Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 160 (2009) 766–777
FUZZY sets and systems
www.elsevier.com/locate/fss

Migrativity of aggregation functions

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Available online 15 October 2008

Abstract

In this paper we introduce a slight modification of the definition of migrativity for aggregation functions that allows useful characterization of this property. Among other things, in this context we prove that there are no t-conorms, uninorms or nullnorms that satisfy migrativity (with the product being the only migrative t-norm, as already shown by other authors) and that the only migrative idempotent aggregation function is the geometric mean. The *k*-Lipschitz migrative aggregation functions are also characterized and the product is shown to be the only 1-Lipschitz migrative aggregation function. Similarly, it is the only associative migrative aggregation function possessing a neutral element. Finally, the associativity and bisymmetry of migrative aggregation functions are discussed.

Keywords: Aggregation functions; Migrativity; Associativity; Bisymmetry; t-Norms; Uninorms; Nullnorms

1. Introduction

In this paper we analyze migrative aggregation functions by means of a clarifying characterization and offer an alternative approach to some recently obtained results [13].

Interest in the study of migrativity is associated with the key role that aggregation functions play in knowledge management whenever partial information from different times, places or circumstances need to be amalgamated into a global summary. These amalgamation processes are quite often associated with decision-making problems as decision aid techniques [20] that produce useful and manageable information for decision-makers [18,19]. Such a wide framework of potential applications requires a detailed analysis of each possible property that can be imposed in each context. In particular, classical aggregation functions require partial and global information in terms of a number within the unit interval and standard properties of monotonicity, commutativity and associativity (see [14] for a critical view). Migrativity refers to a certain ratio interchangeability between coordinates: if the intensity of a particular coordinate is reduced to $100 \cdot \alpha$ per cent, global evaluation will be the same regardless of which of its coordinates is being reduced.

The remainder of the paper is organized as follows. Section 2 outlines some basic definitions. In Section 3 we propose a slightly different definition for migrativity that is characterized in Section 4 to determine the functions that satisfy migrativity or idempotency. In Section 5 we analyze the case when a migrative function has a neutral element.

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0165-0114/$ - see front matter
doi:10.1016/j.fss.2008.09.018

We characterize aggregation functions that satisfy the 1-Lipschitz and migrative properties in Section 6. In Section 7 we address cases when associativity holds under migrativity and establish the relationship between migrativity and bisymmetry in Section 8. Section 9 highlights the main results and open problems for future research.

2. Definitions and properties

Recall that a *triangular norm* (t-*norm* for short) is a commutative, associative, non-decreasing function $T : [0,1]^2 \to [0,1]$ such that $T(1,x) = x$ for all $x \in [0,1]$ [1,4,10,12,15,21]. Three basic t-norms are the *minimum* t-norm

$$T_M(x,y) = \min(x,y) \quad \text{for all } x, y \in [0,1],$$

the *product* t-norm

$$T_P(x,y) = xy \quad \text{for all } x, y \in [0,1],$$

and the *Lukasiewicz* t-norm

$$T_L(x,y) = \max(x+y-1,0) \quad \text{for all } x, y \in [0,1].$$

A t-norm T is said to be *continuous* if T is a continuous function.

A *triangular conorm* (t-*conorm* for short) is a commutative, associative, non-decreasing function $S : [0,1]^2 \to [0,1]$ such that $S(0,x) = x$ for all $x \in [0,1]$.

Uninorms were introduced by Yager and Rybalov [22] (see also [8,9]) as any commutative, associative and non-decreasing function $U : [0,1]^2 \to [0,1]$ with a *neutral* element $e \in]0,1[$ (a formal definition of a neutral element is given below, but note that the excluded extreme values $e = 0$ and $e = 1$ correspond to t-conorms and t-norms, respectively).

Nullnorms were introduced by Calvo et al. [5] as any binary commutative, associative and non-decreasing function $V : [0,1]^2 \to [0,1]$ such that there exists a value $a \in]0,1[$ such that $V(x,0) = x$ for all $x \in [0,a]$ and $V(x,1) = x$ for all $x \in [a,1]$ (note that in this case the excluded extreme values $a = 0$ and $a = 1$ correspond to t-norms and t-conorms, respectively).

