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WEAK SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTION

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Abstract. Existence of a weak solution to the n-dimensional system of stochastic differential equations driven by a fractional Brownian motion with the Hurst parameter $H \in (0,1) \setminus \{\frac{1}{2}\}$ is shown for a time-dependent but state-independent diffusion and a drift that may by split into a regular part and a singular one which, however, satisfies the hypotheses of the Girsanov Theorem. In particular, a stochastic nonlinear oscillator driven by a fractional noise is considered.

Keywords: fractional Brownian motion, Girsanov theorem, weak solutions

MSC 2000: 60H10

1. Introduction

Let $B^H = \{B^H_t, t \geqslant 0\}$ be an n-dimensional *fractional Brownian motion* (fBm) with the Hurst parameter $H \in (0,1)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. B^H is a centered Gaussian process with the covariance matrix

$$\mathbb{E}\,[(B^H_s)^i (B^H_t)^j] = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H})\,\delta_{ij}, \quad s,t \geqslant 0,\ i,j = 1,\dots,n,$$

where $(B^H)^k,\ k = 1,\dots,n$, are the components of B^H.

For $H = \frac{1}{2}$ the process B^H is a standard Brownian motion. For $H \neq \frac{1}{2}$ it has stationary increments and is self-similar, i.e. $B^H_{\alpha t}$ and $\alpha^H B^H_t$ have the same distribution for all $\alpha > 0$. For any $0 < \delta < H$ the process B^H has also a version with Hölder continuous trajectories of order δ. However, B^H is neither a semimartingale nor a Markov process, hence standard methods of stochastic integration are not applicable.

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In this paper, we study the stochastic differential equation

$$X_t = \tilde{x} + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H \tag{1.1}$$

in $\mathbb{R}^n$ driven by an n-dimensional fractional Brownian motion B^H.

Recently, several papers devoted to the equation (1.1) were written assuming that

$$n = 1,\ \sigma \equiv 1 \tag{1.2}$$

holds. The proofs of existence of weak solutions in these papers are based on the Girsanov Theorem. D. Nualart and Y. Ouknine in [23] show existence of a weak solution of (1.1) supposing that either $H < \frac{1}{2}$ and b is a Borel function of linear growth, or $H > \frac{1}{2}$ and b is Hölder continuous of order α in x and Hölder continuous of order β in t for some $\alpha \in (1 - \frac{1}{2}H^{-1}, 1)$ and $\beta \in (H - \frac{1}{2}, 1)$. Later they proved in [24] that in the case $H < \frac{1}{2}$ it is sufficient to assume that $b^2 \leqslant K + F(t,x)$ for almost all (t,x), where F is a non-negative Borel function such that

$$\int_0^T \left(\int_{\mathbb{R}} |F(t,x)|^p \,\mathrm{d}x \right)^{\frac{\beta}{p}} \mathrm{d}x < +\infty$$

for some $p > 1$, $\beta > p(p-H)^{-1}$. In the case $H > \frac{1}{2}$, Y. Mishura and D. Nualart ([20]) considered an equation with time independent drift which satisfies the corresponding hypothesis from [23] up to a finite number of jumps and proved the existence of a weak solution for $H < \frac{1}{4}(1+\sqrt{5})$. B. Boufoussi and Y. Ouknine ([2]) for $H > \frac{1}{2}$ found a weak solution to an equation with a drift $b = b_1 + b_2$, where the function b_1 satisfies the assumptions from [23] and $b_2(s,\cdot)$ is left-continuous and nondecreasing (or continuous and nonincreasing) for each s.

We follow the same strategy as in the cited articles, but for a system of equations ($n > 1$) and time-dependent diffusion σ. We assume that the drift b may be split into two terms b_1 and b_2, where b_1 is a Borel function of linear growth, the equation

$$Y_t = \tilde{x} + \int_0^t b_1(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H \tag{1.3}$$

has a solution and b_2 is a Borel function such that $\sigma^{-1}b_2$ satisfies the hypotheses of the Girsanov Theorem, where $\sigma^{-1}(t)$ is an inverse of $\sigma(t)$ for all $t \in [0,T]$ which we suppose to exist. In the case of a standard Brownian motion, such a generalization is rather straightforward, however, in the case of a fractional Brownian motion the proofs involve heavy calculations which have to be carefully modified if (1.2) is not satisfied. As may be expected, the result depends on the Hurst parameter substantially. In the regular case $H > \frac{1}{2}$ a less limited integration theory is available, on the other hand, in the singular case $H < \frac{1}{2}$ the Girsanov transform is applicable under less restrictive conditions. More precisely, we prove the following theorem (for definitions and notation, see the subsequent sections):

