CONTROL OF A SIMPLE NON-LINEAR SYSTEM USING PROBABILISTIC MIXTURES

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Abstract: The paper shows application of a new mixture-based approach to control on a simple non-linear system. The general algorithm can be described in 3 steps. First the probabilistic mixture model must be obtained from the process data. Second, the desired values on the system must be convert to the form of probabilistic density function (pdf). Then the approximation of fully probabilistic design is performed, which results into the pdf describing the control variables. Mixture-based control can be interpreted as constructing several linear controllers and then switching them during the process. The successful usage of the mentioned approach by controlling a SIMULINK model is presented. The result is compared with control of the same model with an adaptive LQ controller.

Keywords: control, probabilistic mixtures, non-linear model

1. INTRODUCTION

The paper deals with control of nonlinear system. The control of nonlinear systems is very difficult task. There exist a lot of approaches with relatively good results. For example adaptive LQ controller (Bobál *et al.* 1998). Theory related to general non-linear control can be found e.g. in (Krstić *et al.* 1995). Similar approaches as the one discussed below can be found for example in (Anderson *et al.* 2001) or (Hespanha *et al.* 2001).

The headstone of each approach to the system control is to build a suitable model of the system, which is close to the reality, but not very complex so that it can be in a certain manner analytically solvable. Project (Kárný *et al.* 2003), to which this work contributes, deals with non standard dynamic mixture models and subsequent fully probabilistic design (Kárný 1996). This promising approach is quite new and therefore has very few applications on realistic systems. The aim of this paper is to verify, on a nonlinear realistic model, that the mentioned approach has a chance to be successful in real cases.

The mixture model can be interpreted as a switching of linear regression models. Estimation of parameters of such a model is very difficult task, which is described e.g. in (Nedoma *et al.* 2000). With the estimated model, the approximation of the fully probabilistic design is performed. It can be interpreted as constructing of ideal controller for each linear regression model and then switching of these controllers. The mixture model as well as the result of design can be updated during the control process, thereby an adaptive version of control is achieved. Here, however, a non-adaptive version is considered.

Next section sketches the main ideas of control algorithm. Then the example of controlling helicopter model is presented.

2. BASIS OF THE APPROACH

The aim of this section is to outline the basic ideas of the mixture-based control. At first, the probabilistic mixture model will be defined. Then the main control ideas will be explained.

First of all, some basic notion and notation must be specified.

f probability density function (pdf) The meaning of the pdf is given through its argument.
x^* set of all possible values of quantity x
$\mathring{x}$ number of entries in the vector x or elements in x^*
$d(t) = (d_1, \cdots, d_t)$
Kullback-Leibler distance $\mathcal{D}(f||g) = \int f \ln\left(\frac{f}{g}\right)$ measures well proximity of a pair of pdfs f, g.
chain rule

$$f(\alpha, \beta|\gamma) = f(\alpha|\beta, \gamma) f(\beta|\gamma) \tag{1}$$

2.1 *Mixture model*

The quantities related to the system are called channels. The channels, which can not be influenced directly are called innovations, while the others are called actions. Let's denote the values of channels in time t as d_t. Then d_t has the form $d_t = (\Delta_t, u_{o;t})$=(innovations, actions).

The general probabilistic model of the system is based on pdf on all trajectories of the system from time 1 up to a horizon $\mathring{t}$. Formally: we are looking for a joint pdf $f(d_1, \cdots, d_{\mathring{t}}) \equiv f(d(\mathring{t}))$. This pdf must be assumed in a certain form. The following discussion describes the adopted dynamic mixture.

The pdf $f(d(\mathring{t}))$ can be factorized according to the chain rule (1)

$$f(d(\mathring{t})) = \prod_{t \in t^*} f(d_t|d(t-1)).$$

It is reasonable to consider, that the system is not influenced by too old data. Thus, the pdf $f(d_t|d(t-1))$ needs not to be conditioned with all historical data $d(t-1)$, but just with a finite dimensional vector $\phi_{t-1} \subset d(t-1)$.

Using this assumption , the model gets the form

$$f(d(\mathring{t})) = \prod_{t \in t^*} f(d_t|\phi_{t-1}).$$

The vector ϕ_t is an observable state. We deal with models containing ϕ_t, that can be evaluated in a recursive manner $\phi_t = \Phi(\phi_{t-1}, d_t)$ with the known function Φ.