All the above definitions are particular cases of aggregation rules, which play a key role in any amalgamation process: from a unique binary function, thanks to associativity, we can obtain an aggregated value for any finite number of pieces of information. An *aggregation rule* was formally defined by Calvo et al. [6] (see also [16]) as a function

$$A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$$

such that

(1) $A(x_1,\ldots,x_n) \leqslant A(y_1,\ldots,y_n)$ whenever $x_i \leqslant y_i$ for all $i \in \{1,\ldots,n\}$.
(2) $A(x) = x$ for all $x \in [0,1]$.
(3) $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.

Observe that since an aggregation rule A can be canonically represented [3] by a family $\{A_n\}_{n\in\mathbb{N}}$ of n-ary functions where each

$$A_n : [0,1]^n \to [0,1]$$

is given by

$$A_n(x_1,\ldots,x_n) = A(x_1,\ldots,x_n),$$

item (2) above means that $A_1(x) = x$. A consistent constructive approach to aggregation rules based on the recursive application of binary functions can be found elsewhere [2,7].

This paper focuses on a discussion of binary aggregation functions, understood as non-decreasing functions

$$A : [0,1]^2 \to [0,1]$$

such that $A(0,0) = 0$ and $A(1,1) = 1$. Basic results in this context should be then translated to the whole aggregation operation via associativity or recursivity [2,7].

Moreover, recall that:

- An aggregation function A is called *bisymmetric* if

$$A(A(x,y), A(z,t)) = A(A(x,z), A(y,t)) \quad \text{for all } x, y, z, t \in [0,1].$$

- An element $e \in [0,1]$ is called a *neutral element* of the aggregation function A if

$$A(x,e) = A(e,x) = x \quad \text{for all } x \in [0,1].$$

- An element $a \in [0,1]$ is called an *annihilator* of the aggregation function A if

$$A(a,x) = A(x,a) = a \quad \text{for all } x \in [0,1].$$

3. Migrativity

Durante and Sarkoci [11], as reported by Fodor and Rudas [13], introduced the term α-migrative for a class of binary operations having the following property, already described by Mesiar and Novák [17], Problem 1.8(b): given α in $]0,1[$, a binary function

$$G : [0,1]^2 \to [0,1]$$

is said to be α-*migrative* if

$$G(\alpha x, y) = G(x, \alpha y) \quad \text{for all } x, y \in [0,1].$$

Note that such a definition applies to each fixed value $\alpha \in]0,1[$, but not to values $\alpha \notin]0,1[$.

We then propose the following definition.

Definition 1. A function $G : [0,1]^2 \to [0,1]$ is called migrative if and only if

$$G(\alpha x, y) = G(x, \alpha y) \quad \text{for all } x, y \in [0,1] \tag{1}$$

and every $\alpha \geqslant 0$ such that $\alpha x, \alpha y \in [0,1]$.

This definition of migrativity is equivalent to imposing

$$G(\alpha x, \beta y) = G(\beta x, \alpha y) \quad \text{for all } x, y \in [0,1]$$

for any $\alpha, \beta \in [0,1]$ such that $\alpha x, \beta y, \beta x, \alpha y \in [0,1]$.

Nevertheless, this definition of migrativity holds when every possible α-migrativity holds, including $\alpha = 0$ and even values $\alpha > 1$ whenever this is possible. Obviously, 1-migrativity is a trivial property (all binary functions are 1-migrative).

In the following proposition we present a useful graphical characterization of 0-migrativity, to be compared with a subsequent characterization of migrativity introduced in the next section.

Proposition 1. *A function* $G : [0,1]^2 \to [0,1]$ *is* 0-*migrative if and only if*

$$G(x, 0) = G(0, y) \quad \textit{for all } x, y \in [0,1].$$

Proof. From 0-migrativity it follows that

$$G(0, y) = G(0 \cdot x, y) = G(x, 0 \cdot y) = G(x, 0),$$

which clearly holds for every choice of x and y. The other implication is trivial. □

Of course, in case G is a continuous aggregation function, 0-migrativity is implied from the rest of the α-migrativities, $\alpha > 0$. However, this implication is not true in general, as shown by the following discontinuous aggregation function.

