COMBINED INVARIANTS TO LINEAR FILTERING AND ROTATION

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A new class of moment-based features invariant to image rotation, translation, scaling, contrast changes and also to convolution with an unknown PSF are introduced in this paper. These features can be used for the recognition of objects captured by a nonideal imaging system of unknown position and blurring parameters.

Keywords: Complex moments, convolution invariants, rotation invariants, combined invariants, invariant basis.

1. INTRODUCTION

In scene analysis, we often obtain the input information in a form of an image captured by a nonideal imaging system. Most real cameras and other sensors can be modeled as a *linear space-invariant* system, where the relationship between the input $f(x,y)$ and the acquired image $g(x,y)$ is described as

$$g(\tau(x,y)) = a(f*h)(x,y) + n(x,y)\,. \tag{1}$$

In the above model, $h(x,y)$ is the point-spread function (PSF) of the system, $n(x,y)$ is an additive random noise, a is a constant describing the overall change of contrast, τ stands for a transform of spatial coordinates due to projective imaging geometry and $*$ denotes 2D convolution.

In many application areas, it is desirable to find a description of the original scene that does not depend on the imaging system without any prior knowledge of its parameters. Basically, there are two different approaches to this problem: image normalization or direct description by invariants.

Image normalization consists of two major steps: geometric registration, that eliminates the impact of imaging geometry and transforms the image into some "standard" form, and blind deconvolution, that removes or suppresses the blurring. Both these steps have been extensively studied in the literature, we refer to the recent surveys on registration[4,7] and on deconvolution/restoration techniques.[10,16] Generally, image normalization is an ill-posed problem whose computing complexity can be extremely high.

In the *invariant approach* we look for image descriptors (features) that do not depend on $h(x,y)$, $\tau(x,y)$ and a. In this way we avoid a difficult inversion of Eq. (1). In many applications, the invariant approach is much more effective than the normalization. Typical examples are the recognition of objects in the scene against a database, template matching, etc.

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Much effort has been spent to find invariants to imaging geometry, particularly to linear and projective transformations. Moment invariants,[1,3,5,8,9,18,19] Fourier-domain invariants,[2,12] differential invariants[15,20,21] and point sets invariants,[11,13,14,17] are the most popular groups of them. On the other hand, only few invariants to convolution have been described in the literature. A consistent theory has been published recently in Ref. 6 where two sets of convolution invariants were constructed in spatial as well as Fourier domains. Unfortunately, those features are not invariant to rotation and therefore their practical utilization is limited.

This paper performs the first attempt to find *combined invariants* that are invariant simultaneously to convolution and linear transform of spatial coordinates.

The rest of the paper is organized as follows. In Sec. 2, some basic definitions and propositions are given to build up the necessary mathematical background. Sections 3 and 4 perform the major contribution of the paper. In Sec. 3, the invariants to convolution composed from the complex moments are introduced. In Sec. 4 we present a derivation of the combined invariants. We also show how to select a complete and independent system of them. In Sec. 5, the previous results are extended to get additional invariance to image scaling and/or contrast changes. Finally, Sec. 6 describes numerical experiments performed on simulated data.

2. MATHEMATICAL BACKGROUND

In this section, we introduce some basic terms and propositions that will be used later in the paper.

Definition 1. *By image function (or image) we understand any real function* $f(x,y) \in L_1$ *which is nonzero on a bounded support and whose integral is positive.*

Definition 2. *Complex moment* $c_{pq}^{(f)}$ *of order* $(p+q)$ *of the image* $f(x,y)$ *is defined as*

$$c_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+iy)^p (x-iy)^q f(x,y)\,dx\,dy \tag{2}$$

where i *denotes the imaginary unit.*

In polar coordinates, (2) becomes the form

$$c_{pq} = \int_{0}^{\infty}\int_{0}^{2\pi} r^{p+q+1} e^{i(p-q)\theta} f(r,\theta)\,dr\,d\theta\,. \tag{3}$$

It follows immediately from (3) that $c_{pq} = c_{qp}^*$ (the asterix denotes a complex conjugate).