The model of particular process is obtained by identification of model parameterized by an unknown parameter Θ

$$f(d(\mathring{t})|\Theta) \equiv \prod_{t \in t^*} f(d_t|\phi_{t-1}, \Theta) \; .$$

Hypothesis on existence of several operation modes implies that we have to search for multiple-mode models. It is known that multiple-mode probabilistic models can almost always be approximated by a finite mixture of uni-modal models, called component*s*.

In this way, we set the final form of the model.

$$f(d(\mathring{t})|\Theta) \equiv \prod_{t \in t^*} f(d_t|\phi_{t-1}, \Theta)$$

where each $f(d_t|\phi_{t-1}, \Theta)$ is a finite mixture

$$f(d_t|\phi_{t-1}, \Theta) \equiv \sum_{c \in c^*} \alpha_c f(d_t|\phi_{c;t-1}, \Theta_c, c),$$

$$c^* = \{1, \ldots, \mathring{c}\}, \; \mathring{c} < \infty$$

$$\phi_{c;t} = \Phi_c(\phi_{c;t-1}, d_t)$$

$$\Theta \in \Theta^* \equiv \left\{ \{\Theta_c \in \Theta_c^*\}_{c \in c^*}, \; \alpha \equiv [\alpha_1, \ldots, \alpha_{\mathring{c}}] \in \alpha^* \right.$$

$$\left. \alpha^* \equiv \{\alpha_c \geq 0, \; \sum_{c \in c^*} \alpha_c = 1\} \right\}$$

$\phi_{c;t}$ is a subvector of vector $[\phi_t, 1]$ called state vector of c-th component.
$f(d_t|\phi_{c;t-1}, \Theta_c, c)$ is called parameterized component of a mixture
α_c is weight of the parameterized component.

For completing the description of the model, the functional form of the parameterized component must be specified. The common Gaussian distribution is used.

$$f(d_t|\phi_{c;t-1}, \Theta_c, c) = \mathcal{N}_{d_t}(\theta_c' \phi_{c;t-1}, r_c) \tag{2}$$

In this case the parameter $\Theta_c = [\theta_c, r_c]'$ consists of the matrix of regression coefficients θ_c and the covariance matrix r_c.

Further on we assume that model is known or well estimated in the off-line mode.

2.2 *Control design with mixture model*

According to the general fully probabilistic design, the control task is solved by finding some ideal pdf ${}^{[I]}f(d(\mathring{t}))$, which is close to the desired user target pdf ${}^{[U]}f(d(\mathring{t}))$ and simultaneously it is accessible i.e. has relation to the identified model $f(d(\mathring{t}))$.

The need to specify the desired value as a joint pdf on all data doesn't bring big restrictions. The pdf could be selected in the way that the marginal pdf on quantities without exact restriction is very flat. On other hand, exact requirements could invoke sharp marginal. Our approach deals with ${}^{[U]}f(d_t| {}^{[U]}\phi_{t-1})$ in the form

$${}^{[U]}f(d_t|\phi_{t-1}, {}^{[U]}\Theta) = \mathcal{N}_{d_t}({}^{[U]}\theta' \, {}^{[U]}\phi_{t-1}, {}^{[U]}r),$$

which can well represent both mentioned modalities. ${}^{[U]}\phi_{t-1}$ is a subvector of $[\phi_{t-1}, 1]$ and is called the state vector of target pdf.

One simple idea for choosing ${}^{[I]}f(d(\mathring{t}))$ is to obtain it with replacing weights of components α_c in $f(d(\mathring{t}))$ in each time t.

For formal formulation of this idea it is reasonable to consider new random variable c_t and generalize the considered pdf ${}^{[I]}f(d(\mathring{t}))$ to ${}^{[I]}f(d(\mathring{t}), c(\mathring{t}))$. The random variable c_t can be interpreted as a pointer to a component used in time t.

Now we use the fact, that some channels can be directly influenced i.e. $d_t = (\Delta_t, u_{o;t})$.

