\Psi_{c;t+1}))\right]\, d\Theta_c}{\int A^{\nu_{c;t}}(\Theta_c) \exp\left[B'(\Theta_c) V_{c;t}\right]\, d\Theta_c}. \end{aligned} \tag{6}$$

2.3. Recursive estimation with forgetting

Quality of the designed adaptive controller depends on validity of the model learned. To respect this, the model should not only reflect and preserve all significant time-invariant relationships of real system, but also allow to track slow changes in system dynamics. Thus, the case of slowly varying parameters, that admits time variations of Θ_t while assuming $\Theta_{t+1} \approx \Theta_t$, is to be supposed.

The *stabilized forgetting* technique, for the detailed description see [13], is employed in the work. According to that, the updating at the next time step is chosen as a compromise between pdf $f(\Theta_{t+1} = \Theta_t = \Theta|d(t))$, which does not assume parameter changes, and an alternative pdf $f_A(\Theta_{t+1} = \Theta|d(t))$ that does. The degree of a compromise is controlled by parameter $\lambda \in (0, 1]$, which expresses the belief that no parameter changes happened after measuring d_t and before processing d_{t+1}. Thus, the time updating step becomes

$$f(\Theta_{t+1} = \Theta|d(t)) \propto \left[f(\Theta_{t+1} = \Theta_t = \Theta|d(t))\right]^{\lambda} \left[f_A(\Theta_{t+1} = \Theta|d(t))\right]^{1-\lambda}. \tag{7}$$

Quasi-Bayes estimation with stabilized forgetting, applied to the mixture model and its alternative (marked by $_A$) belonging to the exponential family, reduces to the simple recursion

$$\begin{aligned} V_{c;t} &= \lambda \left(V_{c;t-1} + w_{c;t} C(\Psi_{c;t})\right) + (1-\lambda) V_{A\,c;t-1}, \\ \nu_{c;t} &= \lambda \left(\nu_{c;t-1} + w_{c;t}\right) + (1-\lambda)\nu_{A\,c;t-1}, \\ \kappa_{c;t} &= \lambda \left(\kappa_{c;t-1} + w_{c;t}\right) + (1-\lambda)\kappa_{A\,c;t-1}. \end{aligned} \tag{8}$$

The formulas (8) are equivalent to a weighted least squares for normal ARX components [12].

3. CONTROL DESIGN

The joint pdf $f(d(\mathring{t})) = \prod_{t \in t^*} f(d_t|d(t-1))$ describes closed-loop behavior reflected in data $d(\mathring{t})$ till horizon $\mathring{t}$. Control design tries to make this behavior close to a desired one under some restrictions. Both desired behavior and restrictions are given externally by the user. The design is performed for the known mixture model. The standard substitution of unknown parameters by their newest point estimates then provides the constructed adaptive controller.

3.1. Classification of control tasks

Discussion of possible optimization extent uses the following expansion of the model (4)

$$f(d_t|d(t-1)) = \sum_{c_t \in c^*} \alpha_{c_t} f(y_t|\psi_{c_t;t}, c_t) f(u_t|\phi_{c_t;t-1}, c_t), \quad \psi_{c_t;t} \equiv \begin{bmatrix} u_t \\ \phi_{c_t;t-1} \end{bmatrix} \tag{9}$$

that distinguishes the controlled system and controller. Notation is simplified by omitting of known parameters. Further on, to distinguish the optimized pdfs from the pdfs resulting from estimation, the *superscript* $^{[E]}$ is attached to the latter one.

The component weights $^{[E]}\alpha_{c_t}$ determine probability of pointers c_t to the active components. The design can generally replace them by the optimized probabilities $f(c_t|d(t-1))$.

The pdfs $^{[E]}f(y_t|u_t, d(t-1), c_t) = {}^{[E]}f(y_t|\psi_{c_t;t}, c_t)$ predicting outputs y_t reflect reactions of the controlled system on the past data. Controller influences them through input values only.

The pdfs $^{[E]}f(u_t|d(t-1), c_t) = {}^{[E]}f(u_t|\phi_{c_t;t-1}, c_t)$ predicting inputs u_t describe the controller used during estimation. It can be replaced by another controller $f(u_t|d(t-1), c_t)$.