Example 1. The strongest aggregation function

$$G_s(x, y) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise} \end{cases}$$

is not 0-migrative: by definition $G_s(1, 0) = 1$, but

$$G_s(1, 0) = G_s(1, 0 \cdot 0) = G_s(0 \cdot 1, 0) = G_s(0, 0) = 0.$$

However, it is α-migrative for every $\alpha > 0$:

$$G_s(\alpha \cdot 0, 0) = G_s(0, \alpha \cdot 0) = G(0, 0) = 0$$

and

$$G_s(\alpha x, y) = G_s(x, \alpha y) = 1$$

whenever $x > 0$ or $y > 0$.

Note also that if $G : [0, 1]^2 \to [0, 1]$ is α-migrative for all $\alpha > 0$, this guarantees that $G(x, 0) = G(1, 0)$, for all $x \in]0, 1]$. In fact,

$$G(x, \alpha \cdot 0) = G(\alpha \cdot x, 0) \quad \text{for all } \alpha \in]0, 1]$$

in such a way that αx takes any value within $]0, 1]$ whenever $x > 0$. Analogously, $G(0, y) = G(0, 1)$, for all $y \in]0, 1]$ holds. And since we show below that migrativity implies symmetry, it must be true that

$$G(x, 0) = G(1, 0) = G(0, 1) = G(0, y) \quad \text{for all } x, y \in]0, 1]$$

whenever α-migrativity holds for all $\alpha > 0$.

4. Characterization of migrativity

The key representation result for migrative aggregation functions is a consequence of the next two lemmas.

Lemma 1. *A function $G : [0, 1]^2 \to [0, 1]$ is migrative if and only if there exists a function $g : [0, 1] \to [0, 1]$ such that $G(x, y) = g(xy)$ for all $x, y \in [0, 1]$.*

Proof. On one hand, if the function g exists,

$$G(\alpha x, y) = g(\alpha x y) = G(x, \alpha y).$$

On the other hand, given a migrative G, since

$$G(x, y) = G(1, xy) \quad \text{for all } x, y \in [0, 1],$$

we have that $G(x, y) = G(u, v)$ whenever $xy = uv$. Hence, such a function g is well and univocally defined. □

In other words, a mapping $G : [0, 1]^2 \to [0, 1]$ verifies migrativity if and only if

$$G(x, y) = G(1, xy) \quad \text{for all } x, y \in [0, 1]$$

and the function $g(\cdot)$ is nothing other than the partial function $G(\cdot, 1) = G(1, \cdot)$. That is, g is determined by the upper bound of G.

Moreover, it is then immediate that migrativity implies symmetry:

$$G(x, y) = G(1, xy) = G(y1, x) = G(y, x)$$

for all $x, y \in [0, 1]$, and if we say that $G : [0, 1]^2 \to [0, 1]$ is strictly increasing only if $G(x_1, x_2) < G(y_1, y_2)$ whenever $x_1 \leqslant y_1$, $x_2 \leqslant y_2$, with at least one of these two inequalities being strict, we have the following immediate results.

Lemma 2. *Let $G : [0,1]^2 \to [0,1]$ be a migrative function. Then*

(1) *G is non-decreasing if and only if g is non-decreasing.*
(2) *G is strictly increasing in $]0,1]^2$ if and only if g is strictly increasing.*
(3) *$G(1,1) = 1$ if and only if $g(1) = 1$.*
(4) *$G(0,0) = 0$ if and only if $g(0) = 0$.*
(5) *G is continuous if and only if g is continuous.*
(6) *G is idempotent if and only if $g(x) = \sqrt{x}$, for all $x \in [0,1]$, i.e., G is the geometric mean:*

$$G(x,y) = \sqrt{xy}, \quad \text{for all } x, y \in [0,1]$$

We can therefore reach a direct but relevant conclusion for aggregation functions merely by assuming migrativity and without imposing associativity, commutativity or continuity (see [14] for a discussion of the relevance of standard assumptions for aggregation functions):

Theorem 1. *A function $G : [0,1]^2 \to [0,1]$ is a migrative aggregation function if and only if*

$$G(x,y) = g(xy) \quad \text{for all } x, y \in [0,1]$$

holds for some non-decreasing function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(1) = 1$.