Simple use of the chain rule and assumption of independence the system on old c_t gives:

$$
\begin{aligned}
{}^{[I]}f(d(\mathring{t}), c(\mathring{t})) &= \prod_{t \in t^*} {}^{[I]}f(d_t, c_t | d(t-1), c(t-1)) = \\
&= \prod_{t \in t^*} {}^{[I]}f(\Delta_t, u_{o;t}, c_t | d(t-1)) = \\
&= \prod_{t \in t^*} {}^{[I]}f(\Delta_t | u_{o;t}, c_t, d(t-1)) \\
&\qquad {}^{[I]}f(u_{o;t}, c_t | d(t-1))
\end{aligned}
$$

The part ${}^{[I]}f(\Delta_t | u_{o;t}, c_t, d(t-1))$ can not be directly influenced by the controller (it is influenced by optional inputs only), hence we can not optimize over it. Hence ${}^{[I]}f(\Delta_t | u_{o;t}, c_t, d(t-1)) = f(\Delta_t | u_{o;t}, c_t, d(t-1))=$ known model. The part ${}^{[I]}f(u_{o;t}, c_t | d(t-1))$ is optimized.

The user target ideal pdf ${}^{[U]}f(d(\mathring{t}))$ must be extended to ${}^{[U]}f(d(\mathring{t}), c(\mathring{t}))$.

The part $c(\mathring{t})$ doesn't enter the user's requirements. Hence it can be selected uniform and independent on the past history.

$$
{}^{[U]}f(d(\mathring{t}), c(\mathring{t})) \propto {}^{[U]}f(d(\mathring{t}))
$$

Now the problem can be formulated as an optimization task.

We are looking for such $\left\{ {}^{[I]}f(u_{o;t}, c_t | d(t-1)) \right\}_{t \in t^*}$ minimizing

$$
\mathcal{D}\left({}^{[I]}f(d(\mathring{t}), c(\mathring{t})) || \, {}^{[U]}f(d(\mathring{t}), c(\mathring{t})) \right). \qquad (3)
$$

The desired ideal pdf ${}^{[I]}f(d_t | d(t-1))$ can be then obtained through marginalization.

$$
\begin{aligned}
&{}^{[I]}f(d_t | d(t-1)) = \\
&= \sum_{c_t \in c^*} {}^{[I]}f(u_{o;t}, c_t | d(t-1)) f(\Delta_t | u_{o;t}, c_t, d(t-1)).
\end{aligned}
$$

The details of deriving the approximation of the fully probabilistic design can be found in (Kárný 1996).

The design results in some finite dimensional characteristics, which are then, together with the new data record d_t, simply used for the control.

If the parameters of the target pdf ${}^{[U]}f$ have the form

$$
{}^{[U]}\theta = [\mu_1, \cdots, \mu_{\mathring{d}}], \text{ and } {}^{[U]}\phi_{t-1} = [1] \qquad (4)
$$

the mentioned characteristics can be evaluated in off-line phase, which provides tremendoes acceleration. Current software solution implements just this possibility, therefore the general algorithm described below also uses this assumption.

The control algorithm using the result of optimization can be formulated as follows.

Off-line phase

- Identify good mixture model representing the system.
- Specify parameters of target distribution ${}^{[U]}\theta'$, ${}^{[U]}r$.
- Select the design horizon $\mathring{t}$.
- Make the approximation of the fully probabilistic design, which results into following characteristics:

$$
k_c, \; Q_c, \; {}^{[I]}\theta_c, \; {}^{[I]}r_c, c \in c^*.
$$

- Set the time counter $t = 0$

On-line phase

- increase the time counter $t = t + 1$
- acquire data record d_t and update the state vector ϕ_t .
- Evaluate the auxiliary discrete pdf ${}^{[I]}f(c_{t+1} | \phi_t)$ using this relation:

$$
\begin{aligned}
&{}^{[I]}f(c_{t+1} | \phi_t) \propto \\
&\exp\left[-0.5\left(k_{c_{t+1}} + \phi_t' Q_{c_{t+1}} \phi_t\right)\right].
\end{aligned}
$$

- Then the ideal pdf is given by

$$
\begin{aligned}
&{}^{[I]}f(d_{t+1} | \phi_t) = \\
&\sum_{c_{t+1} \in c^*} {}^{[I]}f(c_{t+1} | \phi_t) \mathcal{N}_{d_{t+1}}\left({}^{[I]}\theta_{c_{t+1}} \phi_{t+1}, \; {}^{[I]}r_{c_{t+1}} \right).
\end{aligned}
$$

- marginalize the pdf ${}^{[I]}f(d_{t+1} | \phi_t)$ and obtain ${}^{[I]}f(u_{o;t+1} | \phi_t)$.
- Use mean values of the previous pdf for control.
- Go to the beginning of the on-line phase.

3. EXAMPLE

This section applies the theoretical solution to control of a realistic helicopter model. First the model is described, then the use of theory is discussed. The third part compares controlling with adaptive LQ controller and mixture-based controller with the same prior data.