Optimization of probabilities $f(c_t|d(t-1))$, selecting active pointers c_t, is referred as *academic design*. Resulting feedback switches between $\mathring{c}$ closed loops described by the fixed components.

Optimization of individual controllers $f(u_t|d(t-1), c_t)$ without changing the component weights $^{[E]}\alpha_c$ is referred to as *industrial design*. It has to be used whenever the weights have an objective meaning that cannot be influenced by the optional controllers used.

Optimization of both of probabilities $f(c_t|d(t-1))$ selecting active pointers c_t and individual controllers $f(u_t|d(t-1), c_t)$ is referred as *simultaneous design* and optimizes the joint pdf $f(c_t, u_t|d(t-1)) \equiv f(c_t|d(t-1)) f(u_t|d(t-1), c_t)$. It takes the model (9) as an approximation of a non-linear dynamic model and searches for the proper system inputs while respecting changes of operation mode.

Approximate designs are elaborated for all types. The paper [10] provides an example of the design classified here as academic. The industrial design with quadratic loss and normal components is described in [18]. Here, the most general simultaneous fully probabilistic control design [19] applied to the known mixture model (9) is presented.

3.2. Simultaneous fully probabilistic design

The supposed simultaneous design selects among joint strategies

$$\{f(c_t, u_t|d(t-1))\}_{t \in t^*} \tag{10}$$

the optimal joint strategy. It is distinguished by the *superscript* $^{[I]}$. Any strategy of the form (10) complemented by pdfs resulted from estimation, $\left\{ {}^{[E]}f(y_t|u_t, d(t-1), c_t) \right\}_{t \in t^*}$, and reflecting controlled system dynamics, describe the optimized closed loop

$$f(d(\mathring{t}), c(\mathring{t})) = \prod_{t \in t^*} f(d_t, c_t|d(t-1)) = \prod_{t \in t^*} {}^{[E]}f(y_t|u_t, d(t-1), c_t) f(c_t, u_t|d(t-1)). \tag{11}$$

Adopted fully probabilistic design expresses the desired behavior through the *user ideal pdf*

$${}^{[U]}f(d(\mathring{t}), c(\mathring{t})) = \prod_{t \in t^*} \underbrace{{}^{[U]}f(y_t|u_t, d(t-1), c_t)\, {}^{[U]}f(u_t|d(t-1), c_t)}_{{}^{[U]}f(d_t|d(t-1), c_t)} {}^{[U]}f(c_t|d(t-1)). \tag{12}$$

Often, the *observed behavior* is only of a direct user's interest. Then, the user ideal pdf ${}^{[U]}f(d_t|d(t-1),c_t)$ is independent of c_t. It is assumed here, i.e.

$$ {}^{[U]}f(y_t|u_t,d(t-1),c_t) \equiv {}^{[U]}f(y_t|u_t,d(t-1)), \quad {}^{[U]}f(u_t|d(t-1),c_t) \equiv {}^{[U]}f(u_t|d(t-1)). \quad (13) $$

The first equality in (13) says where the system output should be with a high probability. Typically, its mean value is required to be equal to a given set point and covariance to the (non-reducible) covariance of innovations. Similarly, the second equality in (13) expresses where the system input should be. Hard restrictions on inputs are naturally respected by restricting the support of the pdf ${}^{[U]}f(u_t|d(t-1))$ to the set determined by these restrictions. The pf ${}^{[U]}f(c_t|d(t-1))$ is a tuning knob of the design as far their choice allows to set design preferences on various components. The following approach applies generally to any quantity: if a part of ${}^{[U]}f(\cdot)$ is out of user's interests, the designer can either use it as a tuning knob or fix it as uniform pdf.

