THE EQUATIONS OF STOCHASTIC
NONLINEAR OSCILLATOR DRIVEN
BY FRACTIONAL BROWNIAN MOTION

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Abstract: Existence of a weak solution to the $n$-dimensional equation of stochastic nonlinear oscillator driven by a fractional Brownian motion with Hurst parameter $H \in (0,1) \setminus \{ \frac{1}{2} \}$ has been shown if a diffusion matrix is time-dependent but state-independent and a drift may be singular but has to satisfy conditions of Girsanov Theorem.

Keywords: Fractional Brownian Motion, Girsanov Theorem, Weak Solution, Stochastic Nonlinear Oscillator.

1 Introduction

In this paper, we prove an existence of a weak solution to the $n$-dimensional equation of stochastic oscillator

$$\frac{d^2}{dt^2}x_t + F(t, x_t, \frac{d}{dt}x_t) = \sigma(t) \frac{d}{dt}B^H_t,$$

where $B^H = ((B^H)^i)_{i=1}^n$ is an $n$-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$, i.e. $B^H$ is a centered Gaussian process with covariance function

$$\mathbb{E}[(B^H_s)^i(B^H_t)^j] = \frac{1}{2} (s^{2H} + t^{2H} - |s-t|^{2H}) \delta_{ij}, \quad s,t \geq 0, \ i,j = 1, \ldots, n.$$

Process $B^H$ is a standard Wiener process for $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$ $B^H$ has a version with Hölder continuous trajectories of order $\gamma$ for any $0 < \gamma < H$ and has stationary increments. Nevertheless, $B^H$ is neither Markov process nor a semimartingale, hence standard methods of integration are not applicable. This example of a stochastic oscillator and other results about existence of weak solutions with full proofs will appear in [11]. There are many papers devoted to equations driven by a fractional Brownian motion, e.g. in [2], [8] the strong existence and uniqueness of solutions to one-dimensional SDE’s for any $H \in (0,1)$ is established. In the case $H > \frac{1}{2}$ the existence and uniqueness of solutions is also proved in [4], [5] (by the rough path approach, using Young type integrals and the concept on $p$-variation) and in [10]. An existence of weak solutions in one-dimensional case is studied in [8], [9], [7] and [1] using Girsanov Theorem for fractional Brownian motion. Stochastic equations in Hilbert spaces driven by a fractional Brownian motion are studied in [3], [6].
2 Preliminaries

Consider an $n$-dimensional stochastic differential equation

$$X_t = \tilde{x} + \int_0^t b(s, X_s) \, ds + \int_0^t \sigma(s) \, dB_s^H,$$  \hfill (2.1)

where $\tilde{x} \in \mathbb{R}^n$ is deterministic initial condition, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a drift which can be divided into a regular part and a singular part and $\sigma : [0, T] \to \mathcal{L}(\mathbb{R}^n)$ is a diffusion, which is time-dependent but state-independent. $\mathcal{L}(\mathbb{R}^n)$ is a space of all linear bounded operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. Furthermore, consider another simpler equation

$$Y_t = \tilde{x} + \int_0^t b_1(s, Y_s) \, ds + \int_0^t \sigma(s) \, dB_s^H.$$  \hfill (2.2)

**Definition 2.1.** An adapted process with continuous trajectories is a solution to the equation (2.2) if $\{Y_t, t \in [0, T]\}$ satisfies the equation (2.2). The solution to the equation (2.2) is pathwise unique if

$$\mathbb{P}\{Y_t = \tilde{Y}_t \ \forall \ t \in [0, T]\} = 1$$

holds for any two solutions $\{Y_t, t \in [0, T]\}, \{\tilde{Y}_t, t \in [0, T]\}$.

By a weak solution to the equation (2.1) we mean a couple of adapted processes $(B^H, X)$ with continuous trajectories on a complete probability space $(\Omega, F, \mathbb{P})$ such that $B^H$ is an $n$-dimensional fractional Brownian motion on the interval $[0, T]$ and $X$ and $B^H$ satisfy (2.1).

