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On Models of Partial Repairs

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Abstract

Models of imperfect repairs are mostly based on the reduction of the cumulated hazard rate, either directly or indirectly (by reducing the virtual age of the system). Other set of models connects the repair with the change of time scale after such a virtual age shift. If the state of the system is characterized by a process of deterioration, the repair degree can be connected with the reduction of deterioration level. Such a view actually transforms the time scale (or the scale given by the cumulated hazard rate) to the scale of growing deterioration. From the problems connected with such models (consistency of statistical analysis, model fit assessment etc.) we shall discuss mainly the question of the repair schemes, their consequences, and possibilities of an 'optimal' repair policy leading to the hazard rate stabilization. Further, we shall consider the case that the hazard rate model consists of more components ('repairable' or not) and examine the same questions, namely that of the repair scheme and of the impact of repairs under certain models for intensity of failures.

1 Introduction

In reliability models it is often assumed that the intensity of failure of a technical device is influenced by a process of degradation. The degradation level is either observed directly, or just indirectly, through statistical data. Further, let us assume that the component corresponding to the device deterioration can be controlled. Hence, it is possible to search for the relationship between the extent of repair (taken as the reduction of certain value characterizing the damage) and the "repair level" in the sense of Kijima models (Kijima, 1989), i.e. taken as the reduction of virtual age of the object, or, in other words, as the increase of its survival time.

In the contribution, we concentrate to the case when the degradation is modeled via a non-decreasing function or a random process, for instance the step-wise random shocks process (actually a compound Poisson process or its generalization), with known or estimable statistical characteristics. The effect of degradation level reduction will be studied and the prolonged expected life-time after such actions evaluated. Finally, repair strategies optimal with respect to certain requirements will be considered. Hence, the following sequence of problems will be studied:

1. Models of repairs, in general, will be introduced.
2. A model proposed in Dorado et al (1997) considering the time shift and acceleration as a consequence of a repair will be recalled.
3. Simple Kijima II type model for preventive repair will be considered and its scheme applied to the case of a model with degradation process. Repair then will be connected with the reduction of the level of degradation.
3. The case will be generalized to the scheme where the degradation will be modeled via the compound Poisson process.
4. Regression models of intensity will be used in cases when just a part of the system is repaired.
5. Finally, the results of repair schemes will be displayed in several randomly generated examples.

2 Basic scheme of repairs

Let us first recall briefly the most common schemes of repair of a repairable component and the relationship with the distribution of the time to failure. The renewal means that the component is repaired completely, fully (e.g. exchanged for a new one) and that, consequently, the successive random variables – times to failure – are distributed identically and independently. The resulting intensity of the stream of failures is called the renewal density, and has the meaning

$$h(t) = lim_{d\to 0^+} \frac{P(failure\, occurs\, in\, [t, t+d))}{d}.$$

Its integral (i.e. cumulated intensity) is then $H(t) = E[N(t)] = \sum_{k=0}^{\infty} k \cdot P(N(t) = k)$, where $N(t)$ is the number of failures in (0,t].

Let $f(t), F(t)$denote the density and distribution function of the time to failure. Then so called renewal equation $h(t) = f(t) + \int_0^t h(t-u)f(u)du$ holds provided the 'renewal' occurs just after each failure, consequently also $H(t) = F(t) + \int_0^t H(t-u)f(u)du$.

2.1 Kijima's models

There are several natural ways how the notion of complete repairs can be generalized to repairs partial, incomplete. One of basic contribution to it is in the paper of M. Kijima (1989).

Let F be again the failure distribution of a new system. Assume that at each time the system fails, after a lifetime T_n from the preceding failure, a maintenance activity takes place (executed in negligible time) such that reduces the *virtual age* to some value $V_n = y, y \in [0, T_n + V_{n-1}]$ immediately after the n-th repair ($V_0 = 0$). The distribution of the n-th failure-time T_n is then

$$P[T_n \le x | V_{n-1} = y] = \frac{F(x+y) - F(y)}{1 - F(y)}.$$

M. Kijima then specified several sub-models of imperfect repairs. Denote by A_n the degree of the n-th repair (a random variable taking values between 0 and 1). Then in Model I the n-th repair cannot remove the damages incurred before the (n-1)th repair, $V_n = V_{n-1} + A_n \cdot T_n$.

On the contrary, the Model II allows for such a reduction of the virtual age, namely $V_n = A_n \cdot (V_{n-1} + T_n)$. Special cases contain the perfect repair model with $A_n = 0$, minimal repair model, $A_n = 1$, and frequently used variant with constant degree $A_n = a$.

Naturally, there are many others different generalizations, e.g. we can consider a randomized degree of repair, or the regressed degree (based on the system history). A set of variant models is also due M.S. Finkelstein (2000), who actually 'accelerated' the virtual time after each 'renewal' repair. It means that the distributions of T_i, the time-to-failure after i-th repair, differ. A reasonable assumption is that T_i is stochastically non-increasing with i, $T_{i+1} \le_{st} T_i$, i.e. $F_{i+1}(t) \ge F_i(t)$.

A simplest example assumes that $F_i(t) = F(u^{i-1}t), u > 1$, then a generalization can consider an accelerated time model with time-dependent functions $W_i(t)$, i.e. $F_{i+1}(t) = F_i(W_i(t))$, where usually $W_i(t) \ge t, W_i'(t) \ge 1$. It follows that $F_i(t) = F_0(W_0(W_1(..(W_{i-1}(t))..)$. The interpretation is straightforward, values of $W(t)$ measure (reflect) a relative speed of degradation.

3 Model with shift and change of scale

Dorado (1997) derived very useful generalization of Kijima virtual age models which allows us not only to shift the age of the system after a repair but also to change the distribution of the forthcoming cycle, i.e. the shape of the failure intensity after a repair.

For any CDF F, $\theta \in (0, 1]$ and $v \in [0, \infty)$, consider the family of distribution functions

$$\bar{F}_v^{\theta}(t) = \frac{\bar{F}(\theta t + v)}{\bar{F}(v)}, \quad t > 0. \tag{1}$$

The family of distributions $\left\{F_v^{\theta}\right\}$ are stochastically ordered in θ, that is, $\theta \le \theta'$ implies $F_v^{\theta}(t) \le F_v^{\theta'}(t)$, for all v and t. Then the survival function $\bar{F}_v^{\theta}(t)$ can be viewed as the life of a functioning item of age v which has been scaled by a factor θ, with lower values of θ representing longer remaining life. Authors mentioned above refer to $F_v^\theta(t)$ as the life distribution of an item with an *effective age* v and a *life supplement* θ.