The probabilistic design selects the optional pdfs, describing the system inputs and pointers, to make the joint pdf $f(d(\mathring{t}),c(\mathring{t}))$ as close as possible to the chosen user ideal pdf ${}^{[U]}f(d(\mathring{t}),c(\mathring{t}))$. Closeness of these pdfs is measured by the Kullback-Leibler (KL) distance [14]

$$ \begin{aligned} \mathcal{D}\left({}^{[I]}f||{}^{[U]}f\right) &\equiv \sum_{c(t)\in c^*(t)} \int {}^{[I]}f(d(\mathring{t}),c(\mathring{t})) \ln\left(\frac{{}^{[I]}f(d(\mathring{t}),c(\mathring{t}))}{{}^{[U]}f(d(\mathring{t}),c(\mathring{t}))}\right) dd(\mathring{t}) = \qquad (14) \\ &= \sum_{t\in t^*} \sum_{c(t-1)\in c^*(t-1)} \int f(d(t-1),c(t-1)) \times \\ &\times \left\{ \sum_{c_t\in c^*} \int {}^{[I]}f(c_t,u_t|d(t-1)) \left[\ln\left(\frac{{}^{[I]}f(c_t,u_t|d(t-1))}{{}^{[U]}f(c_t,u_t|d(t-1))}\right) + \right.\right. \\ &+ \left.\left. \underbrace{\int f(y_t|u_t,d(t-1),c_t) \ln\left(\frac{{}^{[I]}f(y_t|u_t,d(t-1),c_t)}{{}^{[U]}f(y_t|u_t,d(t-1),c_t)}\right) dy_t \Bigg]}_{\omega(c_t,u_t,d(t-1))} du_t \right\} dd(t-1). \end{aligned} $$

with the property $\mathcal{D}\left({}^{[I]}f||{}^{[U]}f\right) \geq 0, \;\; \mathcal{D}\left({}^{[I]}f||{}^{[U]}f\right) = 0 \;\Leftrightarrow\; {}^{[I]}f = {}^{[U]}f,$ a.s. (15)

The property (15) used in the optimization assures that the function $\omega(c_t,u_t,d(t-1)) \geq 0$. Optimization-preparing manipulations in (14) use the chain rule for pdfs, Fubini theorem on multiple integration [20] and marginalization of pdfs.

Proposition 3.1 (Simultaneous design minimizing the KL distance) *The simultaneous strategy, selected among strategies (10) and minimizing the KL distance (14) of the closed-loop description (11) to the user ideal pdf (12), is described by the following formulas, solved for $t = \mathring{t}, \mathring{t}-1, \ldots, 1$,*

$$ \begin{aligned} {}^{[I]}f(c_t,u_t|d(t-1)) &\propto {}^{[U]}f(c_t,u_t|d(t-1)) \exp[-\omega_\gamma(c_t,u_t,d(t-1))], \quad \textit{where} \qquad (16) \\ \omega_\gamma(c_t,u_t,d(t-1)) &\equiv \int {}^{[E]}f(y_t|u_t,d(t-1),c_t) \ln\left[\frac{{}^{[E]}f(y_t|u_t,d(t-1),c_t)}{\gamma(d(t))\, {}^{[U]}f(y_t|u_t,d(t-1))}\right] dy_t \\ \gamma(d(\mathring{t})) &\equiv 1 \\ \gamma(d(t-1)) &\equiv \sum_{c_t\in c^*} \int {}^{[U]}f(c_t,u_t|d(t-1)) \exp[-\omega_\gamma(c_t,u_t,d(t-1))]\, du_t. \end{aligned} $$

Systematic replacement of $\omega_\gamma(c_t, u_t, d(t-1))$ *by* $\omega_{\bar{\gamma}}(c_t, u_t, d(t-1))$, *with* $\bar{\gamma}(d(\mathring{t})) \equiv \gamma(d(\mathring{t})) = 1$ *and with the following definition of* $\bar{\gamma}(d(t-1))$ *for* $t < \mathring{t}$

$$\bar{\gamma}(d(t-1)) \equiv \int {}^{[U]}f(u_t|d(t-1)) \exp\left[-\sum_{c_t \in c^*} {}^{[U]}f(c_t|d(t-1))\omega_{\bar{\gamma}}(c_t, u_t, d(t-1))\right] du_t, \quad (17)$$

gives the strategy minimizing an upper bound on the KL distance. The controller samples system inputs from ${}^{[I]}f(u_t|d(t-1)) = \sum_{c_t \in c^*} {}^{[I]}f(c_t, u_t|d(t-1))$.