In this way, the function *g generates* the aggregation function *G*, which is univocally characterized by *g*. Moreover, from item 6 of Lemma 2 we realize that *the only function G that is migrative and idempotent is the geometric mean aggregation function*. Note also that:

(1) The weakest migrative aggregation function is merely the weakest aggregation operation; that is,

$$G_w(x,y) = \begin{cases} 1 & \text{if } xy = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and its migrative generator is

$$g_w(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(2) The strongest migrative aggregation function is the strongest aggregation function with annihilator 0; that is,

$$G_{sa}(x,y) = \begin{cases} 0 & \text{if } xy = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and its migrative generator is

$$g_{sa}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

(3) The strongest aggregation function,

$$G_s(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise} \end{cases}$$

is not migrative, as explained above.

5. Migrativity and neutral elements

In this section we see how the existence of the neutral element of migrative non-decreasing functions determines the shape of aggregation functions.

Note first that since

$$G(0,0)=G(x,0)=G(0,y) \quad \text{for all } x,y\in[0,1]$$

follows from 0-migrativity, this guarantees that $e\neq 0$ for any migrative function G possessing a neutral element e. Moreover, since G is migrative, if such a neutral element $e\in]0,1]$ exists, it must be

$$G(x,e)=G(e,x)=g(ex)=x \quad \text{for all } x\in[0,1]$$

whenever $ex\in[0,1]$. Consequently, $g(e)=G(e,1)=1$ and thus $G(x,y)=1$ whenever $xy\geqslant e$. On the other hand, if $u=xy<e$, then

$$g(u)=G(x,y)=G\left(\frac{xy}{e},e\right)=\frac{xy}{e}=\frac{u}{e}.$$

Hence,

$$g(x)=\min\left(1,\frac{x}{e}\right) \quad \text{for all } x\in[0,1] \tag{2}$$

whenever the existence of a neutral element is assumed.

Theorem 2. *Let $G:[0,1]^2\to[0,1]$ be a non-decreasing migrative function with neutral element $e\in]0,1]$. Then G is a continuous commutative aggregation function given by*

$$G(x,y)=\min\left(1,\frac{xy}{e}\right) \quad \textit{for all } x,y\in[0,1]. \tag{3}$$

Proof. This is a direct consequence of Eq. (2). Hence, $g(x)=\frac{x}{e}$ for all $x\in[0,e]$ and $g(x)=1$ for all $x\in[e,1]$. It is then easy to check that G is commutative and continuous, with $G(0,0)=0$ and $G(1,1)=1$. □

Note that a function $G:[0,1]^2\to[0,1]$ has 0 as annihilator if and only if G is 0-migrative and $G(0,0)=0$. Hence, according to our definition, 0 is an annihilator for every migrative aggregation function G. Moreover, if $G(x,0)=0$ for all $x\in[0,1]$, then such an annihilator is unique: if $G(a,x)=a$ for all $x\in[0,1]$, then it must be $G(a,0)=a$ and therefore $a=0$.

Moreover we must point out that the existence of a neutral element cannot be guaranteed for a non-decreasing and migrative function. In fact, the aggregation functions G generated by the following functions g are non-decreasing and migrative, but have no neutral element:

- $g(x)=x/2$, for all $x\in[0,1]$.
- $g(x)=x^2$, for all $x\in[0,1]$.
- $g(x)=1-(1-x)^2$, for all $x\in[0,1]$.
- $g(x)=\sqrt{x}$, for all $x\in[0,1]$.

6. Migrativity and the k-Lipschitz property

Recall that a binary aggregation function $G:[0,1]^2\to[0,1]$ is k-Lipschitz if and only if

$$|G(x,y)-G(x',y')|\leqslant k(|x-x'|+|y-y'|)$$

for all $x,x',y,y'\in[0,1]$ and some $k\in]0,\infty[$.

We then prove that the only 1-Lipschitz migrative binary aggregation function is the product.