Theorem 1.1. *Let $b_1, b_2\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ be Borel mappings such that $b = b_1 + b_2$ on $[0,T] \times \mathbb{R}^n$ and assume that $\sigma(t)$ is regular for all $t \in [0,T]$. Suppose that*

$$\exists K_b > 0\ \forall t \in [0,T]\ \forall x \in \mathbb{R}^n \quad \|b_1(t,x)\| \leqslant K_b(1 + \|x\|),$$

and let there exist a solution Y to the equation (1.3). *Set $u(t) = \sigma^{-1}(t) b_2(t, Y_t)$, $t \in [0,T]$. Assume that $u \in L^\infty([0,T]; \mathbb{R}^n)$ $\mathbb{P}$-almost surely and either $H < \frac{1}{2}$, $\sigma \in \mathcal{C}^{\delta^*}([0,T]; \mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$ and*

$$\exists K > 0\ \forall t \in [0,T]\ \forall x \in \mathbb{R}^n \quad \|\sigma^{-1}(t)\, b_2(t,x)\| \leqslant K(1 + \|x\|),$$

or $H > \frac{1}{2}$, $\sigma \in L^\infty([0,T]; \mathcal{L}(\mathbb{R}^n))$ and

$$\exists \alpha \in \Big(1 - \frac{1}{2H}, 1\Big)\ \exists \beta \in \Big(H - \frac{1}{2}, 1\Big)\ \exists C > 0\ \forall s,t \in [0,T]\ \forall x,y \in \mathbb{R}^n$$
$$\|\sigma^{-1}(t)\, b_2(t,x) - \sigma^{-1}(s)\, b_2(s,y)\| \leqslant C(\|x - y\|^\alpha + |t - s|^\beta).$$

Then there exists a weak solution to the equation (1.1).

Let us note that pathwise uniqueness holds for equations discussed in [23], [24] and, consequently, there exists a strong solution. (Cf. also the paper [6] devoted to thorough discussion of a strong solution to (1.1) under the hypothesis (1.2).) The methods employed in these articles seem to depend on the fact that $n = 1$ and we do not know whether pathwise uniqueness holds for system of equations considered in our paper. Further, results on the existence of solutions to equations with a state-dependent diffusion σ are worth being mentioned (see e.g. [16], [17], [22], [26]). In the case of a multiplicative noise, various approaches to stochastic integration with respect to B^H (based on rough paths theory, fractional calculus, white noise theory or Malliavin calculus) need not be equivalent. For our equation (1.1), this problem disappears and all the theories yield the same stochastic integral. Finally, while most of the available results on (1.1) concern only scalar euqations, there are some results available about infinite-dimensional systems resembling (1.1) (cf. e.g. [7], [18]).

The paper is organised as follows. In Section 2, some necessary preliminaries about fractional Brownian motion can be found. Hölder continuity of trajectories of a solution to (1.3) is proved in Section 3. In Section 4, Girsanov's Theorem and some of its consequences are discussed. In Section 5, our main results on the existence of weak solutions are proved. The last section is devoted to a study of a stochastic nonlinear oscillator

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x + F\Big(t, x, \frac{\mathrm{d}}{\mathrm{d}t}x\Big) = \overline{\sigma}(t)\frac{\mathrm{d}}{\mathrm{d}t}B_t^H,$$

that is, rigorously, a system

$$\begin{aligned} x_t &= x_0 + \int_0^t v_s \,\mathrm{d}s, \\ v_t &= v_0 - \int_0^t F(t, x_s, v_s)\,\mathrm{d}s + \int_0^t \overline{\sigma}(s)\,\mathrm{d}\overline{B}_s^H. \end{aligned} \tag{1.4}$$

In particular, it is proved that for $\overline{\sigma}$ constant and invertible the law of $(x_t, v_t)^{\mathrm{T}}$ is equivalent to the Lebesgue measure on $\mathbb{R}^{2n}$ for all $H \in (0,1) \setminus \{\frac{1}{2}\}$, $t > 0$ and initial data $(x_0, v_0)^{\mathrm{T}} \in \mathbb{R}^{2n}$. (Results on the absolute continuity of the law of solutions to stochastic differential equations driven by a fractional Brownian motion may be found in [21] and [12], but none of them applies to (1.4).)