Proof: The result is implied by backward induction (dynamic programming). For $t = \mathring{t}$ and arbitrary $d(t-1), c(t-1)$, the non-negativity of $f(d(t-1), c(t-1))$ implies that the minimization of (14) with respect to the optional pdf $f(c_t, u_t|d(t-1))$ concerns only the term

$$\sum_{c_t \in c^*} \int f(c_t, u_t|d(t-1)) \left[\ln\left(\frac{f(c_t, u_t|d(t-1))}{{}^{[U]}f(c_t, u_t|d(t-1))}\right) + \omega_\gamma(c_t, u_t, d(t-1))\right] du_t =$$
$$\sum_{c_t \in c^*} \int f(c_t, u_t|d(t-1)) \ln\left(\frac{\gamma(d-1)f(c_t, u_t|d(t-1))}{{}^{[U]}f(c_t, u_t|d(t-1)) \exp[-\omega_\gamma(c_t, u_t, d(t-1))]}\right) du_t -$$
$$\ln(\gamma(d-1)).$$

This equality is implied by
– exploiting equality $\omega(c_t, u_t, d(t-1))) = \omega_\gamma(c_t, u_t, d(t-1)))$ for $\gamma(d(t)) = 1$, see (14),
– adding and subtracting the normalizing term $\ln(\gamma(d-1))$,
– noticing independence of $\ln(\gamma(d-1))$ on c_t, u_t over which the conditional expectation is taken.

The listed properties of the KL distance (15) imply that the minimizing pdf ${}^{[I]}f(\cdot)$ has the claimed form (16). The partial minimum transferred to the further minimization step (Bellman function) is $-\ln(\gamma(d-1))$. It means, however, that we arrive at the formally identical situation as for $t = \mathring{t}$ but with a non-unit function $\gamma(d(t-1))$. This verifies that the strategy is optimal.

The function $\gamma(d-1)$ has the form of a weighted arithmetic mean. It is bounded from below by the function that uses the corresponding weighted geometric mean. This gives, more numerically feasible, upper bound on the Bellman function $-\ln(\gamma(d(t-1)))$

System inputs alone can be fed into the controlled system so that they have to be sampled from the marginal pdf derived from the optimal strategy ${}^{[I]}f(c_t, u_t|d(t-1))$ found. □

4. APPLICATION TO MIXTURES OF NORMAL ARX MODELS

Here, simultaneous design is specialized to the mixtures with normal components and the normal user ideal pdf. Combination of an approximation of the type (17) and the simultaneous design extends classical linear-quadratic design with the resulting optimization reduced to numerically tractable manipulations on quadratic forms. This design, complemented by mixture modelling capable of approximating a wide range of non-linear controlled systems, provides practically applicable and effective control algorithms. The fact, that the considered normal mixtures generalize popular neural networks based on radial-basis functions [2], confirms the approximation power of the adopted modelling.

4.1. Design conditions

The mixture model (4) with normal components written in the matrix form is

$$\begin{aligned} f(d_t|d(t-1),\Theta_c,c) &= f(y_t|u_t,d(t-1),\Theta_c,c)f(u_t|d(t-1),\Theta_c,c) \equiv \\ & \mathcal{N}_{y_t}(\theta_c^{y'}\psi_t, r_c^y)\mathcal{N}_{u_t}(\theta_c^{u'}\phi_{t-1}, r_c) \\ \mathcal{N}_d(\bar{d},r) &\equiv |2\pi r|^{-0.5}\exp\left[-0.5(d-\bar{d})'r^{-1}(d-\bar{d})\right], \quad \psi_t \equiv [u_t', \phi_{t-1}']'. \end{aligned} \tag{18}$$

The parameter Θ_c consists of matrix regression coefficients θ_c^y, θ_c^u and covariance matrices r_c^y, r_c^u of appropriate dimensions. The regression vector $\psi_{c;t}$, used in the prediction of the system output y_t, attaches the current system input u_t to the phase state vector $\phi_{t-1} \equiv [d_{t-1}', \ldots, d_{t-\partial}', 1]'$ with the delayed data items. If need be, regression coefficients $\theta_c^{y'}$, $\theta_c^{u'}$ are complemented by zeros so that the common state vector for all component can be considered.