The following lemma describes an important rotation property of the complex moments.

Lemma 1. *Let* f' *be a rotated version (around the origin) of* f*, i.e.* $f'(r,\theta) = f(r,\theta+\alpha)$ *where* α *is the angle of rotation. Let us denote the complex moments of* f' *as* c'_{pq}*. Then*

$$c'_{pq} = e^{-i(p-q)\alpha} \cdot c_{pq}\,. \tag{4}$$

Using Eq. (3), the proof of Lemma 1 is straightforward. The next lemma shows how the complex moments are affected by convolution.

Lemma 2. *Let $f(x,y)$ and $h(x,y)$ be two image functions and let $g(x,y) = (f*h)(x,y)$. Then $g(x,y)$ is also an image function and we have, for its moments,*

$$c_{pq}^{(g)} = \sum_{k=0}^{p}\sum_{j=0}^{q}\binom{p}{k}\binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)}$$

for any p and q.

The assertion of Lemma 2 can be easily proven just using the definition of complex moments and convolution, respectively.

In the following text, we assume that the PSF $h(x,y)$ is centrally symmetric (i.e. $h(x,y) = h(-x,-y)$) and that the imaging system is energy preserving, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)dxdy = 1.$$

The centrosymmetry implies that $c_{pq}^{(h)} = 0$ if $p+q$ is odd. The assumption of centrosymmetry is not a significant limitation of practical utilization of the method. Most real sensors and imaging systems, both optical and nonoptical ones, have the PSF with certain degree of symmetry. In many cases they have even higher symmetry than the central one, such as axial or radial symmetry. Thus, the central symmetry is general enough to describe almost all practical situations.

3. INVARIANTS TO CONVOLUTION FROM THE COMPLEX MOMENTS

In this section, invariants to convolution based on complex moments are introduced.

Theorem 1. *Let $f(x,y)$ be an image function. Let us define the following function $K^{(f)}$: $\mathbf{Z} \times \mathbf{Z} \to \mathbf{C}$.*
If $(p+q)$ is even then

$$K(p,q)^{(f)} = 0\,.$$

If $(p+q)$ is odd then

$$K(p,q)^{(f)} = c_{pq}^{(f)} - \frac{1}{c_{00}^{(f)}}\sum_{n=0}^{p}\sum_{\substack{m=0\\0<n+m<p+q}}^{q}\binom{p}{n}\binom{q}{m}K(p-n,q-m)^{(f)}\cdot c_{nm}^{(f)}. \quad (5)$$

Then $K(p,q)^{(f*h)} = K(p,q)^{(f)}$ for any p and q and for any centrosymmetric $h(x,y)$. The number $r = p+q$ is called the order of the invariant.

Proof. The statement of the theorem is trivial for any even r. Let us prove the statement for odd r by induction.

- $r = 1$

$$K(0,1)^{(g)} = c_{01}^{(g)} = c_{01}^{(f)}c_{00}^{(h)} + c_{00}^{(f)}c_{01}^{(h)} = c_{01}^{(f)} = K(0,1)^{(f)}$$

and similarly for $K(1,0)^{(g)}$.

