The Projective Invariants for Polygons

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Abstract. The paper deals with features of a general polygon which are invariant with respect to projective transform. First, some properties of the area of a triangle under projective transform are discussed. New projective triangular invariants of polygons are derived as the quotient of two different products of the areas of the triangles formed by the vertices of the polygon. The features are proved to be invariant to numbering of the vertices of the polygon. The number of projective triangular invariants for polygons with the given number of vertices is discussed. Numerical experiments dealing with three octagons deformed by projective transforms are described.

1 Introduction

One of the important tasks in image processing and computer vision is a recognition of objects on images captured under different viewing angles. If the scene is planar, then the distortion between two frames can be described by *projective transform*

$$\begin{aligned} x' &= (a_0 + a_1x + a_2y)/(1 + c_1x + c_2y) \\ y' &= (b_0 + b_1x + b_2y)/(1 + c_1x + c_2y) \ , \end{aligned} \tag{1}$$

where x, y are the coordinates in the first frame and x', y' are the coordinates in the second one.

Feature-based recognition of such objects requires features that are invariant to projective transform. Several different approaches to this problem have been published in the recent works. One of them is based on the assumption that non-linear term of the eq. 1 is relatively slight and the projective transform can be approximated by the affine one. This assumption is true, if the distance from the object to the sensor is much greater than the size of the object. In such cases, various affine invariants can be applied such as moment invariants [1,2] or Fourier descriptors [3].

However, in some cases the distortion cannot be approximated by the affine transform and therefore the use of exact projective invariants is required. The

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invariants which have been developed for this purpose can be categorized into two groups: differential invariants and point-based ones e.g. [4].

This paper deals with point-based features for recognition of projectively deformed polygons,which are also invariant to the starting point of vertices numbering.

2 The Area of a Transformed Triangle

Let's denote (i,j,k) the area of a triangle with vertices $[x_i, y_i]$, $[x_j, y_j]$ and $[x_k, y_k]$, i.e.

$$\begin{aligned}(i,j,k) &= (x_i y_j - x_j y_i + x_j y_k - x_k y_j + x_k y_i - x_i y_k)/2 \\ &= \frac{1}{2}\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}. \end{aligned} \tag{2}$$

If the vertices are numbered clockwise, we will consider the area negative. If $J(x,y)$ is a Jacobian of the projective transform (1) at the point $[x,y]$

$$J(x,y) = \frac{\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}}{(1 + c_1 x + c_2 y)^3}, \tag{3}$$

then the area of the triangle after the transform can be expressed as

$$\begin{aligned}(i,j,k)' &= \frac{\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix}}{(1 + c_1 x_i + c_2 y_i)\,(1 + c_1 x_j + c_2 y_j)\,(1 + c_1 x_k + c_2 y_k)} \cdot (i,j,k) = \\ &= (i,j,k)\,\sqrt[3]{J(x_i,y_i)\,J(x_j,y_j)\,J(x_k,y_k)}. \end{aligned} \tag{4}$$

The proof of eq. (4) can be found in [5].

As it can be seen from (4), the area of the triangle after the transform is given by the area of the original triangle multiplied by some power of Jacobian of the projective transform. Therefore the triangle area is so called *relative invariant* [6]. This property can be utilized for a construction of absolute projective invariants, i.e. invariants, which don't change value under projective transform at all.

The simplest absolute invariant is the cross-ratio of five points

$$\rho = \frac{(1,2,3)(1,4,5)}{(1,2,4)(1,3,5)}. \tag{5}$$

3 The Projective Triangular Invariants of Polygons

A polygon is given by n vertices, where each vertex is given by its coordinates $[x_i, y_i]$. Although the order of the vertices is given, the choice of the starting vertex is arbitrary and therefore our features should be invariant to the choice of the starting vertex.

Equation (4) implies that each product of the areas of triangles is a relative invariant to projective transform. If point $[x_a, y_a]$ occurs n_a times in the product, then the product is only multiplied by coefficient $(\sqrt[3]{J(x_a, y_a)})^{n_a}$ after projective transform.

To ensure the invariance to the choice of the starting point new invariants were developed

$$R_{k_1,k_2,k_3} = \prod_{i=1}^{n} (i, (i + k_1 - 1) \bmod n + 1, (i + k_1 + k_2 - 1) \bmod n + 1), \qquad (6)$$

where $n = k_1 + k_2 + k_3$ and "mod" denotes modulo operation. The commas between the indices will be omitted, if it doesn't lead to misunderstanding.