Basic Scheme: Consider two sequences $\{V_i\}_{i\geq 0}$ and $\{\theta_i\}_{i\geq 0}$ called the effective ages and life supplements, respectively, satisfying

$$V_0 = 0,\ \Theta_0 = 1,\ V_i \geq 0,\ \Theta_i \in (0,1] \quad \text{and}$$
$$V_i \leq V_{i-1} + \Theta_{i-1}T_i \quad \text{for} \quad i > 0. \tag{2}$$

The general model of virtual age defines the joint distributions of the inter-failure times T_i as follows

$$P(T_i \leq t | V_{i-1},\ \Theta_{i-1},\ T_1, \ldots, T_{i-1}) = F_{V_{i-1}}^{\Theta_{i-1}}(t) \tag{3}$$

for $t > 0,\ i \geq 1$, where $F_{V_{i-1}}^{\Theta_{i-1}}(t)$ is defined according to (1).

It is easy to see that T_j defined by distribution $F_{V_{i-1}}^{\Theta_{i-1}}(t)$ is stochastically larger than T_j defined by $F_{V_{i-1}}^1$, i.e. better than the working item of age V_{j-1}. Furthermore, we can see that for each $i \geq 1$ the effective age V_i of the system after the i-th repair is less than its effective age $X_i := V_{i-1} + \Theta_{i-1}T_i$ just before the i-th repair, which in turn is less than the actual age S_i. Thus the general repair model defined above can be considered as a better-than-minimal repair model.

Special Cases:

1. The perfect repair model. Consider the case when $\Theta_i = 1$ and $V_i = 0$ for all $i \geq 0$. Then (2) is automatically satisfied and (3) reduces to $P(T_i \leq t | T_1, \ldots, T_{i-1}) = F_0^1(t) = F(t)$ for all $i \geq 1$. From this we see that T_i's are independent identically distributed with common CDF F and that corresponds to the perfect repair model.

2. The minimal repair model. Consider the case when $\Theta_i = 1$ and $V_i = S_i$ for each i. Then $V_i = S_i = V_{i-1} + \Theta_{i-1}T_i$ and, hence, (2) is satisfied. Under these conditions (3) then reduces to $P(T_i \leq t | S_{i-1}) = F_{S_{i-1}}^1(t)$ and this corresponds to the minimal repair model.

3. Kijima's Model I. Let $\{\xi_i\}_{i\geq 1}$ be a sequence of random variables independently distributed on $[0,1]$ and independent of other processes. Consider the case when $\Theta_i = 1$ for each i and $V_i = \sum_{k=1}^{i} \xi_k T_k$ for $i \geq 1$. Then $V_{i+1} = V_i + \xi_i T_i$ and $\xi_i \leq 1 = \Theta_i$ for each i and, hence, (2) is satisfied. (3) reduces to $P(T_i \leq t | T_k,\ \xi_k,\ 1 \leq k \leq i-1) = F_{V_{i-1}}^1(t)$. This is Kijima's Model I.

4. Kijima's Model II. Consider the case when $\Theta_i = 1$ for each i and $V_i = \sum_{k=1}^{i} (\prod_{l=k}^{i} \xi_l) T_k$ for $i \geq 1$. Since $V_i = \xi_i(V_{i-1} + T_i)$ and $\xi_i \leq 1 = \Theta_i$ for each i, then (2) is satisfied and (3) reduces again to $P(T_i \leq t | T_k,\ \xi_k,\ 1 \leq k \leq i-1) = F_{V_{i-1}}^1(t)$. This is Kijima's Model II.

5. The supplemented life repair model. Up to this point we have restricted the Θ_i's to be identically equal to 1. Lets consider other choices of the life supplement sequence $\{\Theta_i\}$. We know that smaller values of Θ_i corresponds to the larger operation time of the forthcoming cycle of the system. Hence, we can use Θ_i as a measure of how ith repair supplements the expected remaining life of the system. This explain the use of the term "life supplement". Dorado (1997) restricted Θ_i's to be in $(0,1]$ but one can also consider $\Theta_i > 1$. This case will reduce the expected remaining life of the system after i-th repair, i.e. aging will accelerate, like in the setting of Finkelstein (2000).

   Let us assume that a minimal repair was performed at the time of the first failure, then T_2 would have the distribution $F_{T_1}^1$. If we want a longer expected life for T_2 than we can use the distribution $F_{T_1}^{\Theta_1}$ for some $\Theta_1 \in (0,1)$. Starting with the distribution $F_{T_1}^{\Theta_1}$ for T_2 and using minimal repair upon the second failure, the random variable T_3 would have the distribution $F_{V_2}^{1}$ where $V_2 = T_1 + \Theta_1 T_2$. Again, if we want a longer expected lifetime for T_3 we can use the distribution $F_{V_2}^{\Theta_2}$ for some $\Theta_2 \in (0,1)$. In general the supplemented life repair model is defined as follows: $\Theta_0 := 1$, $V_0 := 0$, $V_i := \sum_{k=0}^{i-1} \Theta_k T_{k+1}$ for $i \geq 1$. It is easy to see that $V_{i+1} = V_i + \Theta_i T_{i+1}$ for $i \geq 0$ and the condition (2) is satisfied. The condition (3) reduces to $P(T_j \leq t | T_k,\ \Theta_k,\ 1 \leq k \leq i-1) = F_{V_{i-1}}^{\Theta_{i-1}}$ and for failure intensity we have $h_1(t) = h(t)$ and $h_i(t) = \Theta_{i-1} h(\Theta_{i-1} t + V_{i-1}),\ t \in (0, T_i]$ for $i \geq 2$.