- We assume the theorem valid for all invariants of orders $1, 3, \cdots, r-2$. Using Lemma 2 we get

$$\begin{aligned}
K(p,q)^{(g)} &= c_{pq}^{(g)} - \frac{1}{c_{00}^{(g)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} K(p-n,q-m)^{(g)} \cdot c_{nm}^{(g)} \\
&= \sum_{k=0}^{p}\sum_{j=0}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)} \\
&\quad - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} K(p-n,q-m)^{(f)} \\
&\quad \times \sum_{k=0}^{n}\sum_{j=0}^{m} \binom{n}{k}\binom{m}{j} c_{kj}^{(h)} c_{n-k,m-j}^{(f)}.
\end{aligned}$$

Using the identity

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c},$$

we get by changing the order of the summation and by shifting the indices

$$\begin{aligned}
K(p,q)^{(g)} &= K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)} \\
&\quad - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \sum_{k=0}^{n} \sum_{\substack{j=0 \\ 0<k+j}}^{m} \binom{p}{n}\binom{q}{m}\binom{n}{k}\binom{m}{j} \\
&\quad \times K(p-n,q-m)^{(f)} c_{kj}^{(h)} c_{n-k,m-j}^{(f)} \\
&= K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)} \\
&\quad - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \sum_{k=0}^{n} \sum_{\substack{j=0 \\ 0<k+j}}^{m} \binom{p}{k}\binom{q}{j}\binom{p-k}{n-k}\binom{q-j}{m-j} \\
&\quad \times K(p-n,q-m)^{(f)} c_{kj}^{(h)} c_{n-k,m-j}^{(f)} \\
&= K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)} c_{p-k,q-j}^{(f)} \\
&\quad - \frac{1}{c_{00}^{(f)}} \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \sum_{n=k}^{p} \sum_{\substack{m=j \\ n+m<p+q}}^{q} \binom{p}{k}\binom{q}{j}\binom{p-k}{n-k}\binom{q-j}{m-j} \\
&\quad \times K(p-n,q-m)^{(f)} c_{kj}^{(h)} c_{n-k,m-j}^{(f)}
\end{aligned}$$

$$= K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)}$$

$$\left(c_{p-k,q-j}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=k}^{p} \sum_{\substack{m=j \\ n+m<p+q}}^{q} \binom{p-k}{n-k}\binom{q-j}{m-j} K(p-n,q-m)^{(f)} c_{n-k,m-j}^{(f)}\right)$$

$$= K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)}$$

$$\left(c_{p-k,q-j}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p-k} \sum_{\substack{m=0 \\ n+m<p+q-k-j}}^{q-j} \binom{p-k}{n}\binom{q-j}{m}\right.$$

$$\left.\times\, K(p-n-k,q-m-j)^{(f)} c_{nm}^{(f)}\right),$$

which we can rewrite as

$$K(p,q)^{(g)} = K(p,q)^{(f)} + \sum_{k=0}^{p} \sum_{\substack{j=0 \\ 0<k+j}}^{q} \binom{p}{k}\binom{q}{j} c_{kj}^{(h)} \cdot D_{kj} \tag{6}$$

where

$$D_{kj} = c_{p-k,q-j}^{(f)} - \frac{1}{c_{00}^{(f)}} \sum_{n=0}^{p-k} \sum_{\substack{m=0 \\ n+m<p+q-k-j}}^{q-j} \binom{p-k}{n}\binom{q-j}{m} K(p-n-k,q-m-j)^{(f)} c_{nm}^{(f)}.$$

If $k+j$ is odd then $c_{kj}^{(h)} = 0$. If $k+j$ is even then

$$K(p-k,q-j) = c_{p-k,q-j} - \frac{1}{c_{00}} \sum_{n=0}^{p-k} \sum_{\substack{m=0 \\ 0<n+m<p+q-k-j}}^{q-j} \binom{p-k}{n}\binom{q-j}{m}$$

$$\times\, K(p-k-n,q-j-m) \cdot c_{nm}\,.$$

Consequently,

$$D_{kj} = K(p-k,q-j)^{(f)} - \frac{1}{c_{00}^{(f)}} K(p-k,q-j)^{(f)} c_{00}^{(f)} = 0\,.$$

Thus, Eq. (6) implies $K(p,q)^{(g)} = K(p,q)^{(f)}$ for every p and q. □

A similar theorem can be found in our recent work.[6] The essential difference is that here we employ the complex moments instead of the standard ones as we did in Ref. 6. This makes possible to construct combined invariants, as will be shown in Sec. 4.