The invariants (6) are called relative projective triangular invariants (RPTI).

Only one RPTI exists for $n = 3$

$$R_{111} = (1, 2, 3)^3$$

and for $n = 4$

$$R_{112} = (1, 2, 3)(2, 3, 4)(3, 4, 1)(4, 1, 2).$$

Any absolute invariants cannot be constructed from R_{111} and R_{112}.

There are two RPTIs for $n = 5$

$$\begin{aligned} R_{113} &= (1, 2, 3)(2, 3, 4)(3, 4, 5)(4, 5, 1)(5, 1, 2), \\ R_{122} &= (1, 2, 4)(2, 3, 5)(3, 4, 1)(4, 5, 2)(5, 1, 3) \end{aligned}$$

and four RPTIs for $n = 6$

$$\begin{aligned} R_{114} &= (1, 2, 3)(2, 3, 4)(3, 4, 5)(4, 5, 6)(5, 6, 1)(6, 1, 2), \\ R_{123} &= (1, 2, 4)(2, 3, 5)(3, 4, 6)(4, 5, 1)(5, 6, 2)(6, 1, 3), \\ R_{132} &= (1, 2, 5)(2, 3, 6)(3, 4, 1)(4, 5, 2)(5, 6, 3)(6, 1, 4), \\ R_{222} &= (1, 3, 5)^3(2, 4, 6)^3. \end{aligned}$$

RPTI is not affected by the cyclic shift of the indices, i.e. $R_{123} = R_{231} = R_{312}$, but the interchange of two indices gives different RPTI; R_{123} is generally different from R_{132}.

The quotient of two general relative triangular invariants is an absolute invariant, if each vertex occurs with the same frequency in the numerator and in the denominator. It is satisfied for RPTIs defined by (6) ever. The proof is implied by (4). If a point $[x_a, y_a]$ occurs n_a times both in the numerator and in the denominator, the coefficient $(\sqrt[3]{J(x_a, y_a)})^{n_a}$ is eliminated by reducing. Note: in our case $n_a = 3$.

In the denominator it is suitable to use $R_{1,1,n-2}$, which is constructed from the areas of the triangles from the adjacent vertices of the polygon. If some three adjacent vertices lie on one straight line, the medium vertex can be omitted and the polygon has got less than n vertices. If no three adjacent vertices lie on one straight line, $R_{1,1,n-2}$ is different from zero and it can be used for the normalization of the invariants.

Absolute projective triangular invariant (PTI) after this normalization is denoted as I_{k_1,k_2,k_3}:

$$I_{k_1,k_2,k_3} = \frac{R_{k_1,k_2,k_3}}{R_{1,1,n-2}}. \tag{7}$$

The number of the PTIs is

$$\begin{array}{l} \ell = \ (n-1)(n-2)/6 - 1 \ \text{ if } n \bmod 3 \neq 0, \\ \ell = (n-1)(n-2)/6 - 1/3 \text{ if } n \bmod 3 = 0. \end{array} \tag{8}$$

4 Numerical Experiments

In order to demonstrate the behaviour of the PTIs in practical tasks, the following experiments were carried out.

The objects in Figs. 1a, 2a and 3 were created by a graphic editor and then the objects in Figs. 1a and 2a were deformed by projective transforms with various parameters (Figs. b, c, d). The objects in Figs. 1, 2 and 3 are octagons. The acquired values of the PTIs are shown in Tab. 1. They show good stability of the PTIs. Another important property of projective invariants is discriminability, what means how the values of the invariants differ for different objects without projective dependency. The object in Fig. 3 is quite different and its values of the PTIs are very distinct. The objects in Figs. 1 and 2 are similar and their values of the PTIs are also similar, but the values of I_{223} are distinct and the objects can be recognized according to them. The discriminability is satisfactory.