The factors determining the user ideal pdf ${}^{[U]}f(d(\mathring{t}), c(\mathring{t}))$ (12) are supposed to be normal

$$ {}^{[U]}f(d_t|d(t-1),c_t) \equiv \overbrace{\mathcal{N}_{y_t}\left({}^{[U]}\theta^{y'}\psi_t,\ {}^{[U]}r^y\right)}^{{}^{[U]}f(y_t|u_t,d(t-1),c_t)\equiv {}^{[U]}f(y_t|u_t,d(t-1))} \times \overbrace{\mathcal{N}_{u_t}\left({}^{[U]}\theta^{u'}\psi_t,\ {}^{[U]}r^u\right)}^{{}^{[U]}f(u_t|d(t-1),c_t)\equiv {}^{[U]}f(u_t|d(t-1))} \tag{19}$$

and ${}^{[U]}f(c_t|d(t-1)) \propto {}^{[U]}f(c_t)$ with ${}^{[U]}f(c_t)$ is an optional pf on c^* allowing to suppress a priori undesirable components. The over-braced expressions in (19) emphasize that the user ideal pdf is independent of the components, see (13).

4.2. Simultaneous design for normal mixture

The conditions listed in Section 4.1 and application of Proposition 3.1 give simultaneous control design for normal mixtures. It uses the following matrices and simple relationships allowing evaluation of *expectation* $\mathcal{E}$, and *covariance* cov, in a compact way.

$$\begin{aligned} \phi_t &\equiv \begin{bmatrix} I_{\mathring{\phi}-1} & 0_{\mathring{\phi}-1,\mathring{d}} & 0_{\mathring{\phi}-1,1} \\ 0_{1,\mathring{\phi}-1} & 0_{1,\mathring{d}} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ \psi_t \end{bmatrix} \equiv \mathcal{K} \begin{bmatrix} y_t \\ \psi_t \end{bmatrix}, \ I_k \equiv (k,k)\text{-unit matrix} \\ 0_{k,l} &\equiv (k,l)\text{-zero matrix}, \ \mathcal{E}\left[\begin{bmatrix} y_t \\ \psi_t \end{bmatrix} |\psi_t, c\right] = \begin{bmatrix} \theta_c^{y'} \\ \mathcal{K} \end{bmatrix} \psi_t \equiv \mathcal{K}_c\psi_t. \end{aligned} \tag{20}$$

The quadratic forms $x'Qx$, with symmetric kernels Q, use the splitting of the kernel according to the used vector split $x = [a', b']'$, i.e.

$$Q \equiv \begin{bmatrix} Q^a & Q^{ab'} \\ Q^{ab} & Q^b \end{bmatrix}. \ \text{With that,} \ \mathcal{E}[x'Qx|b] = \mathcal{E}'[x|b]Q\mathcal{E}[x|b] + \mathrm{tr}\left[\mathrm{cov}(a|b)Q^a\right] \tag{21}$$

Proposition 4.1 (Simultaneous control design for normal mixtures) *Let us make the simultaneous control design selecting strategy (10) for the system described by the normal mixture with components (18). The parameters* $\Theta_c \equiv \{\theta_c^y, r_c^y, \theta_c^u, r_c^u\}, c \in c^*$ *of the mixture model as well those determining the user ideal pdf (19)* ${}^{[U]}\Theta \equiv \left\{{}^{[U]}\theta^y, {}^{[U]}r^y, {}^{[U]}\theta^u, {}^{[U]}r^u\right\}$ *are* *known (estimated). Let us define, for* $t = \mathring{t}, \mathring{t}-1, \ldots, 1$ *and* $Q_{\mathring{t}} = 0_{\mathring{\phi},\mathring{\phi}}$, *the strategy*

$$
{}^{[I]}f(c_t, u_t|\phi_{t-1}) \propto \alpha_{c_t;t} \exp\left[-0.5\psi_t' Q_{c_t;t-1}\psi_t\right], \quad \textit{where} \tag{22}
$$