4 Kijima II model for preventive repairs

Let us consider the following simple variant of the Kijima II model with constant degree $1-\delta$ and assume that it is used for the description of the system virtual age change after preventive repairs. Further, let us assume that after the failure the system is repaired just minimally, or that the number of failures is much less than the number of preventive repairs. Let Δ be the (constant) time between these repairs, V_n, V_n^* the virtual ages before and after $n-$th repair, and:

$$V_n = V_{n-1}^* + \Delta,\ \ V_n^* = \delta \cdot V_n.$$

If we start from time 0, then $V_1 = \Delta,\ V_1^* = \delta\Delta,\ V_2 = \delta\Delta + \delta = \Delta(\delta+1), V_2^* = \Delta(\delta^2+\delta),\ V_3 = \Delta(\delta^2+\delta+1)$ etc. Consequently, $V_n \to \frac{\Delta}{1-\delta}$, i.e. it 'stabilizes', for each δ and Δ there is a limit meaning that the actual intensity of failures $h(t)$ 'oscillates' between $h_0(\frac{\delta\Delta}{1-\delta})$ and $h_0(\frac{\Delta}{1-\delta})$, where $h_0(t)$ is the hazard rate of the time-to-failure distribution of the non-repaired system.

Simultaneously, the cumulated intensity increases regularly through intervals of length Δ by $dH = H(\frac{\Delta}{1-\delta}) - H(\frac{\delta\Delta}{1-\delta})$,, i.e. 'essentially' with the constant slope $a = dH/\Delta$.

Example: Let us consider the Weibull model, with $H_0(t) = \alpha \cdot \exp^{\beta}$, ($\beta > 1$, say). In that case

$$dH = \alpha\Delta^{\beta}\frac{1-\delta^{\beta}}{(1-\delta)^{\beta}},\ \ a = \alpha\Delta^{\beta-1}\frac{1-\delta^{\beta}}{(1-\delta)^{\beta}}.$$

As the special cases, again the perfect repairs, $\delta = 0$, minimal repairs with $\delta \sim 1$, and the exponential distribution case with $\beta = 1$ can be considered.

Figure 1 shows a graphical illustration of such a stabilization in the case that the hazard rate $h_0(t)$ increases exponentially.

Remark 1. If the model holds (with constant times between repairs Δ) it is always possible to stabilize the intensity by selecting the upper value of H^* and repair always when $H(t)$ should reach that value. Then $V_n = V = H^{-1}(H^*),\ V_n^* = \delta V_n$ again, and the interval between repairs should be $\Delta = V(1-\delta)$.

On the contrary, if we can reduce just the last time increment, (KIJIMA I model), for degrees δ_n and intervals Δ_n of repairs we get that $V_n = \sum_{k=1}^{n} \delta_k \Delta_k$, in the constant Δ case we have to decrease δ_k to 0 in order to keep V_n stabilized. Similarly in the case of accelerated model of repairs, there has to be a deal between the acceleration and the decrease of inter-repairs intervals.

4.1 An optimal selection of repair interval and degree

If we consider the stabilized case, and moreover the failures are much less frequent than preventive repairs, then there quite naturally arises the problem of selection of δ to given repair interval Δ (or optimal selection of both). By optimization we mean here the search for values

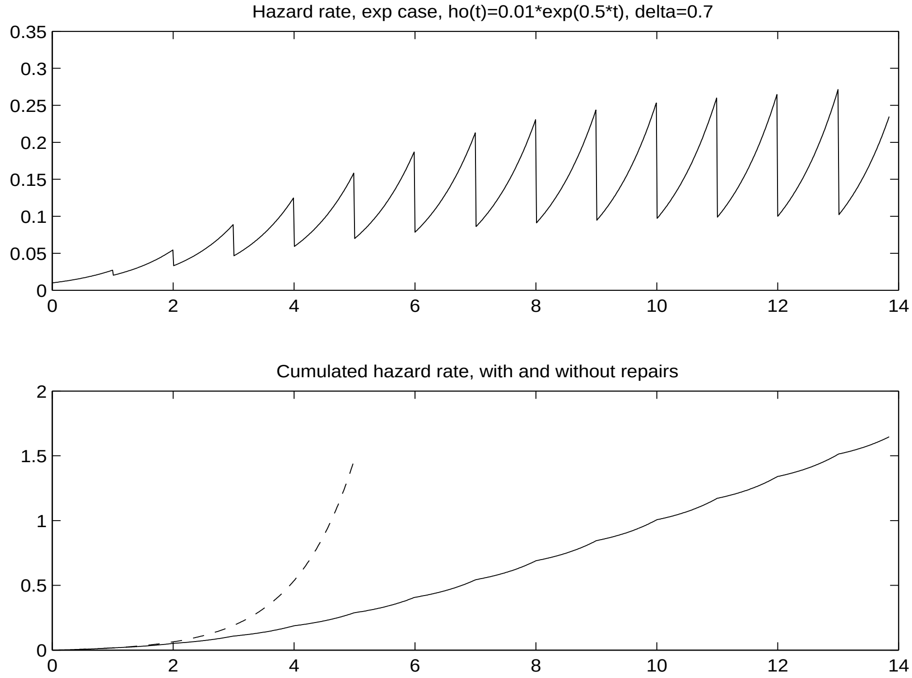

Figure 1: Case of exponentially increasing $h_0(t) = 0.01 * \exp(0.5 * t), \delta = 0.7, \Delta = 1$

yielding the minimal costs of repairs, which has a sense especially in the case when the repairs after failures are too expensive.

Let C_0 be the cost of failure (and its repair), $C_1(\delta, \Delta)$ the cost of the preventive repair. Then the mean costs to a time t can be written as

$$C \approx C_0 \cdot E(N(t)) + \frac{t}{\Delta} \cdot C_1(\delta, \Delta),$$

where $E(N(t))$ is the mean number of failures up to t, which is actually $H(t)$, H is the cumulated intensity of failures under our repairs sequence. Namely $E(N(t)) \approx \frac{t}{\Delta} \cdot dH$. For instance, in the Weibull model with $H_0(t) = \alpha \cdot \exp^{\beta}$ we already have seen that $dH = \alpha \Delta^{\beta}(1 - \delta^{\beta})/((1 - \delta)^{\beta})$.

The problem is the selection of function C_1, it should reflect the extent of repair. It leads us to the idea to evaluate the level of system damage, deterioration, and connect the repair with its reduction.