The next theorem is a main result in [11]. Remark that $\mathcal{C}^\delta([0, T]; \mathcal{L}(\mathbb{R}^n))$ is a space of all Hölder continuous mappings of order $\delta$, $0 < \delta < 1$, from the interval $[0, T]$ to the space $\mathcal{L}(\mathbb{R}^n)$.

**Theorem 2.2.** Let $b_1, b_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0, T] \to \mathcal{L}(\mathbb{R}^n)$ be Borel mappings such that $b = b_1 + b_2$ on $[0, T] \times \mathbb{R}^n$ and assume that there exists a Borel measurable inverse $\sigma^{-1}$ of $\sigma$. Suppose that

$$\exists K_b > 0 \ \forall t \in [0, T] \ \forall x \in \mathbb{R}^n \ \|b_2(t, x)\| \leq K_b (1 + \|x\|)$$ \hfill (2.3)

and there exists a solution $Y$ to the equation (2.2). Set $u(t) = \sigma^{-1}(t)b_2(t, Y_t)$, $t \in [0, T]$. Assume that $u \in L^\infty([0, T]; \mathbb{R}^n)$ $\mathbb{P}$-almost surely and either $H < \frac{1}{2}$, $\sigma \in \mathcal{C}^{\delta*}([0, T]; \mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$ and $\exists K > 0$ $\forall t \in [0, T] \ \forall x \in \mathbb{R}^n$

$$\|\sigma^{-1}(t)b_2(t, x)\| \leq K(1 + \|x\|),$$

or $H > \frac{1}{2}$, $\sigma \in L^\infty([0, T]; \mathcal{L}(\mathbb{R}^n))$ and $\exists \alpha \in (1 - \frac{1}{2H}, 1)$, $\exists \beta \in (H - \frac{1}{2}, 1)$ $\exists C > 0$ $\forall s, t \in [0, T] \ \forall x, y \in \mathbb{R}^n$

$$\|\sigma^{-1}(t)b_2(t, x) - \sigma^{-1}(s)b_2(s, y)\| \leq C(\|x - y\|^\alpha + |t - s|^\beta).$$

Then there exists a weak solution to the equation (2.1).
Sketch of the proof (cf. [11] for details) First we show that there exists a version of the stochastic integral \( \int_0^t \sigma(s) dB_H^s, t \in [0,T] \) with Hölder continuous trajectories of order \( \gamma, 0 < \gamma < H \), using Kolmogorov-Chentsov Theorem. As \( H \) increases, conditions on integrand \( \sigma \) are less restrictive and computations in the proof of Hölder continuity are easier for \( H > \frac{1}{2} \) then for \( H < \frac{1}{2} \). Next we prove that if we have solution \( Y_t \) to the equation (2.2), where \( b_1 \) is a Borel function satisfying condition (2.3), then there exists a version of \( \{ Y_t, t \in [0,T] \} \) with Hölder continuous trajectories of order \( \gamma, 0 < \gamma < H \).

The next aim is to use Girsanov Theorem which for a fractional Brownian motion takes the following form

**Theorem 2.3.** Let \( B_H^t = \{ B_H^t, t \in [0,T] \} \) be an \( n \)-dimensional fractional Brownian motion with Hurst parameter \( H \) on the interval \([0,T]\). Consider an adapted \( n \)-dimensional process \( u = \{ u_t, t \in [0,T] \} \) with integrable trajectories. Set

\[
v(s) = K_H^{-1} \left( \int_0^s u_r \, dr \right)(s), \quad s \in [0,T],
\]

where \( K_H \) is an isometry between the space \( (\mathcal{H}, \| \cdot \|_H) \) all deterministic time-dependent integrable functions with respect to the fractional Brownian motion and a space \( L^2(\Omega, L^2(\mathbb{R}^n)) \) (for precise definitions see [11]). The space \( L^2([0,T]; \mathbb{R}^n) \) is an image of \( L^2([0,T]; \mathbb{R}^n) \) under the operator \( I_{0+}^{H+\frac{1}{2}} \) defined by the formula

\[
(I_{0+}^{H+\frac{1}{2}} \varphi)(t) := \frac{1}{\Gamma(H + \frac{1}{2})} \int_0^t (t - s)^{H - \frac{1}{2}} \varphi(s) \, ds
\]

for \( \varphi \in L^1([0,T]; \mathbb{R}^n) \). The above integral is well-defined for a.e. \( t \in [0,T] \) and \( \Gamma \) denotes Gamma function.
The equations of stochastic nonlinear oscillator