Theorem 3. *A migrative binary aggregation function G is k-Lipschitz if and only if $k\geqslant 1$ and the corresponding migrative generator g is k-Lipschitz, i.e., $|g(x)-g(x')|\leqslant k|x-x'|$ for all $x,x'\in[0,1]$.*

Proof. Recall that the migrative generator $g : [0, 1] \to [0, 1]$ of G is given by $g(x) = G(1, x)$ and thus its k-Lipschitzivity follows from that of G. Moreover, $g(1) - g(0) = 1$ and thus necessarily $k \geqslant 1$. Conversely, if a migrative aggregation function is generated by a k-Lipschitz migrative generator g, then owing to the symmetry of G it is enough to show that for any fixed $y \in [0, 1]$ the horizontal section $G(\cdot, y)$ is k-Lipschitz. However, this easily follows from the k-Lipschitz property of g. Indeed, $|G(x, y) - G(x', y)| = |g(xy) - g(x'y)| \leqslant k|xy - x'y| \leqslant k|x - x'|$. □

Observe that if a function $g : [0, 1] \to [0, 1]$ fulfills $g(0) = 0$ and $g(1) = 1$, then the weakest function with these bounds that is k-Lipschitz (with $k \geqslant 1$) is given by $g_{(k)}(x) = \max(0, kx - k + 1)$. Similarly, the strongest k-Lipschitz function with these properties is given by $g^{(k)}(x) = \min(kx, 1)$. Evidently, for each migrative generator g of a k-Lipschitz migrative aggregation function G, $g_{(k)} \leqslant g \leqslant g^{(k)}$ holds. For $k = 1$, we have $g_{(1)} = g^{(1)} = \mathrm{id}$, i.e., there is a unique 1-Lipschitz migrative aggregation function.

Corollary 1. *The only* 1-*Lipschitz and migrative binary aggregation function is the product*.

7. Migrativity and associativity

In this section we analyze associativity within migrative aggregation functions.

Theorem 4. *Let* $G : [0, 1]^2 \to [0, 1]$ *be a non-decreasing migrative function with neutral element* e. *Then G is associative if and only if* $e = 1$, *i.e.*,

$$G(x, y) = xy, \quad \textit{for all } x, y \in [0, 1].$$

Proof. On one hand, if $e < 1$, and since $e > 0$, we can take $x, z \in [0, 1]$ such that $x < e < z$ with $xz = e^2$. Then it follows that:

$$G(G(x, 1), z) = G\left(\frac{x}{e}, z\right) = 1$$

but

$$G(x, G(1, z)) = G(x, 1) = \frac{x}{e}.$$

Therefore, associativity would not hold. On the other hand, associativity follows directly for $e = 1$. □

Please note the similarity of the above result to the characterization of the product t-norm reported by Fodor and Rudas [13] in their Proposition 4. Nevertheless, since the above product t-norm is the only migrative aggregation function that is associative and has a neutral element, and because the only possible annihilator for a migrative function is 0, we can conclude the following:

Corollary 2. *There are no migrative t-conorms*, *uninorms or nullnorms*.

Theorem 5. *Let* $G : [0, 1]^2 \to [0, 1]$ *be a migrative function that is strictly increasing on* $]0, 1]^2$. *Then G is associative if and only if*

$$G(x, y) = cxy \quad \textit{for all } x, y \in [0, 1],$$

for some $0 < c \leqslant 1$.

Proof. Owing to migrativity, the associativity of G means that

$$g(xg(yz)) = g(g(xy)z) \quad \text{for all } x, y, z \in [0, 1].$$

Since g is strictly increasing, the inverse g^{-1} is well defined on the image of g in such a way that

$$xg(yz) = g(xy)z$$

holds whenever $x, y, z \neq 0$. Taking $y = 1$, we have that

$$xg(z) = g(x)z$$

in such a way that

$$\frac{g(x)}{x} = \frac{g(z)}{z} \quad \text{for all } x, z \neq 0.$$

Hence, there exists a value c such that

$$\frac{g(x)}{x} = c \quad \text{for all } x \neq 0$$

and it must follow that $g(x) = cx$ for all $x \in [0, 1]$, with $0 < c \leqslant 1$ (otherwise $g(1) > 1$). □

The next two results show a pattern for migrative aggregation functions that are associative (more research is needed before a general characterization is possible).