The two following lemmas show that the invariants $K(p,q)$ have the same rotation property and the property of antisymmetry as the complex moments themselves.

Lemma 3. *Let f' be a rotated version (around the origin) of f, i.e. $f'(r,\theta) = f(r, \theta + \alpha)$ where α is the angle of rotation. Let us denote the invariants of f' as $K'(p,q)$. Then*

$$K'(p,q) = e^{-i(p-q)\alpha} \cdot K(p,q). \tag{7}$$

Proof. The statement is trivial for any even r. Let us prove it in the case of odd orders by induction.

- $r = 1$

$$K'(0,1) = c'_{01} = e^{i\alpha} \cdot c_{01} = e^{i\alpha} \cdot K(0,1)$$

  and similarly for $K'(1,0)$.
- We assume the statement valid for all invariants of orders $1, 3, \cdots, r-2$. Then

$$\begin{aligned} K'(p,q) &= c'_{pq} - \frac{1}{c'_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} K'(p-n, q-m) \cdot c'_{nm} \\ &= e^{-i(p-q)\alpha} \cdot c_{pq} - \frac{1}{c_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} e^{-i(p-n-q+m)\alpha} \\ &\quad \times K(p-n, q-m) \cdot e^{-i(n-m)\alpha} \cdot c_{nm} \\ &= e^{-i(p-q)\alpha} \cdot c_{pq} - e^{-i(p-q)\alpha} \frac{1}{c_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} \\ &\quad \times K(p-n, q-m) \cdot c_{nm} = e^{-i(p-q)\alpha} \cdot K(p,q)\,. \end{aligned}$$

□

It follows from Lemma 3 that $|K(p,q)|$ is a rotation invariant for any p and q. However, the magnitudes themselves do not yield a complete system of the invariants. Thus, we propose a better approach in the next section.

Lemma 4. *It holds for any p and q that*

$$K(p,q)^* = K(q,p)\,.$$

Proof. The proof of this lemma goes again through induction. Clearly, $K(1,0)^* = c^*_{10} = c_{01} = K(0,1)$. Provided the lemma has been proven for all $K(p,q)$ where $p + q < r$, we get for $p + q = r$ the following:

$$\begin{aligned} K(p,q)^* &= c^*_{pq} - \frac{1}{c^*_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n}\binom{q}{m} K(p-n, q-m)^* \cdot c^*_{nm} \\ &= c_{qp} - \frac{1}{c_{00}} \sum_{m=0}^{q} \sum_{\substack{n=0 \\ 0<n+m<p+q}}^{p} \binom{q}{m}\binom{p}{n} K(q-m, p-n) \cdot c_{mn} = K(q,p)\,. \end{aligned}$$

□

4. COMBINED INVARIANTS

In this section, we introduce a new class of features invariant simultaneously to convolution and rotation.

Theorem 2. *Let $n \geq 1$ and let k_j, p_j and $q_j; j = 1, \cdots, n$, be non-negative integers such that $(p_j + q_j)$ is odd for each j and that*

$$\sum_{j=1}^{n} k_j (p_j - q_j) = 0 \,.$$

Then

$$I = \prod_{j=1}^{n} K(p_j, q_j)^{k_j} \tag{8}$$

is invariant to rotation around the origin and to convolution with a centrosymmetric PSF.

Proof. The invariance to convolution follows immediately from Theorem 1. Lemma 3 implies the invariance to rotation:

$$\begin{aligned} I' &= \prod_{j=1}^{n} K'(p_j, q_j)^{k_j} = \prod_{j=1}^{n} e^{-i(p_j - q_j)\alpha k_j} \cdot K(p_j, q_j)^{k_j} \\ &= e^{-i\alpha \sum_{j=1}^{n} k_j (p_j - q_j)} \cdot \prod_{j=1}^{n} K(p_j, q_j)^{k_j} = I \,. \end{aligned}$$

□

According to Theorem 2, simple examples of combined invariants are $K(1,0)K(0,1)$, $K(1,0)K(1,2)$, $K(2,1)K(1,2)$, $K(2,1)^3 K(0,3)$, etc. As a rule, most invariants (8) are complex. If one prefers to have real-valued features, one can consider their real and imaginary parts separately.