The operator $K_H^{-1} : I_{0+}^{H+\frac{1}{2}}(L^2([0,T];\mathbb{R}^n)) \rightarrow L^2([0,T];\mathbb{R}^n)$ is the inverse of the linear operator $K_H : L^2([0,T];\mathbb{R}^n) \rightarrow I_{0+}^{H+\frac{1}{2}}(L^2([0,T];\mathbb{R}^n))$ defined by

$$(K_H \varphi)(t) := \int_0^t K_H(t,s) \varphi(s) \, ds,$$

where $K_H(t,s)$ is an integral kernel having a form

$$K_H(t,s) = \begin{cases} \frac{C_H}{H} s^{-H_*} \left[ (t-t^-)^{H_*} - H_* I_H \right], & s < t, \\ 0, & s \geq t, \end{cases}$$

where $H_* = H - \frac{1}{2}$, $I_H = \int_s^t u^{H_*-1}(u-s)^{H_*} \, du$ and

$$C_H = \sqrt{\frac{H_* H^{\ast}}{2B(2-2H, H^*)}}.$$

and $B$ denotes Beta function.

To verify conditions (i), (ii) of Girsanov Theorem it is sufficient to show that

$$\mathbb{E} \exp \left\{ \int_{t_1}^{t_2} \| \nu_s \|^2 \, ds \right\} < +\infty$$

for any $0 \leq t_1 < t_2 \leq T$ enough small. This can be shown using Fernique theorem.

**Theorem 2.4.** Let $(V, \| \cdot \|, \mathcal{B}(V))$ be a separable Banach space with a Borel $\sigma$-field $\mathcal{B}(V)$. Suppose that $G$ is a $V$-valued zero-mean Gaussian random variable. Then there exists $\zeta > 0$ such that

$$\mathbb{E} \exp\{\zeta \|G\|^2_V\} < +\infty.$$

Contrary to the regularity of trajectories of the process $\{\int_0^t \sigma(s) \, dB_H^s, \, t \in [0,T]\}$, conditions on the product $\sigma^{-1}b_2$ are more restrictive and computations are more difficult in the case $H > \frac{1}{2}$ than in the case $H < \frac{1}{2}$. Finally we show that the couple $\{\{B_H^t - \int_0^t \nu_s \, ds, \, t \in [0,T]\}, Y\}$ is a weak solution to the equation (2.1) on the probability space $(\Omega, \mathcal{F}, \tilde{P})$, where $\tilde{P}$ is a changed probability measure defined by a density $\xi_T$ with respect to probability measure $\mathbb{P}$.

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### 3 Equation of stochastic oscillator

Consider the formal equation

$$\frac{d^2}{dt^2} x_t + F(t, x_t, \frac{d}{dt} x_t) = \bar{\sigma}(t) \frac{d}{dt} B^H_t,$$  \hspace{1cm} (3.1)
the weak solution of which is defined as the weak solution to
\[
X_t = y_0 + \int_0^t (b_1(X_s) + b_2(s, X_s)) \, ds + \int_0^t \sigma(s) \, dB^H_s ,
\]
where \( \{B^H_t, t \in [0, T] \} \) is a 2n-dimensional fractional Brownian motion whose first \( n \) components are components of \( B^H \) and where for \( t \in [0, T] \) and \( y = (x, v)^T \in \mathbb{R}^{2n} \)
\[
X_t := \begin{pmatrix} x_t \\ v_t \end{pmatrix}, \\
b_1(y) := \begin{pmatrix} v \\ 0 \end{pmatrix}, \\
b_2(t, y) := \begin{pmatrix} 0 \\ -F(t, y) \end{pmatrix}, \\
y_0 := \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}
\]
and
\[
\sigma(t) := \begin{pmatrix} 0 & 0 \\ 0 & \sigma(t) \end{pmatrix},
\]
\( \sigma(t) \) being a \( 2n \times 2n \)-dimensional matrix.
Moreover, consider the linear equation
\[
Y_t = y_0 + \int_0^t b_1(Y_s) \, ds + \int_0^t \sigma(s) \, dB^H_s .
\]