5 Incomplete repair reducing the system deterioration

Let us therefore consider a function $S(t)$ (or a latent random process) evaluating the level of degradation after a time t of system usage. In certain cases we can imagine $S(t) = \int_0^t s(u)du$ with $s(u) \geq 0$ is a stress at time u. We further assume that the failure occurs when $S(t)$ crosses a random level X. Recall also that (in the non-repaired system) the cumulated hazard rate $H_0(t)$ of random variable T = time-to failure has the similar meaning, namely the failure occurs when $H_0(t)$ crosses a random level given by Exp(1) random variable,

Hence, as $T > t <=> X > S(t)$, i.e. $\bar{F}_0(t) = \bar{F}_X(S(t))$, where by $\bar{F}$ we denote the survival function, then

$$H_0(t) = -log\bar{F}_X(S(t)).$$

We can again have some special cases, for instance:

– $X \sim$ Exp(1), then $H_0(t) = S(t)$,

– $S(t) = c \cdot t^d$, $d \geq 0$, and X is Weibull (a, b), then T is also Weibull ($\alpha = ac^b, \beta = b \times d$), i.e. $H_0(t) = \alpha \cdot t^\beta$.

Let us now imagine that the repair reduces $S(t)$ as in the Kijima II model, to $S^*(t) = \delta \cdot S(t)$. In the Weibull case considered above we are able to connect such a change with the reduction of virtual time from t to some t^*:

$$S(t^*) = S^*(t) => t^* = \delta^{\frac{1}{d}} \cdot t,$$

so that the virtual time reduction follows the Kijima II model, too, with $\delta_t = \delta^{\frac{1}{d}}$. As it has been shown, each selection of δ, Δ leads (converges) to a stable ('constant' intensity) case.

For other forms of function $S(t)$, e.g. if it is of exponential form, $S(t) \sim e^{ct} - 1$, such a tendency to a constant intensity does not hold. Nevertheless, it is possible to select convenient δ and Δ, as noted in Remark 1. The case of growth of the intensity of failures is shown in Figure 2, where $\delta = 0.7$ is the order of reduction of $S(t) = H(t)$, i.e. the case has $X \sim \text{Exp}(1)$ distribution. Figure 3 then shows the case with the same initial distributions, but with larger reduction $\delta = 0.3$ and the result - stabilized intensity. We have to remind, that here we reduce the degradation S by δ while in the preceding part the age was reduced!

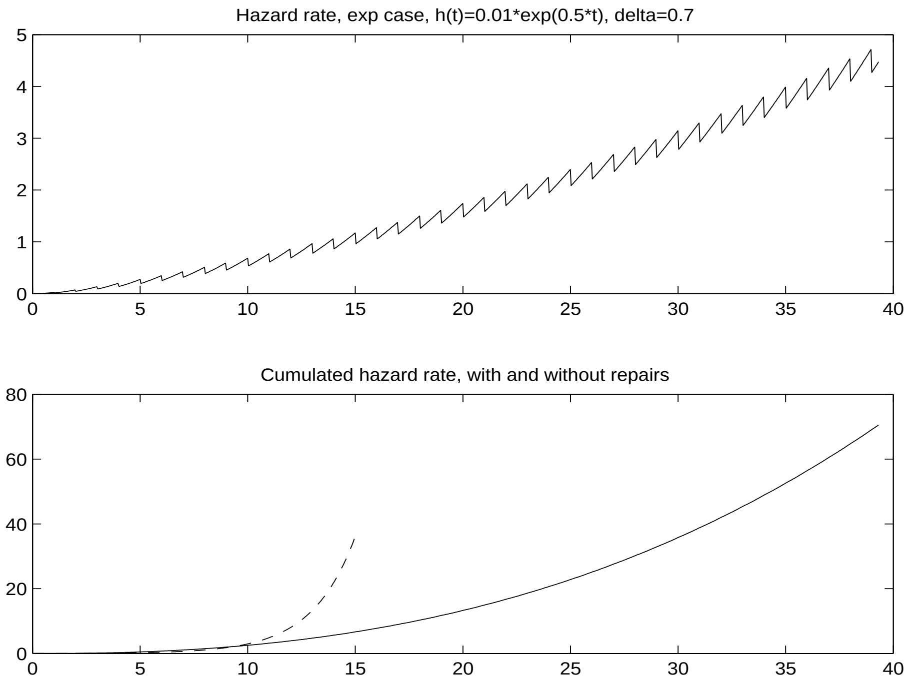

Figure 2: Growing intensity in the case $h_0(t) = 0.01 * exp(0.5 * t), \delta = 0.7$, interval $\Delta = 1$

6 Degradation as a random process

In the case we cannot observe the function $S(t)$ directly, and it is actually just a latent factor influencing the lifetime of the system, it can be modeled as a random process. What is the convenient type of such a process? There are several possibilities, for instance:

1. $S(t) = Y \cdot S_0(t)$, $Y > 0$ is a random variable,
2. Diffusion with trend function $S_0(t)$ and $B(t)$-the Brown process, $S(t) = S_0(t) + B(t)$.
3. $S(t)$ cumulating a random walk $s(t) \geq 0$.
4. Compound Poisson process (and its generalizations).

Though the last choice, sometimes connected also with the "random shock model", differs from the others, because its trajectories are not continuous, we shall add several remarks namely