$$
\begin{aligned}
\alpha_{c_t;t} &\propto {}^{[U]}f(c_t)\exp(-0.5k_{c_t;t-1}), \quad k_{c_t;t-1} = \ln\left(\frac{\left|{}^{[U]}r^y\right|}{|r^y_{c_t}|}\right) - \mathring{y} + \mathrm{tr}\left[r^y_{c_t}\tilde{Q}^y_t\right] \\
\tilde{Q}_t &\equiv \mathcal{K}'Q_t\mathcal{K} + \left[-I_{\mathring{y}}, {}^{[U]}\theta^{y'}\right]'\left({}^{[U]}r^y\right)^{-1}\left[-I_{\mathring{y}}, {}^{[U]}\theta^{y'}\right] \\
Q_{c_t;t-1} &= \mathcal{K}'_{c_t}\tilde{Q}_t\mathcal{K}_{c_t} + \left[-I_{\mathring{u}}, {}^{[U]}\theta^u\right]'\left({}^{[U]}r^u\right)^{-1}\left[-I_{\mathring{u}}, {}^{[U]}\theta^u\right], \quad \alpha_{c_t;t-1} \propto \alpha_{c_t;t}\left|Q^u_{c_t;t-1}\right|^{0.5} \\
Q_{t-1} &= \sum_{c_t\in c^*}\alpha_{c_t;t-1}\left[Q^{\phi}_{c_t;t-1} - Q^{u\phi}_{c_t;t-1}\left(Q^u_{c_t;t-1}\right)^{-1}Q^{u\phi'}_{c_t;t-1}\right].
\end{aligned}
$$

Then, this strategy minimizes the KL distance (14) of the closed-loop description (11) to the user ideal pdf (12) specialized to the normal case by (19) for the control horizon equal one. For other horizons, it minimizes the upper bound implied by the replacement of weighted arithmetic mean in the correct Bellman function by its geometric counterpart.

Proof: We apply Proposition 3.1. Let us assume that $\gamma(d(t)) = \exp[-0.5\phi_t' Q_t \phi_t]$ with a data independent positive semi-definite kernel Q_t. This assumption holds for $\mathring{t}$ with $Q_{\mathring{t}} = 0$. The following backward induction verifies a generic step and provides the guideline for updating $\gamma(d(t))$. For the considered normal components, the function $\omega_\gamma(c_t, u_t, d(t-1))$, (16), gets the form $2\omega_\gamma(c_t, u_t, d(t-1)) =$

$$
\underbrace{\ln\left(\frac{\left|{}^{[U]}r^y\right|}{|r^y_{c_t}|}\right) - \mathring{y}}_{k_{c_t;t}} + \mathcal{E}\left[\phi_t' Q_t\phi_t + \left(y_t - {}^{[U]}\theta^{y'}\psi_t)\right)'\left({}^{[U]}r^y\right)^{-1}\left(y_t - {}^{[U]}\theta^{y'}\psi_t)\right)|c_t,\psi_t\right] \underbrace{=}_{(20)}
$$

$$
k_{c_t;t} + \mathcal{E}\left[[y_t', \psi_t']\underbrace{\left(\mathcal{K}'Q_t\mathcal{K} + \left[-I_{\mathring{y}}, {}^{[U]}\theta^{y'}\right]'\left({}^{[U]}r^y\right)^{-1}\left[-I_{\mathring{y}}, {}^{[U]}\theta^{y'}\right]\right)}_{\tilde{Q}_t}[y_t', \psi_t']'|c_t,\psi_t\right] \underbrace{\equiv}_{(20),(21)}
$$

$$
\underbrace{k_{c_t;t} + \mathrm{tr}\left[r^y_{c_t}\tilde{Q}^y_t\right]}_{k_{c_t;t-1}} + \psi_t'\underbrace{\mathcal{K}'_{c_t}\tilde{Q}_t\mathcal{K}_{c_t}}_{Q_{c_t;t}}\psi_t.
$$

This form and normality of the pdf ${}^{[U]}f(u_t|d(t-1))$ give the optimizing pdf (16)

$$
{}^{[I]}f(c_t, u_t|\phi_{t-1}) \propto
$$

$$
\underbrace{{}^{[U]}f(c_t)\exp(-0.5k_{c_t;t-1})}_{\alpha_{c_t;t}}\exp\left[-0.5\psi_t'\underbrace{\left(Q_{c_t;t} + \left[-I_{\mathring{u}}, {}^{[U]}\theta^u\right]'\left({}^{[U]}r^u\right)^{-1}\left[-I_{\mathring{u}}, {}^{[U]}\theta^u\right]\right)}_{Q_{c_t;t-1}}\psi_t\right].
$$