Theorem 2 allows us to construct, for any order of the convolution invariants, an infinite number of the combined invariants, but only few of them are mutually independent. For the rest of this section, the attention is paid to the construction of a basis of combined invariants. By the term *basis* we understand the smallest set of combined invariants, by means of which all other ones can be expressed using multiplications, divisions, integer powers and complex conjugations only. The knowledge of the basis is a crucial point in all object recognition tasks, because it provides the same discrimination power as the set of all invariants at minimum computational cost.

Theorem 3. *Let $\mathcal{S}$ be a set of the convolution invariants (5) of any odd orders (not necessarily of all invariants), let $\mathcal{S}^*$ be a set of their complex conjugates and let $K(p_0, q_0) \in \mathcal{S} \cup \mathcal{S}^*$ such that $p_0 - q_0 = 1$ and $K(p_0, q_0) \neq 0$. Let $\mathcal{I}$ be a set of all combined invariants created from the elements of $\mathcal{S} \cup \mathcal{S}^*$ according to (8). Let $\mathcal{B} \subset \mathcal{I}$ be constructed as follows:*

$$(\forall p, q | p > q \wedge K(p,q) \in \mathcal{S} \cup \mathcal{S}^*)(\Phi(p,q) \equiv K(p,q) K(q_0, p_0)^{p-q} \in \mathcal{B}) \,.$$

Then $\mathcal{B}$ is a basis of $\mathcal{I}$.

Proof. The independence of $\mathcal{B}$ follows from the mutual independence of the convolution invariants. Let us prove its completeness.

Let I be an arbitrary element of $\mathcal{I} - \mathcal{B}$. Thus

$$I = \prod_{i=1}^{n} K(p_i, q_i)^{k_i}$$

where $K(p_i, q_i) \in \mathcal{S} \cup \mathcal{S}^*$. The product can be decomposed into two factors according to the relation between p_i and q_i:

$$I = \prod_{i=1}^{n_1} K(p_i, q_i)^{k_i} \cdot \prod_{i=n_1+1}^{n} K(p_i, q_i)^{k_i}$$

where $p_i > q_i$ if $i \leq n_1$ and $p_i < q_i$ if $i > n_1$.

Let us construct another invariant J from the elements of $\mathcal{B}$ only as follows:

$$J = \prod_{i=1}^{n_1} \Phi(p_i, q_i)^{k_i} \cdot \prod_{i=n_1+1}^{n} \Phi(q_i, p_i)^{*k_i}.$$

Grouping the factors $K(q_0, p_0)$ and $K(p_0, q_0)$ together we get

$$\begin{aligned} J &= K(q_0, p_0)^{\sum_{i=1}^{n_1} k_i(p_i - q_i)} \cdot K(p_0, q_0)^{\sum_{i=n_1+1}^{n} k_i(q_i - p_i)} \\ &\quad \cdot \prod_{i=1}^{n_1} K(p_i, q_i)^{k_i} \cdot \prod_{i=n_1+1}^{n} K(p_i, q_i)^{k_i} \\ &= K(q_0, p_0)^{\sum_{i=1}^{n_1} k_i(p_i - q_i)} \cdot K(p_0, q_0)^{\sum_{i=n_1+1}^{n} k_i(q_i - p_i)} \cdot I\,. \end{aligned}$$