2. Preliminaries

Define

$$K_H(t,s) = \begin{cases} \dfrac{C_H}{H-\frac{1}{2}}\bigg[\Big(\dfrac{t}{s}\Big)^{H-1/2}(t-s)^{H-1/2} \\ \qquad -\Big(H-\dfrac{1}{2}\Big)s^{1/2-H}\displaystyle\int_s^t u^{H-3/2}(u-s)^{H-1/2}\,\mathrm{d}u\bigg], & s < t, \\ 0, & s \geqslant t, \end{cases}$$

where

$$C_H = \sqrt{\frac{H(2H-1)}{B(2-2H, H-\frac{1}{2})}},$$

where $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$, $a > 0$, $b > 0$, denotes the Beta function.

The process B^H has an integral representation (see e.g. [5])

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{id}\,\mathrm{d}W_s,\ 0 \leqslant t < +\infty, \tag{2.1}$$

where $W = \{W_t, t \geqslant 0\}$ is an n-dimensional Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and id denotes the identity operator from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Denote by $\mathcal{E}$ the set of $\mathcal{L}(\mathbb{R}^n)$-valued step functions on the interval $[0,T]$, i.e. each $\varphi \in \mathcal{E}$ has the form

$$\varphi = \sum_{k=0}^{N-1} a_k \, I_{[t_k, t_{k+1})}, \tag{2.2}$$

for some $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $a_k \in \mathcal{L}(\mathbb{R}^n)$, $k = 0, \ldots, N$, I_A denoting an indicator function of A. Then we define the integral of a function $\varphi \in \mathcal{E}$ of the form (2.2) with respect to fractional Brownian motin as

$$I(\varphi) \equiv \int_0^T \varphi(s) \, \mathrm{d}B_s^H := \sum_{k=0}^{N-1} a_k (B^H(t_{k+1}) - B^H(t_k)).$$

Define a linear operator $\mathcal{K}_H^* \colon \mathcal{E} \to L^2([0,T]; \mathcal{L}(\mathbb{R}^n))$ by

$$(\mathcal{K}_H^* \varphi)(t) := K_H(T,t)\, \varphi(T) - \int_t^T (\varphi(s) - \varphi(t)) \frac{\partial K_H}{\partial s}(s,t) \, \mathrm{d}s, \;\; \varphi \in \mathcal{E}, \;\; t \in [0,T].$$

Then for all $\varphi, \psi \in \mathcal{E}$ we have (in [1] the one-dimensional case is discussed; the extension to the multi-dimensional case is straightforward)

$$\begin{aligned} \mathbb{E} \left\langle \int_0^T \varphi(s) \, \mathrm{d}B_s^H, \int_0^T \psi(s) \, \mathrm{d}B_s^H \right\rangle_{\mathbb{R}^n} \\ = \langle \mathcal{K}_H^*(\varphi), \mathcal{K}_H^*(\psi) \rangle_{L^2([0,T];\mathcal{L}(\mathbb{R}^n))} =: \langle \varphi, \psi \rangle_{\mathcal{H}}. \end{aligned} \tag{2.3}$$

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be the Hilbert space defined as the completion of $\mathcal{E}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. Denote by $\| \cdot \|_{\mathcal{H}}$ the norm in $\mathcal{H}$ associated with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. From (2.3) it follows that the operator $\mathcal{K}_H^*$ provides an isometry between the spaces $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ and $L^2(\Omega; \mathcal{L}(\mathbb{R}^n))$. Since $\mathcal{E}$ is dense in $\mathcal{H}$ there exists a unique extension $\hat{I} \in \mathcal{L}(\mathcal{H}, L^2(\Omega; \mathcal{L}(\mathbb{R}^n)))$ of the operator I. Hence, we can define

$$\forall \varphi \in \mathcal{H} \quad \hat{I}(\varphi) =: \int_0^T \varphi(s) \, \mathrm{d}B_s^H.$$

The process $W = \{W_t, 0 \leqslant t \leqslant T\}$ defined by

$$W_t = \int_0^T (\mathcal{K}_H^*)^{-1} (I_{[0,t]} \, \mathrm{id})(s) \, \mathrm{d}B_s^H, \quad t \in [0,T], \tag{2.4}$$

is a Wiener proces and with this choice of the Wiener process W, the representation (2.1) holds (cf. [23]).

Definition 2.1. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a right-continuous filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathcal{F}_0$ contains $\mathbb{P}$-null sets. A fractional Brownian motion $B^H = \{B_t^H, t \in [0,T]\}$ is called an $\mathcal{F}_t$-fractional Brownian motion if the process W defined in (2.4) is an $\mathcal{F}_t$-Wiener proces, i.e. W is $\mathcal{F}_t$-adapted and for all $h > 0$ $W_{t+h} - W_t$ is independent of $\mathcal{F}_t$.