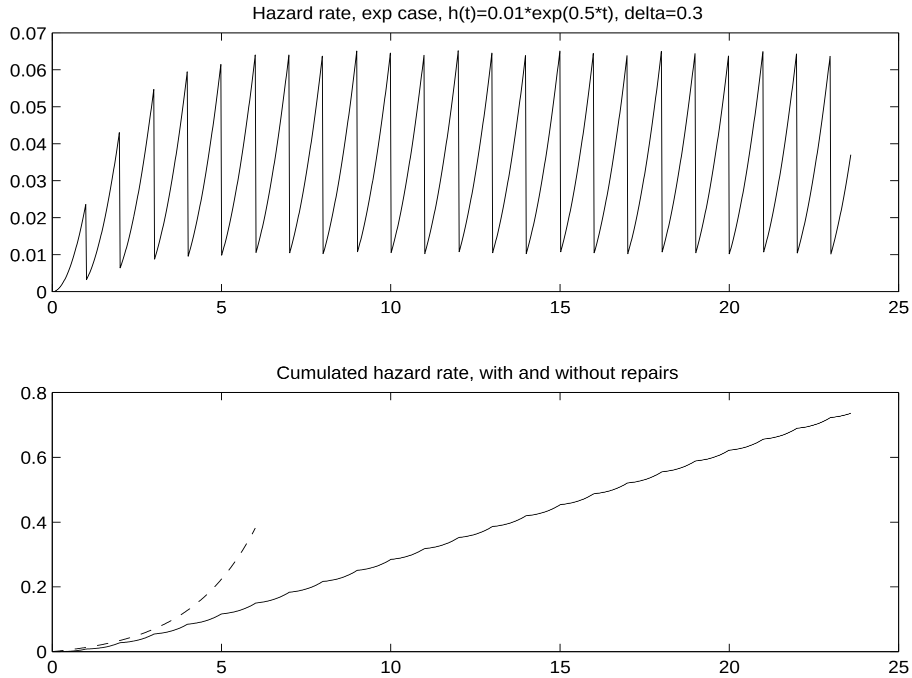

Figure 3: Stabilized intensity in the case $h_0(t) = 0.01 * exp(0.5 * t), \delta = 0.3$, interval $\Delta = 1$

to this case. The compound point process is the following random sum

$$S(t) = \sum_{T_j < t} Y(T_j) = \int_0^t Y(u)dN(u)$$

with the counting process $N(t)$ yielding the random times T_j and random variables $Y(t) > 0$ giving the increments. It holds that

$$ES(t) = \int_0^t \lambda(u) \cdot \mu(u)du, \;\; var(S(t)) = \int_0^t \lambda(u) \cdot (\mu^2(u) + \sigma^2(u))du.$$

Again, it is assumed that the failure occurs when the process $S(t)$ crosses a level x.

Hence, $S(t) < x <=> t < T$, therefore $\bar{F}_0(t) = F_{S(t)}(x)$, where $F_{S(t)}(x)$ is the compound distribution at t. If X is a random level, then the right side has the form $\int_0^\infty F_{S(t)}(x)dF_X(x)$.

The evaluation of the compound distribution is not an easy task, nor in the simplest version of compound Poisson process. There exist approximations (derived often in the framework of the financial and insurance mathematics). Another way how to evaluate it consists in the random generation.

In the next part we shall deal only with the simplest case of constant λ and μ, with exponentially (and independently on the process history) distributed increments and a constant bound x.

Figure 4 displays such a case, with $\mu = 0.2$ and $\lambda = 5$. 1000 trajectories were generated, the upper plot shows only 20 of them, while the lower plot shows resulting (i.e. empirical) distribution of the time of crossing the level $x = 10$. Next Figure 5. shows, how again a regular preventive repair with sufficiently large repair degree – here $\delta = 0.4$ –, leads to stabilized intensity of failures, histogram of failures times has a form of exponential distribution density.

6.1 Partial repairs and their optimization

What occurs when, as in the preceding cases, the repairs of degree $(1-\delta)$ in regular time intervals Δ are applied to the system? It is assumed that when we decide to repair, then we are able to

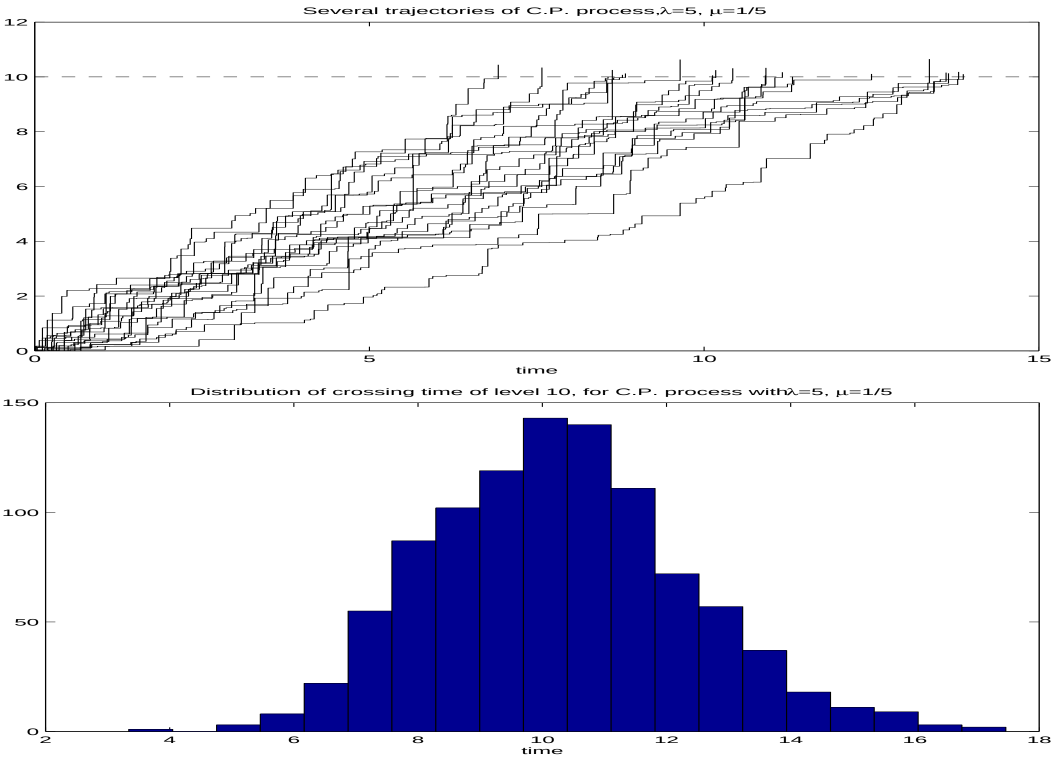

Figure 4: Several randomly generated trajectories of compound Poisson process (above) and generated distribution of time-to failure (below)

observe actual state of $S(t)$. Random generation shows that the system then behaves similarly as in the non-randomized case, and has the tendency to stabilize the intensity.