Since I is assumed to be an element of $\mathcal{I}$, it must hold

$$\sum_{i=1}^{n_1} k_i(p_i - q_i) + \sum_{i=n_1+1}^{n} k_i(p_i - q_i) = 0$$

and, consequently,

$$\sum_{i=1}^{n_1} k_i(p_i - q_i) = \sum_{i=n_1+1}^{n} k_i(q_i - p_i) = L.$$

Now I can be expressed as a function of the elements of $\mathcal{B}$:

$$I = \Phi(p_0, q_0)^{-L} \cdot J.$$

Thus, I has been proven to be dependent on $\mathcal{B}$. □

Using Theorem 3, we can set up for instance a basis of all combined invariants up to the fifth order:

$$\begin{aligned} \mathcal{B}_5 = \{&K(1,0)K(1,2), K(2,1)K(1,2), K(3,0)K(1,2)^3, K(5,0)K(1,2)^5, \\ &K(4,1)K(1,2)^3, K(3,2)K(1,2)\}\,. \end{aligned}$$

Note that Theorem 3 does not guarantee the uniqueness of the basis. Different choices of p_0 and q_0 lead to different bases. For practical reasons it is recommended to choose p_0 and q_0 as small as possible because low-order moments are more robust to noise than the higher-order ones.

5. ADDITIONAL INVARIANCE

In this section, we propose how to make the combined invariants (8) invariant also to translation, scaling and contrast changes.

Translation invariance can be easily reached when using central coordinates in the definition of complex moments. However, $K(1,0) = K(0,1) = 0$ in that case and, consequently, all invariants containing $K(1,0)$ and/or $K(0,1)$ are also zero for any image function $f(x,y)$.

Scaling invariance can be reached by using normalized complex moments

$$\nu_{pq} = \frac{c_{pq}}{c_{00}^{(p+q+2)/2}}$$

when constructing the convolution invariants or, equivalently, by normalizing the basic invariants $\Phi(p,q)$ by c_{00}^{w}, where

$$w = ((p_0 + q_0 + 2)(p - q) + p + q + 2)/2.$$

Invariance to the *contrast changes* can be achieved by normalizing each $K(p,q)$ by c_{00}. Equivalently, $\Phi(p,q)$ should be normalized by c_{00}^{p-q+1}.

It is impossible to achieve simultaneous invariance to scaling and contrast changes in the way of normalization of the invariants by a power of c_{00}. Thus, we propose to employ appropriate ratios of the combined invariants. It can be proven that any normalized ratio Ω of two basic invariants

$$\Omega = \frac{\Phi(p,q)}{\Phi(s,t)} c_{00}^{z}$$

where

$$z = ((p_0 + q_0 + 2)(s - t - p + q) + (s + t - p - q))/2$$

and where

$$(p_0 + q_0 + 1)(p - s) = (p_0 + q_0 - 1)(q - t)$$

is invariant also to scaling and contrast changes. One of the simplest invariants fulfilling the above constraints is, for instance,

$$\Omega = \frac{K(5,4) \cdot c_{00}^2}{K(3,0)K(1,2)^2}.$$

6. NUMERICAL EXPERIMENTS

The aim of the accomplished experiments was to verify the invariance to both convolution and rotation of the combined invariants (8) and also to evaluate their robustness to additive random noise.