The normalizing factor whose negative logarithm is transferred to the next step is

$$\begin{aligned}\gamma(d(t-1)) &= \left|2\pi\, {}^{[U]}r^{u}\right|^{-0.5} \sum_{c_t\in c^*} \alpha_{c_t;t} \int \exp\left[-0.5\psi_t' Q_{c_t;t-1}\psi_t\right] du_t = \sum_{c_t\in c^*} \underbrace{\alpha_{c_t;t} |Q^{u}_{c_t;t-1}|^{0.5}}_{\propto \alpha_{c_t;t-1}} \times \\ & \left|{}^{[U]}r^{u}\right|^{-0.5} \exp\Bigg[-0.5\phi_{t-1}' \underbrace{\left(Q^{\phi}_{c_t;t-1} - Q^{u\phi}_{c_t;t-1}\left(Q^{u}_{c_t;t-1}\right)^{-1} Q^{u\phi'}_{c_t;t-1}\right)}_{\tilde{Q}_{c_t;t-1}} \phi_{t-1}\Bigg] \geq \\ & \mathit{constant} \times \exp\Bigg[-0.5\phi_{t-1}' \underbrace{\left(\sum_{c_t\in c^*} \alpha_{c_t;t-1}\tilde{Q}_{c_t;t-1}\right)}_{Q_{t-1}} \phi_{t-1}\Bigg] \equiv \mathit{constant} \times \bar{\gamma}(d(t-1)).\end{aligned}$$

□

5. ILLUSTRATIVE EXAMPLE

A simple nonlinear system has been chosen to demonstrate the simultaneous design. The system is governed by the following non-linear equations

$$\left.\begin{array}{lcll} y_{1;t} &=& \frac{306}{300+u_t} y_{1;t-1} + e_{1;t}, & e_{1;t} \sim \mathcal{N}_{e_{1;t}}(0,1) \\ y_{2;t} &=& u_t y_{1;t-1} + e_{2;t}, & e_{2;t} \sim \mathcal{N}_{e_{2;t}}(0,0.001) \\ u_t &=& u_{t-1} + e_{3;t}, & e_{3;t} \sim \mathcal{N}_{e_{3;t}}(0,10) \end{array}\right\} \Leftrightarrow \tag{23}$$

$$d_t = \begin{bmatrix} y_{1;t} \\ y_{2;t} \\ u_t \end{bmatrix} \sim \mathcal{N}_{y_{1;t}}\left(\frac{306}{300+u_t} y_{1;t-1}, 1\right) \mathcal{N}_{y_{2;t}}\left(u_t y_{1;t-1}, 0.001\right) \mathcal{N}_{u_t}\left(u_{t-1}, 10\right).$$

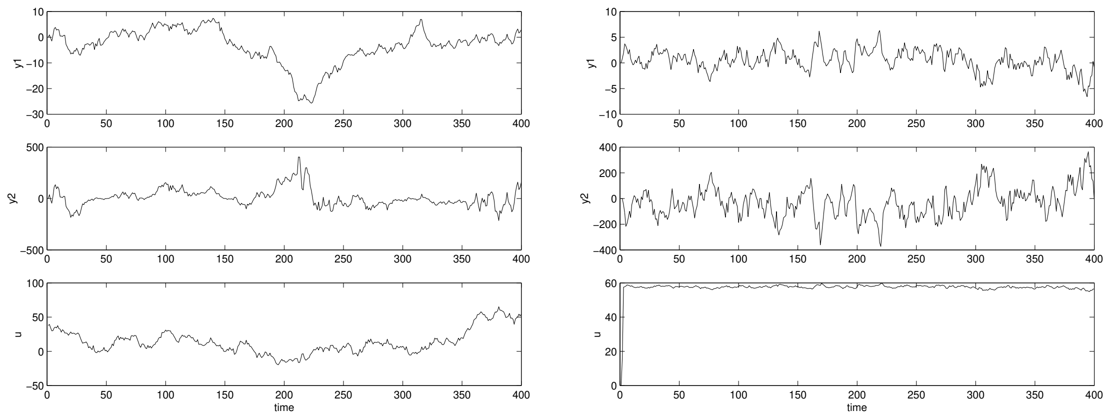
The first entry of data record d_t evolves according non-linear stochastic difference equation stimulated by the system input $u_t = d_{3;t}$. The nominal evolution of the system is documented in Figure 1a. The first equation in (23) implies that the behavior of the first variable $y_{1;t}$ becomes unstable for $u_t < 6$. As the second output $y_{2;t} \equiv d_{2;t}$ represents a noisy product of the output $y_{1;t-1}$ and input u_t, its value reflects a balance of their values.