Let $B^H = \{B_t^H, t \in [0,T]\}$ be an n-dimensional fractional Brownian motion with the Hurst parameter H on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that B^H has representation (2.1) with some Wiener process W. Denote

$$\begin{aligned} \mathcal{N} &= \{F \in \mathcal{F};\ \mathbb{P}(F) = 0\}, \\ \widetilde{\mathcal{F}}_t^{B^H} &= \sigma(B_s^H, 0 \leqslant s \leqslant t) \quad \text{and} \quad \widetilde{\mathcal{F}}_t^W = \sigma(W_s, 0 \leqslant s \leqslant t), \\ \mathcal{F}_t^{B^H} &= \bigcap_{s>t} \sigma(\widetilde{\mathcal{F}}_s^{B^H} \cup \mathcal{N}) \quad \text{and} \quad \mathcal{F}_t^W = \bigcap_{s>t} \sigma(\widetilde{\mathcal{F}}_s^W \cup \mathcal{N}). \end{aligned}$$

Then W is an $\mathcal{F}_t^W$-Wiener process, $(\mathcal{F}_t^W)$ is a complete right-continuous filtration and $\widetilde{\mathcal{F}}_t^{B^H} = \widetilde{\mathcal{F}}_t^W$ holds (see [23]). Hence, $\mathcal{F}_t^{B^H} = \mathcal{F}_t^W$, $(\mathcal{F}_t^{B^H})$ is a complete right-continuous filtration and W is an $\mathcal{F}_t^{B^H}$-Wiener process.

For $\varphi \in L^1([0,T]; \mathcal{L}(\mathbb{R}^n))$, $a > 0$, we define

$$\begin{aligned} (I_{0+}^a \varphi)(t) &:= \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \varphi(s)\,\mathrm{d}s, \\ (I_{T-}^a \varphi)(t) &:= \frac{(-1)^a}{\Gamma(a)} \int_t^T (s-t)^{a-1} \varphi(s)\,\mathrm{d}s, \end{aligned}$$

where $(-1)^a = \mathrm{e}^{-\mathrm{i}\pi a}$ and $\Gamma(b) = \int_0^{+\infty} u^{b-1} \mathrm{e}^{-u}\,\mathrm{d}u$, $b > 0$, denotes the Gamma function. The integrals are well-defined for a.e. $t \in [0,T]$.

For $a \in (0,1)$, $\varphi \in I_{0+}^a(L^p([0,T]; \mathcal{L}(\mathbb{R}^n)))$ and $\psi \in I_{T-}^a(L^p([0,T]; \mathcal{L}(\mathbb{R}^n)))$ we define

$$(D_{0+}^a \varphi)(t) := \frac{1}{\Gamma(1-a)} \left(\frac{\varphi(t)}{t^a} + a \int_0^T \frac{\varphi(t) - \varphi(s)}{(t-s)^{a+1}}\,\mathrm{d}s \right) I_{(0,T)}(t)$$

and

$$(D_{T-}^a \psi)(t) := \frac{(-1)^a}{\Gamma(1-a)} \left(\frac{\varphi(t)}{(T-t)^a} + a \int_t^T \frac{\varphi(t) - \varphi(s)}{(s-t)^{a+1}}\,\mathrm{d}s \right) I_{(0,T)}(t),$$

where the integrals are well-defined for a.e. $t \in [0,T]$.

For $H > \frac{1}{2}$ set

$$\|\varphi\|_{\widetilde{\mathcal{H}}}^2 := H(2H-1) \int_0^T \int_0^T \|\varphi(u)\|_{\mathcal{L}(\mathbb{R}^n)} \|\varphi(v)\|_{\mathcal{L}(\mathbb{R}^n)} |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v$$

and define a space

$$\widetilde{\mathcal{H}} := \{\varphi \in \mathcal{H};\ \|\varphi\|_{\widetilde{\mathcal{H}}}^2 < +\infty\}.$$

Then $(\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}})$ is a Banach space, $\mathcal{E} \subseteq \widetilde{\mathcal{H}}$, $\mathcal{E}$ is dense in $\widetilde{\mathcal{H}}$ and

$$(\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}) \hookrightarrow (\mathcal{H}, \|\cdot\|_{\mathcal{H}}).$$

In [19] we can find the estimate

$$\exists\, \tilde{b}_H > 0\ \forall \varphi \in L^{1/H}([0,T]; \mathcal{L}(\mathbb{R}^n)) \quad \|\varphi\|_{\widetilde{\mathcal{H}}} \leqslant \tilde{b}_H \|\varphi\|_{L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))}.$$