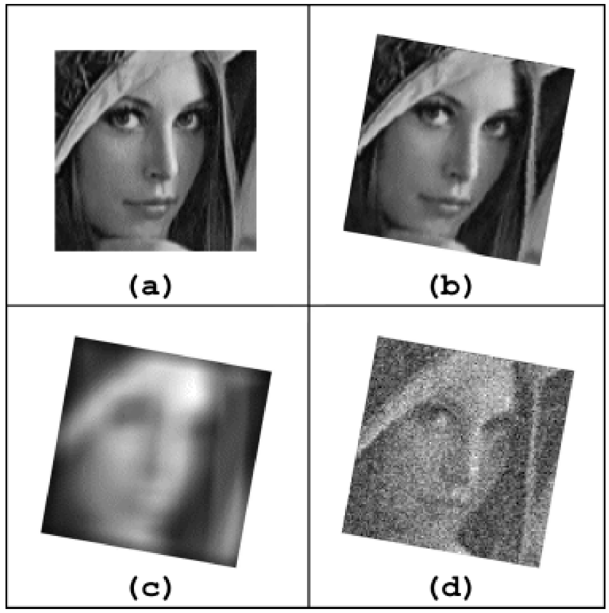

Fig. 1. Examples of the test images: (a) original image, (b) rotated image with no blur, (c) rotated and blurred image, (d) rotated image with an additive noise.

In the first experiment, we took a part of Lena image sized 101×101 pixels with zero border 30 pixels wide [see Fig. 1(a)]. From this image other 19 images were generated by rotating various angles from $0°$ to $90°$. Normalized uniform square masks of different sizes (3×3, 5×5, 7×7, 9×9, 11×11, 13×13 and 15×15) were employed as the blurring filters and every image was convolved with all of them. Figures 1(b) and 1(c) show two examples of the degraded images.

Nine basic combined invariants ($\Phi(2,1)$, $\Phi(3,0)$, $\Phi(5,0)$, $\Phi(4,1)$, $\Phi(3,2)$, $\Phi(7,0)$, $\Phi(6,1)$, $\Phi(5,2)$ and $\Phi(4,3)$, where $p_0 = 2$ and $q_0 = 1$) as well as their relative errors were computed on all 152 test images. Figure 2 shows how the relative errors of the invariants depend on the image blur and rotation. The influence of the image blur is negligible as can be expected from theoretical considerations. The effect of rotation is much more significant which, however, is mainly due to resampling during the rotation. It can be also seen that the higher the order of the combined invariant the more vulnerable the invariant. In all cases under investigation, the relative errors of the invariants were less than 1.5%, that illustrates perfect stability.

The second experiment tested the robustness of the combined invariants when an additive noise is present. Each image from the previous experiment was corrupted by a zero-mean Gaussian noise to get various signal to noise ratios (SNR) from 2 dB to 62 dB [Fig. 1(d)]. On each level of SNR, twenty realizations of noise were generated and the mean value of the particular invariants was used for robustness evaluation. Unlike in the previous experiment, the invariants were computed from the central circular part of the image only. Figure 3 shows the effect of the present noise and image rotation on the relative error of the invariants. The noise influence