Fig. 1. Face of the helicopter model

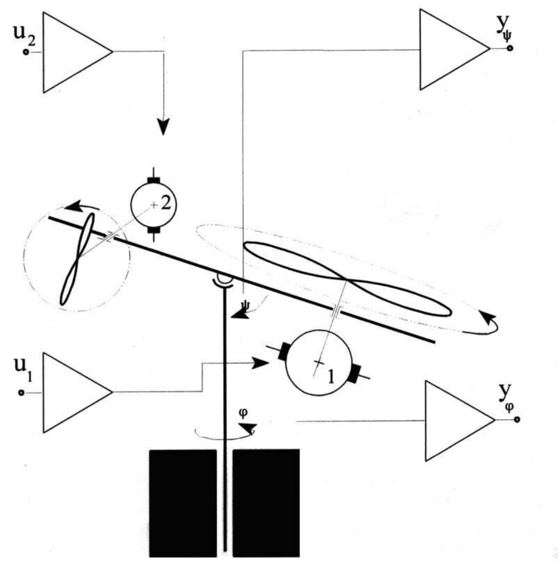

Fig. 2. Main components of the model including control and measured variables

3.1 *Description of the controlled system*

The CE150 Helicopter Model described in this paper is one of a set of products offered by HUMUSOFT Ltd. for teaching system dynamics and control engineering principles. Figure 1 shows the face of the model, whereas figure 2 shows the main components of the model including control and measured variables. The helicopter has two degrees of freedom: pitch $\langle -0.25; 0.27\rangle$ and yaw $\langle -0.74; 0.68\rangle$ The main propeller, which is powered by the first motor, controls the movement of the helicopter in pitch, while the side propeller with the second motor controls the movement in yaw. The span of the control variables is $\langle -1; 1\rangle$ (no propeller rotation for u=0). The whole model is connected to a PC computer through an interface unit. The MATLAB environment together with the Real Time Toolbox is used to implement the control and identification algorithms.

For simplification, just movement in pitch is considered. The pitch and yaw movements are not independent. The rotation of the main propeller is causing the rotation of the helicopter body in yaw and vice versa. We, however, will assume that there is not interaction between pitch and yaw movements. Therefore, while controlling the helicopter in pitch, the helicopter is immobilized in yaw.

Experiments on the physical model are very time consuming. Hence it is very profitable to obtain a software model and to perform the experiments on that model. Fortunately, the distributors of the helicopter distribute such a software model together with the physical model. The mentioned software model is dedicated to run in the SIMULINK environment, which makes it's use very simple.

Although the SIMULINK model is distributed together with the physical model, it had to be slightly modified. The realization of backstops was missing and some constant had to be set up for better relationship to the physical model.

Moreover, the model was adjusted to ascribe the constant 0.48 to its input, so that the input value 0 better correspond with the value when the helicopter starts to move.

As follows from the previous, the resulting model has one input value(the voltage on the main propeller) and one output value (pitch of the helicopter body).

3.2 *Control*

The first step of the control process is identification of the model from the data. The data set must be a realization of the model and should be a "reasonable" realization. Good choice can be data generated by the model controlled with an other approach. Data sample used in this example was obtained by controlling the simulink model with LQ adaptive controller, whose desired value is plotted in figure 3. For a better quality of the identified mixture, the prior information of the desired values was used. The resulting mixture consists of four components representing the four states of the system.

The second step is to specify parameters of the target pdf. According to (4), we need to specify ${}^{[U]}\Theta = [\mu_1, \mu_2]$ and ${}^{[U]}r = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$.

It is simple to express the parameters for constant target value v. ${}^{[U]}\Theta = [v, 0]$, ${}^{[U]}r = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.5 \end{pmatrix}$.

Let's denote the result of design with these parameters as ${}^{[I]}f_v$.