	Fig.1a	Fig.1b	Fig.1c	Fig.1d	Fig.2a	Fig.2b	Fig.2c	Fig.2d	Fig.3
I_{125}	1	0.927	1.16	0.946	1	0.937	1.27	1.16	2051
I_{152}	1	0.901	1.16	0.946	1	1.26	1.03	1.27	1707
I_{134}	0	-0.001	0	0	0	0	-0.01	0	$3.12 \cdot 10^4$
I_{143}	0	0.001	0	0	0	0	0.02	0	$3.18 \cdot 10^4$
I_{224}	564	425	793	587	478	417	563	654	$5.43 \cdot 10^5$
I_{233}	0	0	0	0	502	640	611	665	$2.42 \cdot 10^6$

Table 1: The values of the PTIs of the objects in Figs. 1, 2 and 3

A lot of zero values of the invariants is caused by relatively high degree of symmetry of these objects, many triplets of the vertices lie on one straight line.

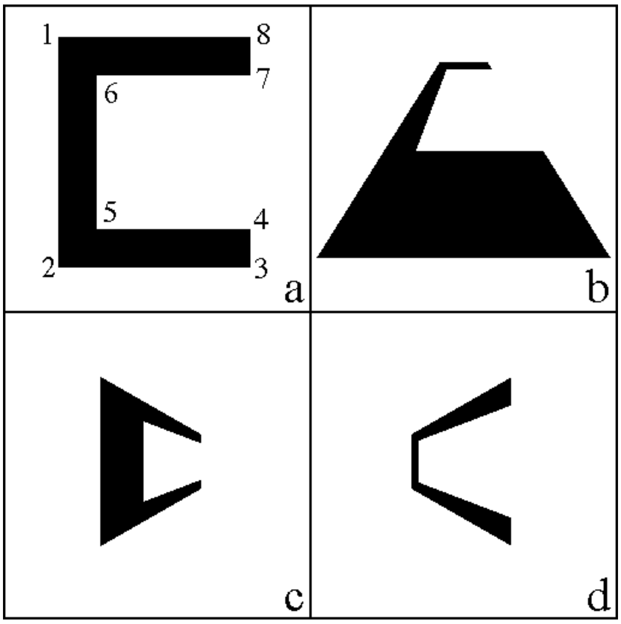

Fig. 1. The octagons I, *a* original, from *b* to *d* projective deformations

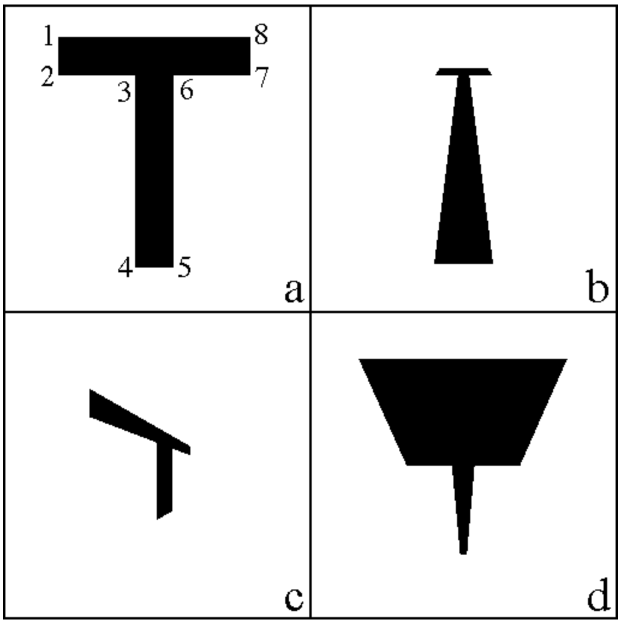

Fig. 2. The octagons II, *a* original, from *b* to *d* projective deformations

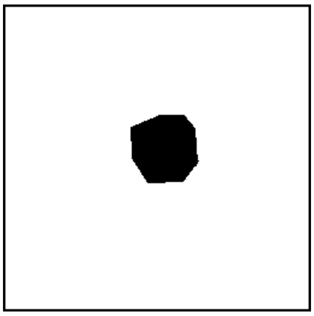

Fig. 3. The octagon III

5 Summary

The projective triangular invariants introduced in this paper are defined on objects from straight line segments, i.e. polygons, where differential invariants fail.

The numerical experiments were carried out to find the numerical stability and discriminability. They both were found satisfactory.

In comparison with previously published point-based methods, our invariants are invariant to the choice of the starting point and they do not need any point matching, on the other side they can only be used for the vertices of the polygons which define the order of the points (they cannot be used for a set of isolated points). The invariants to the choice of labelling presented in [1] or in [7] can be used for arbitrary sets of points, but they are more complicated and therefore their computing complexity is higher.

6 References

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