**Proposition 3.1.** Suppose that \( \sigma : [0, T] \to \mathcal{L}(\mathbb{R}^n) \) is a Borel mapping satisfying either \( H < \frac{1}{2} \) and \( \sigma \in C^{\delta^*}(\mathcal{L}(\mathbb{R}^n)) \) for some \( \delta^* \in (\frac{1}{2} - H, 1) \) or \( H > \frac{1}{2} \) and \( \sigma \in L^\infty(\mathcal{L}(\mathbb{R}^n)) \). Further, let \( b_1 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) be a Borel function satisfying the following conditions:
\[
\forall N \in \mathbb{N} \, \exists K_N > 0 \, \forall t \in [0, T] \, \forall x, y \in \mathbb{R}^n \, \|x\| + \|y\| \leq N \\
\|b_1(t, x) - b_1(t, y)\| \leq K_N \|x - y\|,
\]
and
\[
\exists K_b > 0 \, \forall t \in [0, T] \, \forall x \in \mathbb{R}^n \, \|b_1(t, x)\| \leq K_b (1 + \|x\|) .
\]
Then there exists the pathwise unique solution to the equation \((2.2)\).

**Proof.** Cf. [11].

**Q.E.D.**

Suppose that matrix \( \bar{\sigma} \) is regular for all \( t \in [0, T] \). Let
\[
\Sigma(t) = \begin{pmatrix} 0 & 0 \\ 0 & \bar{\sigma}^{-1}(t) \end{pmatrix}, \quad t \in [0, T],
\]
be a $2n \times 2n$-dimensional matrix. It is easy to see from the proof that the statement of Theorem 2.2 holds if we replace $\sigma^{-1}$ in this theorem by $\Sigma$.

Suppose that $\bar{\sigma}$ is a Borel function satisfying either

(A1) $H < \frac{1}{2}$ and $\bar{\sigma} \in C^\delta([0, T]; \mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$,

or

(A2) $H > \frac{1}{2}$ and $\bar{\sigma} \in L^\infty([0, T]; \mathcal{L}(\mathbb{R}^n))$.

Function $b_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$; $y = (x, v)^T \mapsto (v, 0)^T$ is Lipschitz (consequently $b_1$ satisfies condition (2.3)). Then there exists the pathwise unique solution $\{Y_t, t \in [0, T]\}$ to the equation (3.3) (cf. Proposition 3.1).

Assume that trajectories of the process $\{\bar{\sigma}^{-1}(t)F(t, Y_t), t \in [0, T]\}$ are in $L^\infty([0, T]; \mathbb{R}^n)$ and suppose moreover either

(B1) $H < \frac{1}{2}$ and $\exists K > 0 \ \forall t \in [0, T] \ \forall y \in \mathbb{R}^{2n}$

$$\|\bar{\sigma}^{-1}(t)F(t, y)\| \leq K (1 + \|y\|),$$

or

(B2) $H > \frac{1}{2}$ and $\exists \alpha \in (1 - \frac{1}{2H}, 1) \ \exists \beta \in (H - \frac{1}{2}, 1) \ \exists C > 0 \ \forall s, t \in [0, T] \ \forall y_1, y_2 \in \mathbb{R}^{2n}$

$$\|\bar{\sigma}^{-1}(t)F(t, y_1) - \bar{\sigma}^{-1}(s)F(t, y_2)\| \leq C(\|y_1 - y_2\|^\alpha + |t - s|^\beta).$$

Then assumptions of Theorem 2.2 on a map $(t, y) \mapsto \Sigma(t) b_2(t, y)$, $t \in [0, T]$, $y \in \mathbb{R}^{2n}$, are satisfied because

$$\Sigma(t) b_2(t, y) = \begin{pmatrix} 0 \\ -\bar{\sigma}(t)F(t, y) \end{pmatrix},$$

hence equations (3.2) and thereby (3.1) have weak solutions.

References


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