Thus

$$(L^{1/H}([0,T]; \mathcal{L}(\mathbb{R}^n)), \|\cdot\|_{L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))}) \hookrightarrow (\widetilde{\mathcal{H}}, \|\cdot\|_{\widetilde{\mathcal{H}}}) \hookrightarrow (\mathcal{H}, \|\cdot\|_{\mathcal{H}}),$$

hence

$$\exists\, b_H > 0\ \forall \varphi \in L^{1/H}([0,T]; \mathcal{L}(\mathbb{R}^n)) \quad \|\varphi\|_{\mathcal{H}} \leqslant b_H \|\varphi\|_{L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))}. \tag{2.5}$$

Also (cf. [11]),

$$\mathbb{E}\left\|\int_0^T \varphi(s)\,\mathrm{d}B_s^H\right\|^2 = H(2H-1)\int_0^T\!\!\int_0^T \langle\varphi(u), \varphi(v)\rangle_{\mathcal{L}(\mathbb{R}^n)} |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v \tag{2.6}$$

holds for each $\varphi \in \widetilde{\mathcal{H}}$.

Consider the linear operator $K_H\colon\ L^2([0,T]; \mathcal{L}(\mathbb{R}^n)) \longrightarrow I_{0+}^{H+\frac{1}{2}}(L^2([0,T]; \mathcal{L}(\mathbb{R}^n)))$ defined by

$$(K_H\varphi)(t) := \int_0^t K_H(t,s)\varphi(s)\,\mathrm{d}s.$$

The operator K_H provides an isomorphism between the spaces

$$L^2([0,T]; \mathcal{L}(\mathbb{R}^n)) \quad \text{and} \quad I_{0+}^{H+\frac{1}{2}}(L^2([0,T]; \mathcal{L}(\mathbb{R}^n))).$$

Thus there exists the inverse $K_H^{-1}\colon\ I_{0+}^{H+\frac{1}{2}}(L^2([0,T]; \mathcal{L}(\mathbb{R}^n))) \longrightarrow L^2([0,T]; \mathcal{L}(\mathbb{R}^n))$ of the operator K_H. Set

$$\varphi'_{\frac{1}{2}-H}(t) = t^{\frac{1}{2}-H}\varphi'(t),\ t \in [0,T].$$

For $H < \frac{1}{2}$ and $\varphi \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]; \mathcal{L}(\mathbb{R}^n)))$ almost everywhere differentiable the operator K_H^{-1} has the form (as may be shown analogously to [6] where the case $n = 1$ is studied)

$$(K_H^{-1}\varphi)(t) = t^{H-\frac{1}{2}}(I_{0+}^{\frac{1}{2}-H}\varphi'_{\frac{1}{2}-H})(t),\ t \in [0,T]. \tag{2.7}$$

In the case $H > \frac{1}{2}$, $\varphi \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T]; \mathcal{L}(\mathbb{R}^n)))$, we have (cf. [6])

$$(K_H^{-1}\varphi)(t) = t^{H-\frac{1}{2}}(D_{0+}^{H-\frac{1}{2}}\varphi'_{\frac{1}{2}-H})(t),\ t \in [0,T]. \tag{2.8}$$

We will need the following technical lemma.

Lemma 2.2.

(i) *For any* $H \in (0, \frac{1}{2})$ *there exists a constant* $C(H) > 0$ *depending only on* H *such that*

$$\int_s^T \frac{|r^{H-\frac{1}{2}} - s^{H-\frac{1}{2}}|}{(r-s)^{\frac{3}{2}-H}} \,\mathrm{d}r \leqslant C(H)\, s^{-1+2H} \tag{2.9}$$

holds for each $s \in (0, T)$.

(ii) *For any* $H \in (\frac{1}{2}, 1)$ *there exists a constant* $C(H) > 0$ *depending only on* H *such that*

$$\int_0^s \frac{|s^{\frac{1}{2}-H} - r^{\frac{1}{2}-H}|}{(s-r)^{\frac{1}{2}+H}} \,\mathrm{d}r \leqslant C(H)\, s^{1-2H} \tag{2.10}$$

holds for each $s > 0$.

P r o o f. Using elementary estimates and the mean value theorem. □

3. Equations with regular coefficients

Consider the stochastic differential equation

$$Y_t = \tilde{x} + \int_0^t b_1(u, Y_u)\,\mathrm{d}u + Z_t, \quad t \in [0, T], \tag{3.1}$$

where

$$Z_t = \int_0^t \sigma(s)\,\mathrm{d}B_s^H, \quad t \in [0, T]. \tag{3.2}$$

In this section we show that there exists a solution to (3.1) under suitable regularity conditions on b_1 and σ and we provide an estimate on a Hölder norm of its paths. Let $\mathcal{C}^\delta([0, T]; \mathcal{L}(\mathbb{R}^n))$ be the space of all Hölder continuous functions of order δ from the interval $[0, T]$ to the space $\mathcal{L}(\mathbb{R}^n)$.