We can now return to the 'cost optimization' problem which has been already described in the part 2.1, but without specifying the function $C_1(\delta, \Delta)$. It can be done now for instance as $C_1 \cdot (dS(t))^\gamma + C_2$, where $dS(t) = S(t)(1-\delta) = S(t_{end}) - S(t_{init})$,
C_1 and C_2 are constants, the later evaluating a fixed cost of each repair. Hence the mean costs per time t can be expressed as

$$C_0 \cdot E(N(t)) + \frac{t}{\Delta} \cdot [C_1\{(1-\delta)S(t_{rep})\}^\gamma + C_2]$$

Following figures display several examples of search for optimal repair parameters. Thus, Figure 6 concerns to search for optimal δ, for given fixed virtual age when repair is performed $t_{end} = 5, S(t) = c * td$ is a degradation function (non-random), with $c = 0.5, d = 0.5, X \sim Weib(0.1, 5), gamma = 1, C_0 = 10, C_1 = C_2 = 1$.

Figure 7 shows the case when distribution of $X \sim -Weib(a = 10, b = 7)$, $\mathrm{E}X = 9.35, S = exp(ct) - 1, c = 0.05, \gamma = 1, C_0 = 10, C_1 = C_2 = 1$. We fixed the virtual age to which the component is repaired: $t_{init} = 5$, interval between repairs Δ is optimized.

Of course, a proper selection of costs and function $C1$ in real case is a matter of system knowledge and experience. We performed several randomly generated examples, in some cases it has been possible to find a minimum w.r. to δ and Δ, for given other parameters, mostly a minimum of Δ to fixed δ, while optimal δ to selected Δ lied often close to complete or minimal repair degree. Figure 7 shows one such an example, where degradation is given by a compound Poisson process with $\lambda = 5, \mu = 0.2$, further $C_0 = 20, C_1 = C_2 = 1, \gamma = 1$. Values of mean costs are displayed in a form of contour plot,

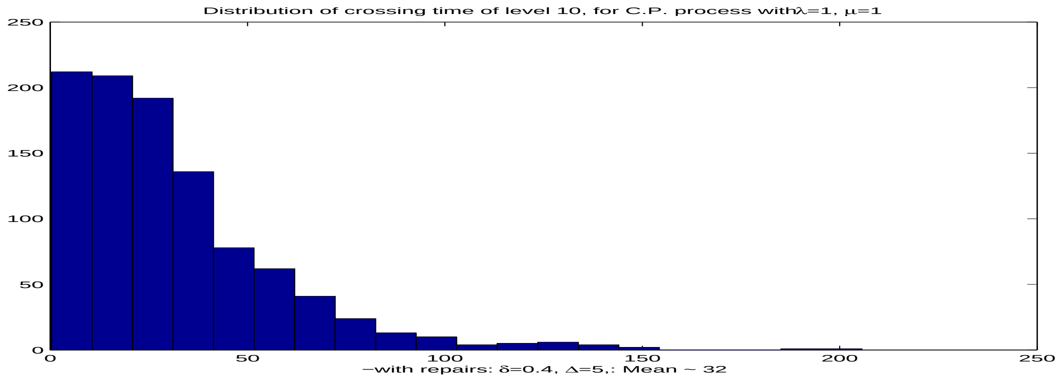

Figure 5: Generated distribution of time-to failure when repairs with $\delta = 0.4$ and $\Delta = 5$ are applied in the case of Fig. 4

7 Degradation process as a part of intensity model

When the degradation process is just one of factors influencing the survival of the system, it is quite natural to use it as a covariate in a regression model of failure intensity, In the situation when such a factor is not observed directly, it is more appropriate to use a model with latent component. In every case, it has a sense to consider the intensity of failure having several parts, one of them expressing the influence of the degradation process of the interest. Moreover, if the additive form of the intensity model is used (as in the case of Aalen regression model for intensity), the components stay separated even when integrated to cumulated intensity. Let us therefore recall several basic regression models for intensities of failures:

1. In the additive (also Aalen's) model, the total intensity is the sum of the intensities of components, e.g. $h(t) = h_1(t) + h_2(t)$.

2. In the multiplicative model $h(t) = h_1(t) \cdot h_2(t)$. The Cox's model uses the form $h(t) = h_0(t) \cdot \exp(B(z(t))$, where $z(t)$ is the regressor, i.e. in our case some characteristics of the deterioration (compare also Bagdonavicius and Nikulin, 1999).

3. Accelerated failure-time model $H(t) = H_0(V(t))$ was already briefly recalled here, too, in the connection with the model of growing virtual age proposed for instance in the paper of Finkelstein (2000).

The schemes of regression mentioned above offer different possibilities how to model the impact of degradation and then of repairs. Let us demonstrate it on the case of the multiplicative model. Namely, let the underlying hazard rate of a non-repaired system be $h_0(t) \cdot \exp(S(t))$, where the function $S(t) > 0$ is non-decreasing and characterizes the degradation of a repairable component. Let us for the simplicity consider just full repairs, in regular time intervals Δ, and follow the system without failure. It starts at time 0, at times $n \cdot \Delta$ its intensity of failure is $h(n \cdot \Delta) = h_0(n \cdot \Delta) \exp(S(\Delta))$, which is by the repair reduced to $h_0(n \cdot \Delta)$ ($S(t)$ is reduced to 0). Thus, we can here speak about a constant degree ($\exp(-S(\Delta))$ of the reduction of intensity, but if h_0 is increasing, the whole $h(t)$ remains increasing by the same trend. In the time interval $s \in ((n-1)\Delta, n\Delta)$ the intensity is then $h(s) = h_0(s) \exp\{S(s - (n-1)\Delta)\}$. Consequently, it yields the case different from the accelerated scheme studied in Finkelstein (2000).

The assumption of additive hazards leads to another set of models. It is also worth to note that the additive model corresponds to certain extent to the case of serial system. In a serial scheme of two independent parts the failure time of the system $T = min(T_1, T_2)$, i.e. $\bar{F}(t) = \bar{F}_1(t) \cdot \bar{F}_2(t)$, so that $H(t) = H_1(t) + H_2(t)$, too.