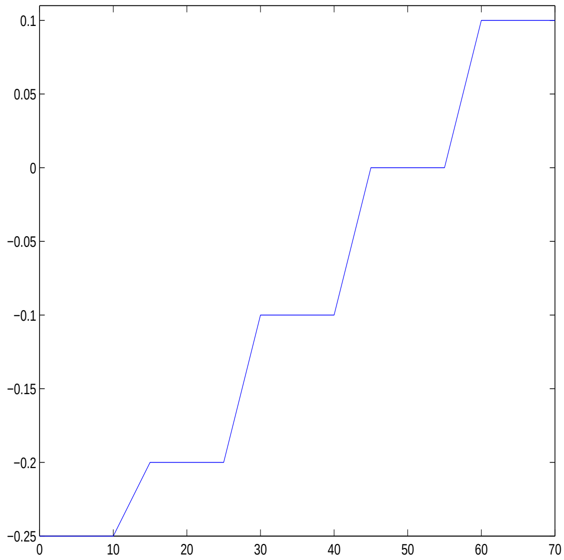

Fig. 3. Desired value of the LQ controller used for learning

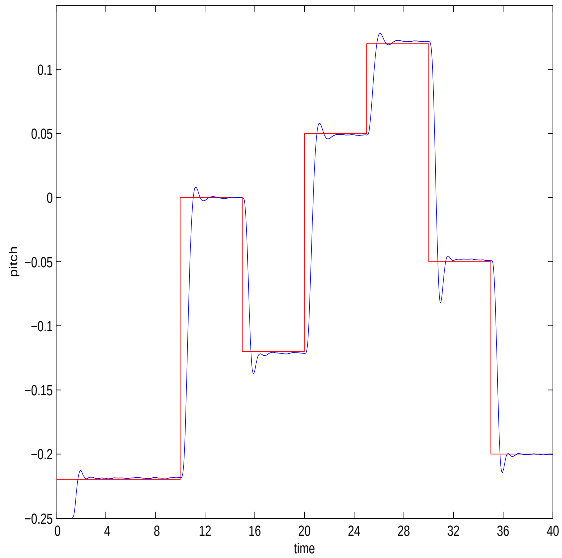

Fig. 4. The result of the mixture control

$t_1 = 10$	v_1 =-0.22
$t_2 = 15$	$v_2 = 0$
$t_3 = 20$	v_3 =-0.12
$t_4 = 25$	$v_4 = 0.05$
$t_5 = 30$	$v_5 = 0.12$
$t_6 = 35$	v_6 =-0.05
$t_7 = 40$	v_7 =-0.2

Table 1. Description of desired values

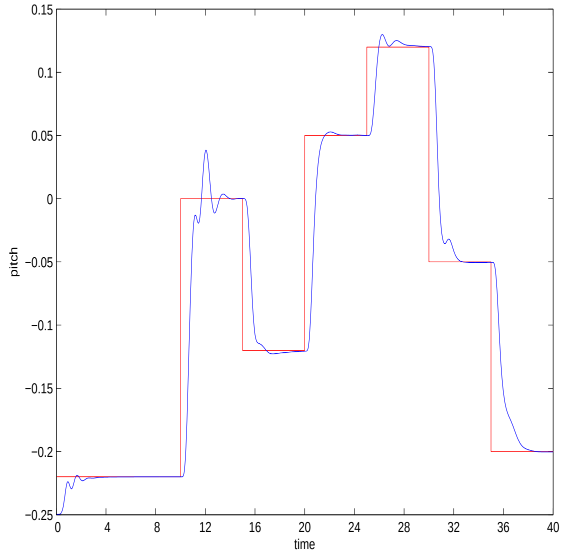

Fig. 5. The result of the adaptive LQ control

More complex targets can be expressed as switching of ${}^{[I]}f_v$ expressing simple targets. More precisely: If the desired values are given by a piecewise constant function

$$J(t) = \begin{cases} v_1, \ t \in (1, t_1 > \\ \vdots \\ v_n, \ t \in (t_{n-1}, t_n > \end{cases} \qquad t_1 < \cdots < t_n,$$

then this target can be achieved by using ${}^{[I]}f_{v_i}$ for control in time $t \in (t_{i-1}, t_i >$.

3.3 *Result*

For comparison of the adaptive LQ controller and mixture-based controller, the desired value in the mentioned form was used. A specific selection is shown in Table 1.

The LQ controller first run with the desired values which are shown in figure 3. Then It run with the desired values specified in table 1.

The results of control in both case are shown in figures 4,5.

It can be seen, that in this case the mixture based controller gives reasonable result and is better, than the adaptive LQ controller.

4. CONCLUSIONS

The paper presents control of a system using probabilistic mixtures. The main aim was to show the possibility to use this new approach in system control, which was fulfilled. It verifies, that the mixture-based approach to system control is promising and hence it deserves to continue with researching it.

As the next step, the discussed approach will be applied to the original physical model. Then it need to be tested on more complex systems. The implementation of adaptive version also remains to be done as well as trying to use more complex target values than the constant.

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