Proposition 3.1. *Let* $H < \frac{1}{2}$ *and* $\sigma\colon [0, T] \to \mathcal{L}(\mathbb{R}^n)$ *be a map satisfying* $\sigma \in \mathcal{C}^{\delta^*}([0, T]; \mathcal{L}(\mathbb{R}^n))$ *for some* $\delta^* \in (\frac{1}{2} - H, 1)$.

Then there exists a version of $\{Z_t, t \in [0, T]\}$ *with Hölder continuous trajectories of order* $\gamma \in (0, H)$, *which is an* n*-dimensional* $\mathcal{F}_t^{B^H}$*-adapted centered Gaussian process.*

P r o o f. In the case $H < \frac{1}{2}$ the inclusion

$$\mathcal{C}^\gamma([0, T]; \mathcal{L}(\mathbb{R}^n)) \subset \mathcal{H}$$

holds for any $1 > \gamma > \frac{1}{2} - H$ (see [25], Section 1.5.2), so $\sigma \in \mathcal{H}$. Thus an aproximation of the integrand σ by step functions in the space $\mathcal{H}$ gives us the second part of the statement. To simplify the notation, set

$$\varphi_{s,H-\frac{1}{2}}(u) = (u-s)^{H-\frac{1}{2}}\varphi(u).$$

Analogously to [1] (where the case $s = 0, t = T$ and $n = 1$ is studied) it can be proved for all $0 \leqslant s < t \leqslant T$ and $\varphi \in \mathcal{H}$ that

$$(\mathcal{K}_H^*\varphi)(u) = C_H(u-s)^{\frac{1}{2}-H}(D_{t-}^{\frac{1}{2}-H}\varphi_{s,H-\frac{1}{2}})(u), \;\; s \leqslant u \leqslant t$$

and

$$\mathbb{E}\Big\|\int_s^t \varphi(u)\,\mathrm{d}B_u^H\Big\|^2 = \|\mathcal{K}_H^*\varphi\|^2_{L^2([s,t];\mathcal{L}(\mathbb{R}^n))}. \tag{3.3}$$

We verify the condition

$$\exists\,\alpha > 0, \beta > 0, K > 0 \;\forall\, t, s \in [0,T] \quad \mathbb{E}\,\|Z_t - Z_s\|^{\beta} \leqslant K\,|t-s|^{1+\alpha}$$

of the Kolmogorov-Chentsov Theorem (see e.g. [13]) using the isometry (3.3).

For $0 \leqslant s < t \leqslant T$ arbitrary we have

$$\begin{aligned}
&\mathbb{E}\,\|Z_t - Z_s\|^2 \\
&= \|\mathcal{K}_H^*\sigma\|^2_{L^2([s,t];\mathcal{L}(\mathbb{R}^n))} = \int_s^t \|C_H(u-s)^{\frac{1}{2}-H}(D_{t-}^{\frac{1}{2}-H}\sigma_{s,H-\frac{1}{2}})(u)\|^2_{\mathcal{L}(\mathbb{R}^n)}\,\mathrm{d}u \\
&= \Big(\frac{C_H}{\Gamma(\frac{1}{2}+H)}\Big)^2 \int_s^t g\|(u-s)^{\frac{1}{2}-H}\Big[\frac{(u-s)^{H-\frac{1}{2}}\sigma(u)}{(t-u)^{\frac{1}{2}-H}} \\
&\quad + \Big(\frac{1}{2}-H\Big)\int_u^t \frac{(v-s)^{H-\frac{1}{2}}\sigma(v) - (u-s)^{H-\frac{1}{2}}\sigma(u)}{(v-u)^{\frac{3}{2}-H}}\Big]\Big\|^2_{\mathcal{L}(\mathbb{R}^n)}\,\mathrm{d}u \\
&\leqslant 2\Big(\frac{C_H}{\Gamma(\frac{1}{2}+H)}\Big)^2\Big[\int_s^t \Big\|\frac{\sigma(u)}{(t-u)^{\frac{1}{2}-H}}\Big\|^2_{\mathcal{L}(\mathbb{R}^n)}\,\mathrm{d}u \\
&\quad + \Big(\frac{1}{2}-H\Big)^2\int_s^t \Big\|(u-s)^{\frac{1}{2}-H}\int_u^t \frac{(v-s)^{H-\frac{1}{2}}\sigma(v) - (u-s)^{H-\frac{1}{2}}\sigma(u)}{(v-u)^{\frac{3}{2}-H}}\,\mathrm{d}v\Big\|^2_{\mathcal{L}(\mathbb{R}^n)}\,\mathrm{d}u\Big] \\
&=: 2\Big(\frac{C_H}{\Gamma(\frac{1}{2}+H)}\Big)^2\Big[I_1 + \Big(\frac{1}{2}-H\Big)^2 I_2\Big].
\end{aligned}$$