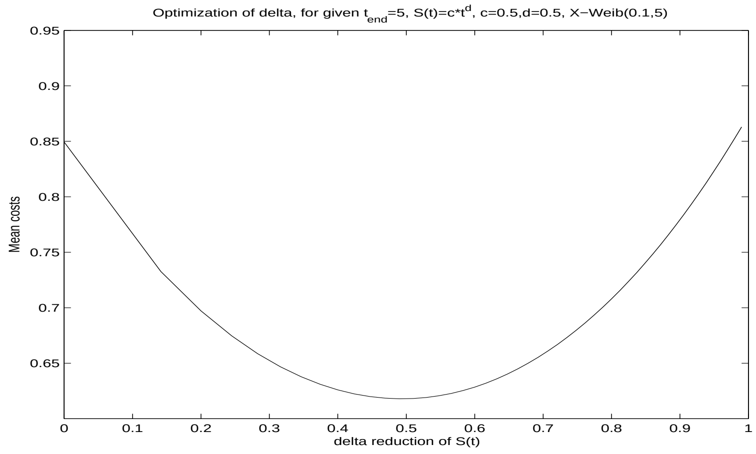

Figure 6: Optimization of δ, for given $t_{end} = 5, S(t) = c * td, c = 0.5, d = 0.5, X \sim Weib(0.1, 5)$

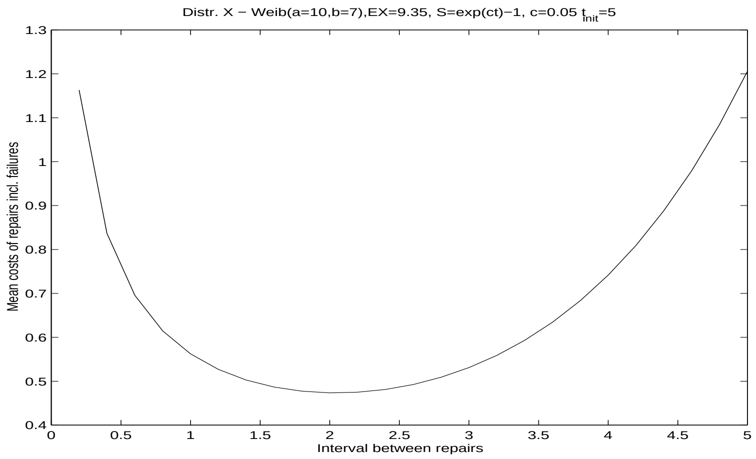

Figure 7: Distr. $X \sim Weib(a = 10, b = 7), S = exp(ct) - 1, c = 0.05t_{init} = 5$

8 Load-sharing System

One of interesting cases when it has a sense to deal with virtual age is such one when the survival distribution is changed (in discrete times) during the lifetime. An example could be the load sharing parallel (or K out of M) system.

Consider a parallel system of M components with the same survival distribution, conditioned by the load. More specifically, let Z be a "global" load (constant one, say) and the survival of components be Weibull ($\alpha(z)$, β). Even more concretely, let us consider actually the AFT model $Z_z = g(z) \cdot e$, $e \sim$ Weibull$(\alpha, \beta) = \alpha(z) = \alpha \cdot g(z)$. One of realistic choices of $g(z)$ is for instance $B \cdot (z + C)^{-A}$, a hyperbolic ("Nelder") curve.

Then, if the load z is divided among (shared by) M components, the load per component is $z_0 = \frac{z}{M}$. Let at time t_1 the first component break, now the load per unit increases to $z_1 = \frac{z}{M-1}$, α decreases from $\alpha_0 = \alpha \cdot g(z_0)$ to $\alpha_1 = \alpha \cdot g(z_1)$ (our $h(z)$ is decreasing). At that moment the actual age of remaining components is t_1, but from that moment they behave as having survival distribution $\overline{F}_1(t)$. How old are they in the scale of this distribution?

In the "standardized" exp(1) scale they just survived $H_0(t_1) = \left(\frac{t_1}{\alpha_0}\right)^\beta$, which corresponds to some $t_1^= H_1(t_1^*) = H_0(t_1)$. Hence, their 'virtual' age (in that sense) is

$$t_1^* = H_1^{-1}(H_0(t_1)) = t_1 \cdot \frac{\alpha_1}{\alpha_0}.$$

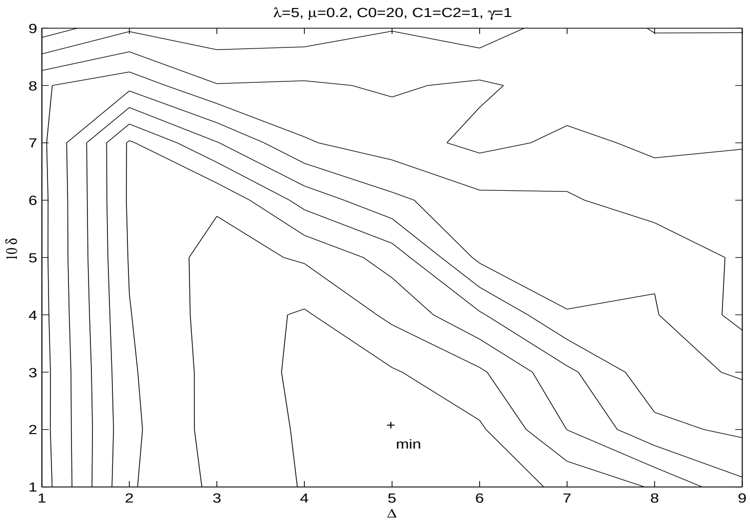

Figure 8: Mean costs corresponding to different Δ (horiz. axis) and δ (vert. axis)

Here, as $H_1 > H_0$, $t_1^* < t_1$.

We then can at t_1 recompute the future survival, taking already survival time t_1^* and distribution $\overline{F}_1(t)$.

After the 2-nd break, at real time t_2, the situation is similar. Components in the world of model $\overline{F}_1$ again can be recomputed to the age corresponding to the future model $\overline{F}_2 \sim$ Weib(α_2, β), $\alpha_2 = \alpha \cdot (z_2)$, $z_2 = z/M - 2$. Namely,

$$t_2^2 = t_2^* \frac{\alpha_2}{\alpha_1}, \quad \text{etc.}$$

Naturally, when a unit is repaired, it starts with its own 'history', and the same model. Hence, in described case both accelerated survival and shift of virtual age can be used.