Let us suppose that the user needs to have $y_{1;t}$ and $y_{2;t}$ close to zero while u_t should be near the value 75. These requirements are expressed by the user ideal pdf in the form

$${}^{[U]}f(d_t|d(t-1)) = \mathcal{N}_{y_{1;t}}\left(0, {}^{[U]}r_1^{y}\right) \mathcal{N}_{y_{2;t}}\left(0, {}^{[U]}r_2^{y}\right) \mathcal{N}_{u_t}\left(75, {}^{[U]}r^{u}\right).$$

Thus, controller, that makes data record $d_t = [y_{1;t}, y_{2;t}, u_t]'$ behave as static component with independent entries and parameters ${}^{[U]}\theta^{y} = [0,\ 0]'$, ${}^{[U]}\theta^{u} = 75$, is to be designed. The variances ${}^{[U]}r \equiv \left[{}^{[U]}r_1^{y},\ {}^{[U]}r_2^{y},\ {}^{[U]}r^{u}\right]$ express the acceptable deviations of variables from their means.

The mixture was identified from the data simulated by (23). The nominal stimulation by inputs as given in 23 was found insufficient to cover possible modes of the system behavior. Thus, an active estimation experiment was performed with u_t randomly switched between several fixed values $u_t \in [0, 6, 18, 30, 45, 60, 75, 90, 105]$. Uniform user pf ${}^{[U]}f(c_t)$ (12) has been assigned to all eight components $\mathring{c} = 8$ of the identified mixture.



a. Nominal trajectories of the system (23) b. Optimum trajectories for ${}^{[U]}r = [1\ 3\ 1]$

Figure 1. Samples of experimental results

Simple analysis of the system and the criterion used shows that the best behavior of the first output $y_{1;t}$ is achieved for higher values of u_t. This is also reflected in the required mean of u_t. On the other hand a large u_t causes $y_{2;t}$ to be high as well, which contradicts the user requirement on it. Optimum behavior is reached as a compromise of all three requirements and depends on the specific choice of the vector ${}^{[U]}r$. An example of such a compromise for ${}^{[U]}r = [1\ 3\ 1]$ is shown in Figure 1b. The systematic offset among u_t and its desired value (75) is its clear manifestation.

6. CONCLUDING REMARKS

Recursively applicable mixture estimation and so called simultaneous fully probabilistic control design for known mixture are presented. Their straightforward certainty-equivalence combination provides a novel class of adaptive controllers suitable for a wide class of control systems. At the same time, it opens a range of problems that should be solved to allow a routine use of such controllers. Let us list some of them.

- The problem of the experiment design vital for any estimation is even more urgent when dealing with multiple models.
- The estimation works with mixture model of a fixed structure and relies on good prior pdfs. This initialization problem has to be solved in off-line mode and existing solutions like [21] are far from being fully satisfactory.
- Constant component weights α_c restrict applicability range. There is a long way from available partial extensions based on hidden Markov models [22] to a real solution.
- The multi-step-ahead design relies on minimization of the upper bound on the KL distance. Detailed analysis of the cost paid for this feasibility as well as consideration of other possible approximations are needed. In adaptive context, the so-called iteration-spread-in-time strategy seems to be an attractive alternative [23]. It usually needs one-stage-ahead optimization after each estimation step.

Even this restricted list covers rather difficult problems. In spite of them, we believe that the discussed approach represents a promising direction with a high application potential [24] and a lot of intellectual challenges.

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