Obviously,

$$I_1 = \int_s^t \Big\|\frac{\sigma(u)}{(t-u)^{\frac{1}{2}-H}}\Big\|^2_{\mathcal{L}(\mathbb{R}^n)}\,\mathrm{d}u \leqslant \frac{\|\sigma\|^2_{\mathcal{C}([0,T];\mathcal{L}(\mathbb{R}^n))}}{2H}(t-s)^{2H} \tag{3.4}$$

holds, where $\mathcal{C}([0,T];\mathcal{L}(\mathbb{R}^n))$ is the space of all continuous functions from the interval $[0,T]$ to the space $\mathcal{L}(\mathbb{R}^n)$, and

$$\begin{aligned}
I_2 &= \int_s^t \left\| (u-s)^{\frac{1}{2}-H} \int_u^t \frac{(v-s)^{H-\frac{1}{2}}\sigma(v) - (u-s)^{H-\frac{1}{2}}\sigma(u)}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \right\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \\
&\leqslant \int_s^t \bigg\| (u-s)^{\frac{1}{2}-H} \\
&\qquad \times \int_u^t \frac{(v-s)^{H-\frac{1}{2}}\sigma(v) \pm (u-s)^{H-\frac{1}{2}}\sigma(v) - (u-s)^{H-\frac{1}{2}}\sigma(u)}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \bigg\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \\
&\leqslant 2\bigg[\int_s^t \left\| (u-s)^{\frac{1}{2}-H} \int_u^t \frac{\sigma(v)[(v-s)^{H-\frac{1}{2}} - (u-s)^{H-\frac{1}{2}}]}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \right\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \\
&\qquad + \int_s^t \left\| \int_u^t \frac{\sigma(v)-\sigma(u)}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \right\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \bigg] =: 2[I_3 + I_4].
\end{aligned}$$

By Lemma 2.2 we get

$$\begin{aligned}
\text{(3.5)} \qquad I_3 &= \int_s^t \left\| (u-s)^{\frac{1}{2}-H} \int_u^t \frac{\sigma(v)[(v-s)^{H-\frac{1}{2}} - (u-s)^{H-\frac{1}{2}}]}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \right\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \\
&\leqslant \|\sigma\|^2_{\mathcal{C}([0,T];\mathcal{L}(\mathbb{R}^n))} \int_s^t (u-s)^{1-2H} \left(\int_{u-s}^T \frac{|r^{H-\frac{1}{2}} - (u-s)^{H-\frac{1}{2}}|}{(r-(u-s))^{\frac{3}{2}-H}} \,\mathrm{d}r \right)^2 \mathrm{d}u \\
&\leqslant \frac{C^2(H)}{2H} \|\sigma\|^2_{\mathcal{C}([0,T];\mathcal{L}(\mathbb{R}^n))} (t-s)^{2H}.
\end{aligned}$$

Using δ^*-Hölder continuity of σ we obtain

$$\begin{aligned}
\text{(3.6)} \qquad I_4 &= \int_s^t \left\| \int_u^t \frac{\sigma(v)-\sigma(u)}{(v-u)^{\frac{3}{2}-H}} \,\mathrm{d}v \right\|^2_{\mathcal{L}(\mathbb{R}^n)} \mathrm{d}u \\
&\leqslant \|\sigma\|^2_{\mathcal{C}^{\delta^*}([0,T];\mathcal{L}(\mathbb{R}^n))} \int_s^t \left(\int_u^t (v-u)^{\delta^* - \frac{3}{2}+H} \,\mathrm{d}v \right)^2 \mathrm{d}u \\
&= \frac{\|\sigma\|^2_{\mathcal{C}^{\delta^*}([0,T];\mathcal{L}(\mathbb{R}^n))}}{2(\delta^*+H)(\delta^*+H-\frac{1}{2})^2} (t-s)^{2\delta^*+2H}.
\end{aligned}$$

Thus (3.4), (3.5) and (3.6) yield

$$\mathbb{E}\,\|Z_t - Z_s\|^2 \leqslant 2\left(\frac{C_H}{\Gamma(\frac{1}{2}+H)}\right)^2 [I_1 + (\tfrac{1}{2}-H)^2 I_2] \leqslant B(t-s)^{2H},$$

where B depends only on T, H, σ.