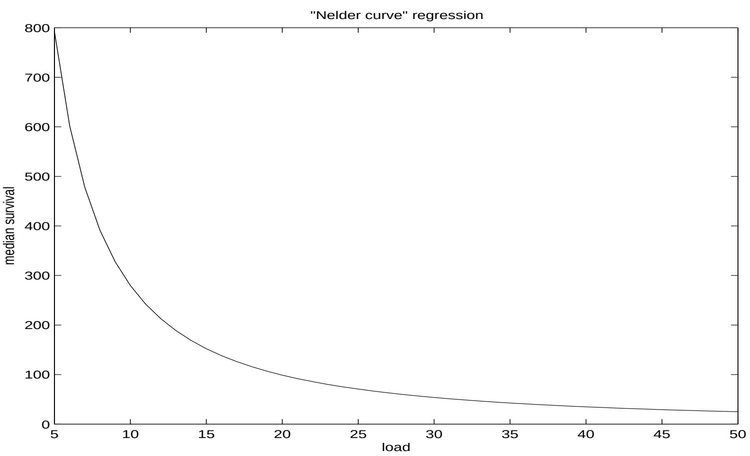

Figure 9:

Let us consider a numerical example, with $M = 5$, $\beta = 3$, $\alpha = 1\,000$, global load $z = 50$, model parameters $A = 1.5$, $C = 0$, $B = 1$. Then Figure 8 shows the medians of distributions Weib(d_z, β), $\alpha_z = \alpha / uz^A$ (loadsh71.eps). We performed simulations of failures and compare real time $t(i)$ and virtual times (ages) $t_V(i)$ recomputed after i-th failure, load $z(i)$ per 1 component and $\alpha(i)$ before i-th failure.

i	1	2	3	4	5
t_i	187	202	232	236	238
$t_V(i)$	134	96	69	25	–
$z(i)$	10	12.5	16.67	25	50
$\alpha(i)$	316	226	147	80	28

No repairs were considered in this "mini-study", however, as it has been said, they can be easily incorporated, too. It is not easy to compute the distribution of such load-shoring system, though there are some approximations available (Daniels, P. Kwam). Therefore we simulated the system survival many times, the resulting histogram follows in Figure 9 above. Figure 9 below shows, for the comparison, the survival distribution of purely parallel system of $M = 5$ components with Weibull ($\alpha = \alpha_0, \beta$) distribution, i. e. distribution of maxima of 5 i.i.d. Weibull variables.

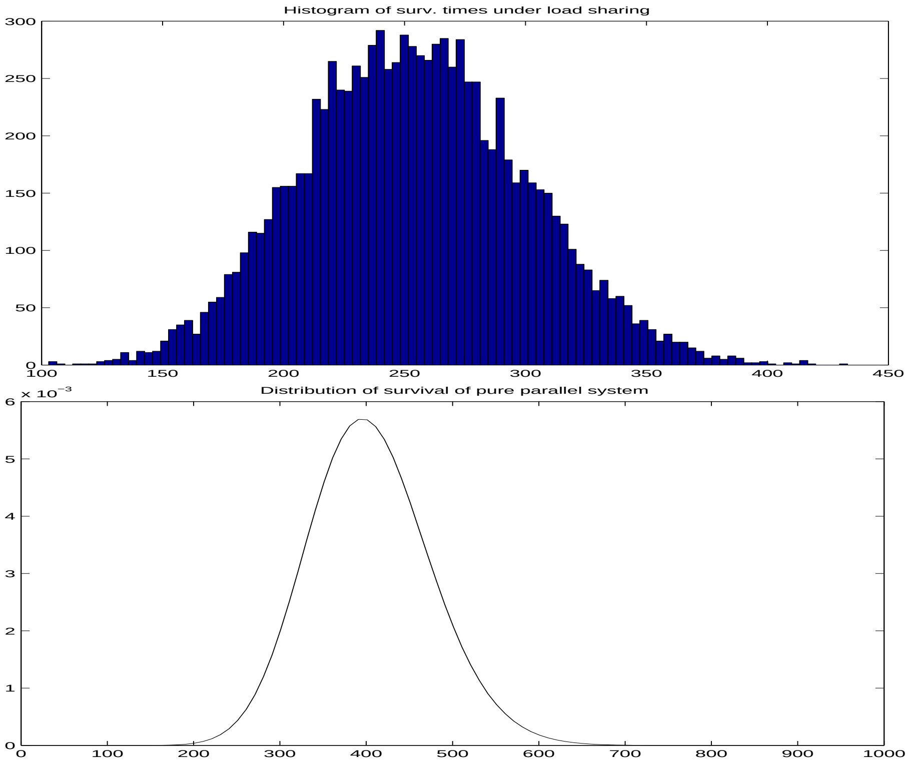

Figure 10: Comparison of final distributions

Recall that minimum has also Weibull distribution with $\alpha = \alpha_0/m^{\frac{1}{\beta}}$, with median about 164.

9 Conclusion and connected problems

The objective of the paper was to propose several new models of (incomplete) repairs based on the process of system deterioration. There are many different real cases corresponding to different models forms. However, especially if the deterioration process is latent, its proper modeling and estimation is crucial for further assessing the system optimal performance and repairs effect. The contemporary statistical techniques based on the Bayes approach and random generation can be very helpful in such analysis and should became the inevitable tool also in the future works on the deterioration and repair schemes modeling.

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References

Bagdonavicius, V.B. and Nikulin, M.S. (1999). Generalized proportional hazards model: estimation. *Lifetime Data Analysis*, No 3.

Dorado, C., Hollander, M. and Sethuraman, J. (1997). Nonparametric estimation for a general repair model. *The Annals of Statistics*, 1140-1160.

Finkelstein, M.S. (2000). Modeling a Process of Non-Ideal Repair. In N. Limnios and M. Nikulin (Eds.), *Recent Advances in Reliability Theory*, Birkhhauser, Series Statistics for Industry and Technology, 41-53.

Kahle, W. and Love, C.E. (2003). Modeling the Influence of Maintenance Actions. In B.H. Lindquist and K.A. Doksum (Eds.), *Mathematical and Statistical Methods in Reliability*, World Scientific Publishing Co., Series on Quality, Reliability and Engeneering Statistics, 387-400.

Kijima, M. (1989). Some results for repairable systems with general repair. *Journal of Applied Probability*, 26, 89-102.