Lemma 3. *Let* $G : [0, 1]^2 \to [0, 1]$ *be an associative and migrative aggregation function generated by* $g : [0, 1] \to [0, 1]$. *Then*

$$g(x) = x \quad \textit{for all } x \in \mathrm{Ran}(g).$$

Proof. From the associativity of G, we have that

$$g(x) = g(x1) = g(xg(11)) = g(g(x1)1) = g(g(x)1) = g(g(x))$$

for all $x \in [0, 1]$. That is, $g(x) = x$ for all $x \in \mathrm{Ran}(g)$. □

In this way, the associativity of a migrative aggregation function G means that

$$\begin{aligned} G(x, G(y, z)) &= G(x, G(yz, 1)) = G(G(x, yz), 1) \\ &= G(G(xyz, 1), 1) = g(g(xyz)) = g(xyz) \end{aligned}$$

for all $x, y, z \in [0, 1]$.

Lemma 4. *Let* $G : [0, 1]^2 \to [0, 1]$ *be a migrative and associative aggregation function generated by a function* $g : [0, 1] \to [0, 1]$ *that is continuous in the open interval* $]a, b[\subset [0, 1]$. *Then g takes the following form for any value within* $]a, b[$:

$$g(x) = \begin{cases} k_1 & \text{if } a < x < k_1, \\ x & \text{if } k_1 \leqslant x \leqslant k_2, \\ k_2 & \text{if } k_2 < x < b, \end{cases}$$

for some $k_1, k_2 \in [g(a), g(b)]$ *such that* $k_1 \leqslant k_2$.

Proof. Assume that g is continuous in $]a, b[$.

- If $g(x) = k$ for all $x \in]a, b[$ and some $k \in [g(a), g(b)]$, then g takes the above form with $k_1 = k_2 = k$.
- Otherwise, since g is continuous but not constant in $]a, b[$, then $g(]a, b[)$ will be an interval that can be denoted as $\langle k_1, k_2 \rangle$, with $k_1 < k_2$. That is,

$$g(]a, b[) = \langle k_1, k_2 \rangle.$$

Therefore, from Lemma 3 it must follow that $g(x) = x$ for all $x \in]k_1, k_2[$. Moreover, because of its non-decreasing property, $g(x) = k_1$ for all $x \in]a, k_1]$ and $g(x) = k_2$ for all $x \in [k_2, b[$ must hold. □

In particular, from Lemma 4 it is therefore immediate that if G is a migrative and associative aggregation function with a generator function g that is continuous in the open interval $]0,1[$, it must follow that

$$g(x)=\begin{cases}0 & \text{if } x=0,\\ k_1 & \text{if } 0<x<k_1,\\ x & \text{if } k_1\leqslant x\leqslant k_2,\\ k_2 & \text{if } k_2<x<1,\\ 1 & \text{if } x=1,\end{cases}$$

for some $k_1,k_2\in[0,1]$ such that $k_1\leqslant k_2$, i.e.,

$$g(x)=\max(k_1,\min(x,k_2))=\min(k_2,\max(x,k_1))$$

for all $x\in[0,1]$. Therefore, G takes the form

$$G_{k_1,k_2}(x,y)=\begin{cases}0 & \text{if } xy=0,\\ k_1 & \text{if } 0<xy<k_1,\\ xy & \text{if } k_1\leqslant xy\leqslant k_2,\\ k_2 & \text{if } k_2<xy<1,\\ 1 & \text{if } xy=1,\end{cases}$$

for some $k_1\leqslant k_2$.

Note that we do not exclude $k_1=0$ or $k_2=1$ (whenever $k_1>0$ then g is discontinuous in $x=0$, and whenever $k_2<1$ then g is discontinuous in $x=1$). However, not all of these aggregation functions are in fact associative, as shown in the next theorem.

Theorem 6. *Let $G:[0,1]^2\to[0,1]$ be a migrative aggregation function generated by a function $g:[0,1]\to[0,1]$ that is continuous in $]0,1[$. Under these conditions the only associative aggregation functions G are either*

$$G_{k,k}(x,y)=\begin{cases}0 & \textit{if } xy=0,\\ k & \textit{if } 0<xy<1,\\ 1 & \textit{if } xy=1,\end{cases}$$

for some $k\in[0,1]$, or

$$G_{k,1}(x,y)=\begin{cases}0 & \textit{if } xy=0,\\ k & \textit{if } 0<xy<k,\\ xy & \textit{if } k\leqslant xy\leqslant 1,\end{cases}$$

for some $k\in[0,1[$.