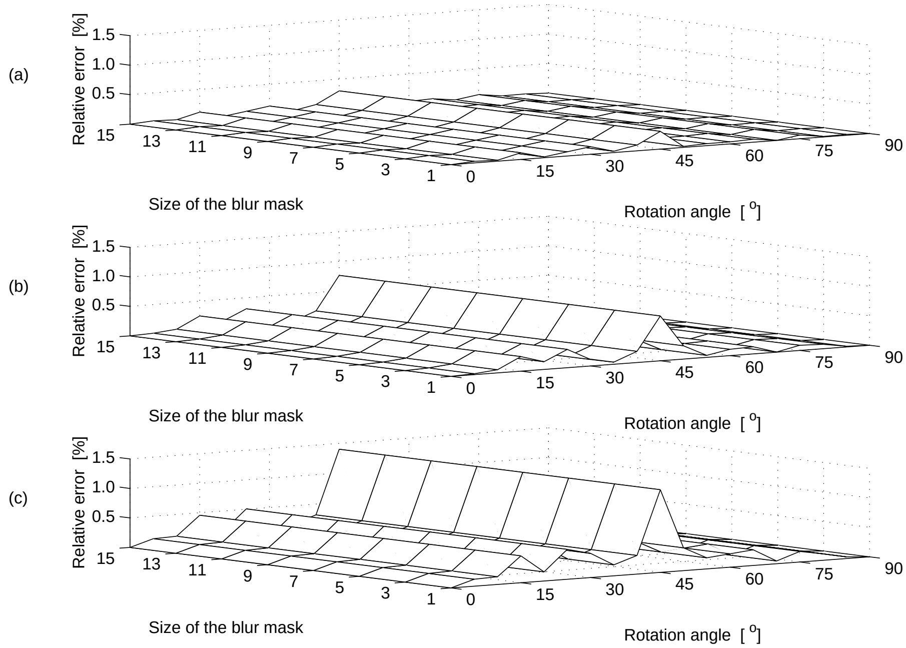

Fig. 2. The relative errors of the invariants for blurred and rotated images. The angle of the image rotation is from 0° to 90°, the size of the blurring filters is from 1 × 1 (no blur) to 15 × 15. (a) $\Phi(2, 1)$, (b) $\Phi(4, 1)$, (c) $\Phi(7, 0)$.

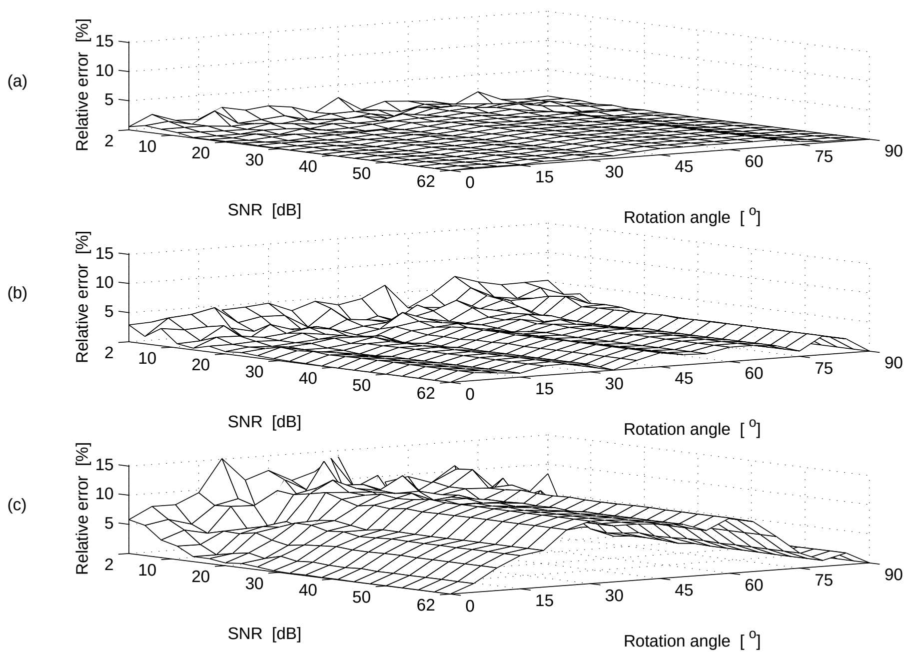

Fig. 3. The relative error of the invariants for blurred and noisy images. The angle of the image rotation is from 0° to 90°, the SNR is from 2 dB to 62 dB. (a) $\Phi(2, 1)$, (b) $\Phi(4, 1)$, (c) $\Phi(7, 0)$.

is more significant in the case of the invariants of higher orders. Comparing the relative errors with those from the first experiment, one can see they are about ten times higher, particularly at low SNR. However, relative errors below 15% are still acceptable. In most practical applications we do not deal with images corrupted so heavily, we usually assume SNR higher than 10 dB. Under such circumstances, the combined invariants show sufficient robustness.

7. CONCLUSION

In this paper, a consistent and well-developed theory of the invariants to image blurring, rotation, scaling and contrast changes is presented. Arbitrarily large systems of the combined invariants of any orders can be constructed such that they are mutually independent and complete. These invariants can be used for object recognition when an unknown rotation and blur are present. In that way, we avoid image deblurring and geometric normalization.

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