Proof. In this case, Lemma 4 applies with $a=0$ and $b=1$. However, owing to associativity this should be

$$g(xg(1z))=g(g(x1)z)\quad \text{for all } x,z\in[0,1]$$

If $k_1<k_2<1$, we can choose $x,z\in]k_2,1[$ such that

(1) $k_1<xz<k_2$ and
(2) $k_1<xk_2<k_2$.

To verify this, take $x\in]\max(k_2,\frac{k_1}{k_2}),1[$, so (2) holds, and note that xz varies continuously on $]xk_2,x[$ when z moves on $]k_2,1[$. Therefore, since $k_1<xk_2<k_2<x$, we can always find a value $z\in]k_2,1[$ such that $xz\in]xk_2,k_2[$ in such a way that (1) holds. However, then

$$xk_2=g(xk_2)=g(xg(z))=g(g(x)z)=g(k_2z)=k_2z.$$

Therefore, either $k_1=k_2$ or $k_2=1$. Then, if $k_1=k_2=k$ we have

$$g(x)=k\quad \text{for all } x\in]0,1[.$$

Otherwise, if $k_1 = k < k_2 = 1$ we have

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ k & \text{if } 0 < x < k, \\ x & \text{if } k \leqslant x \leqslant 1. \end{cases} \quad \square$$

In this way, for every G that is migrative, associative and continuous in $]0, 1[$, we have proved that there exists $k \in [0, 1]$ such that either $g(x) = k$, for all $x \in]0, 1[$ or $g(x) = \max(k, x)$, for all $x \in]0, 1[$. It is then immediate that the only migrative aggregation function that is continuous and associative is the product, i.e., $g(x) = x$ for all $x \in [0, 1]$ (which was, as proven in Corollary 1, the only migrative and 1-Lipschitz aggregation function).
Moreover, in the conditions for Theorem 6 we find some well-known aggregation functions as particular cases:

- If $k_1 = k_2 = 0$, then $G_{0,0} = G_w$.
- If $k_1 = k_2 = 1$, then $G_{1,1} = G_{sa}$.
- If $k_1 = 0$ and $k_2 = 1$, then $G_{0,1} = T_P$.

Moreover, we realize that the behavior in the unit interval can be fully determined from the behavior at these extremes of the unit interval, as now shown.

Theorem 7. *Let $G : [0, 1]^2 \to [0, 1]$ be an associative migrative aggregation function generated by a function $g : [0, 1] \to [0, 1]$ such that $g(x) = x$ for all $x \in [0, k]$ and some $k > 0$. Then the associated G is the product.*

Proof. Taking $0 < z < k$, we know that $xz < k$ and $xg(z) < k$ for any $x \in [0, 1]$. Thus,

$$g(x)z = g(g(x)z) = g(g(x1)z) = g(xg(1z)) = g(xg(z)) = g(xz) = xz$$

and, hence, $g(x)z = xz$ must hold for all $x \in [0, 1]$, i.e., $g(x) = x$ for all $x \in [0, 1]$. $\square$

8. Migrativity and bisymmetry

In this section we consider bisymmetry in relation to migrativity, which implies symmetry, as already pointed out. First we translate the representation obtained above for migrative aggregation functions into this context.

Lemma 5. *Let $G : [0, 1]^2 \to [0, 1]$ be a migrative aggregation function. Then G is bisymmetric if and only if there exists a function $g : [0, 1] \to [0, 1]$ that is non-decreasing with $g(0) = 0$, $g(1) = 1$ and such that for all $x, y, z, t \in [0, 1]$ the following functional equation holds*:

$$g(g(xy)g(zt)) = g(g(xz)g(yt)).$$

Proof. Directly from Theorem 1. $\square$

For example, it is easy to check that G_w, G_{sa} and $G_{k,k}$ are bisymmetric. We now prove that $G(x, y) = (xy)^p$ for all $x, y \in [0, 1]$, where $p > 0$, plays a key role in aggregation functions, verifying migrativity and bisymmetry.

Theorem 8. *Let $G : [0, 1]^2 \to [0, 1]$ be a strictly increasing and continuous aggregation function. Then it is migrative and bisymmetric if and only if*

$$G(x, y) = (xy)^p, \quad \textit{for all } x, y \in [0, 1]$$

for some $p > 0$.

Proof. If G is bisymmetric and migrative, then the function g that generates G verifies that $g(0) = 0$, $g(1) = 1$ and

$$g(g(xy)g(zt)) = g(g(xz)g(yt)) \quad \text{for all } x, y, z, t \in [0, 1].$$

Moreover, we know from Lemma 2 that G is strictly increasing if and only if g is strictly increasing. Therefore, the inverse g^{-1} is defined on the image of g and then we have that

$$g(xy)g(zt) = g(xz)g(yt) \quad \text{for all } x, y, z, t \in [0, 1].$$

If we now take $x = y = 1$, we have $g(zt) = g(z)g(t)$. Since g is continuous, the solution of this functional equation is either a constant function (which is against the hypothesis) or the function $g(x) = x^p$ with $p > 0$. That is, it must be

$$G(x, y) = g(xy) = g(x)g(y) = (xy)^p \quad \text{for all } y \in [0, 1]$$

for some $p > 0$.

Reciprocally, if $G(x, y) = (xy)^p$ for all $x, y \in [0, 1]$ for some $p > 0$, then $G(\alpha x, y) = (\alpha xy)^p = (x\alpha y)^p = G(x, \alpha y)$ for any $\alpha \in [0, 1]$.

Besides,

$$\begin{aligned} G(G(x, y), G(z, t)) &= (G(x, y)G(z, t))^p = ((xy)^p(zt)^p)^p \\ &= ((xz)^p(yt)^p)^p = G(G(x, z), G(y, t)) \end{aligned}$$

for all $x, y, z, t \in [0, 1]$. □

In addition, we know that a symmetric associative function is necessarily bisymmetric. Thus, we can conclude that, since every migrative function is symmetric, every migrative aggregation function that is associative is bisymmetric.

Moreover, from Theorem 8 we know that there exist aggregation functions that are bisymmetric, continuous and migrative but not associative. In addition, neither associativity nor continuity follow from the conjunction of migrativity and bisymmetry: for example,

$$G(x, y) = \begin{cases} 1 & \text{if } xy = 1, \\ \dfrac{(xy)^2}{2} & \text{otherwise,} \end{cases} \tag{4}$$

is bisymmetric and migrative, but is not continuous or associative.

9. Concluding remarks

The characterization of migrative functions

$$G : [0, 1]^2 \to [0, 1]$$

based on a mapping g such that $G(x, y) = g(xy)$ for all $x, y \in [0, 1]$ yielded several interesting results that complement previous approaches to migrativity. Nevertheless, there are quite a number of open questions. In particular, it would be desirable to obtain a full shape characterization of associative or bisymmetric migrative functions, which can only be partially grasped from the results reported in this paper. It would also be interesting to investigate how migrativity behaves under isomorphic transformations.

Moreover, although this paper focused on binary aggregation functions, future research should consider migrativity for aggregation operations $\{A_n\}_{n\in\mathbb{N}}$. A natural extension of migrativity in this context is to impose that

$$\begin{aligned} &A_n(\alpha x_1, x_2, \ldots, x_n) = A_n(x_1, \alpha x_2, \ldots, x_n) \\ &\quad = \cdots = A_n(x_1, x_2, \ldots, \alpha x_n) \quad \text{for all } x_1, \ldots, x_n \in [0, 1] \end{aligned}$$

holds for every $n \in \mathbb{N}$ and all $\alpha \in [0, 1]$. Under this definition, it is clear, for example, that each A_n is characterized by a mapping

$$g_n : [0, 1] \to [0, 1]$$

in such a way that

$$A_n(x_1, \ldots, x_n) = A_n\left(\prod_i^n x_i, 1, \ldots, 1\right) = g_n\left(\prod_i^n x_i\right) \quad \text{for all } x_1, \ldots, x_n \in [0, 1].$$

Acknowledgments

This research was partially supported by Grants TIN2006-06190, TIN2007-65981, APVV-0375-06, VEGA1/4209/07 and GACR 402/08/0618. The authors would like to acknowledge both referees and editors for their useful comments and suggestions. Some of their comments have been reproduced in the text.

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