Since $Z_t - Z_s$ is Gaussian with zero mean for each $k \in \mathbb{N}$ there exists a constant $C(k) > 0$ such that

$$\mathbb{E}\,\|Z_t - Z_s\|^{2k} \leqslant C(k)(t-s)^{2Hk}$$

holds, hence by the Kolmogorov-Chentsov Theorem the process $\{Z_t, t \in [0,T]\}$ has a Hölder continuous version of order $\gamma < (2kH-1)/2k$ for all $k \in \mathbb{N}$ satisfying $2kH > 1$. Taking $k \to +\infty$ completes the proof. □

In the case $H > \frac{1}{2}$ we prove a similar statement.

Proposition 3.2. *Let $H > \frac{1}{2}$ and $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ be a map satisfying $\sigma \in L^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$. Then the conclusion of Proposition* 3.1 *holds true.*

Proof. The second part of the assertion can be shown in the same way as in Proposition 3.1 ($\sigma \in \mathcal{H}$ because $L^\infty([0,T];\mathcal{L}(R^n)) \subset L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n)) \subset \widetilde{\mathcal{H}} \subset \mathcal{H}$). For the remaining part take arbitrary $s,t \in [0,T]$, $s < t$, and set

$$\overline{\sigma}(u) = \begin{cases} \sigma(u), & s \leqslant u \leqslant t,\\ 0, & u \in [0,T] \setminus [s,t]. \end{cases}$$

By the equality (2.6) we obtain

$$\begin{aligned}
\mathbb{E}\,\|Z_t - Z_s\|^2 &= \mathbb{E}\left\| \int_s^t \sigma(u)\,\mathrm{d}B^H(u) \right\|^2 = \mathbb{E}\left\| \int_0^T \overline{\sigma}(u)\,\mathrm{d}B^H_u \right\|^2\\
&= H(2H-1)\int_0^T \int_0^T \langle \overline{\sigma}(u), \overline{\sigma}(v)\rangle_{\mathcal{L}(\mathbb{R}^n)} |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v\\
&= H(2H-1)\int_s^t \int_s^t \langle \sigma(u), \sigma(v)\rangle_{\mathcal{L}(\mathbb{R}^n)} |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v\\
&= H(2H-1)\int_0^{t-s} \int_0^{t-s} \langle \sigma(u+s), \sigma(v+s)\rangle_{\mathcal{L}(\mathbb{R}^n)} |u-v|^{2H-2}\,\mathrm{d}u\,\mathrm{d}v\\
&= \mathbb{E}\left\| \int_0^{t-s} \sigma(u+s)\,\mathrm{d}B^H_u \right\|^2.
\end{aligned}$$

Using the notation $\sigma_s(u) = \sigma(u+s)$, $s \in [0,T]$, $u \in [s,T]$, and (2.5) we get

$$\begin{aligned}
\mathbb{E}\,\|Z_t - Z_s\|^2 &= \mathbb{E}\left\| \int_0^{t-s} \sigma(u+s)\,\mathrm{d}B^H_u \right\|^2 = \mathbb{E}\left\| \int_0^{t-s} \sigma_s(u)\,\mathrm{d}B^H_u \right\|^2\\
&\leqslant b_H \|\sigma_s\|^2_{L^{1/H}([0,t-s];\mathcal{L}(\mathbb{R}^n))} \leqslant b_H \|\sigma\|^2_{L^\infty([0,T];\mathcal{L}(\mathbb{R}^n))}(t-s)^{2H} = B(t-s)^{2H},
\end{aligned}$$

where B is a constant depending only on T, H, σ. The proof is completed in the same way as in Proposition 3.1. □

Remark 3.3. For fixed $0 < \gamma < H$ we identify the process $\{Z_t, t \in [0,T]\}$ with its version having Hölder continuous trajectories of order γ for σ satisfying the assumptions of Proposition 3.1 (case $H < \frac{1}{2}$) or Proposition 3.2 (case $H > \frac{1}{2}$).

Definition 3.4. An $\mathcal{F}_t^{B^H}$-adapted process with continuous trajectories is a *solution* to the equation (3.1) if $\{Y_t, t \in [0,T]\}$ satisfies the equation (3.1) for all $t \in [0,T]$ $\mathbb{P}$-a.s. *Pathwise uniqueness* holds for (3.1) if

$$\mathbb{P}\{Y_t = \widetilde{Y}_t \quad \forall\, t \in [0,T]\} = 1$$

holds for any two solutions $\{Y_t, t \in [0,T]\}$, $\{\widetilde{Y}_t, t \in [0,T]\}$.

Proposition 3.5. *Suppose that either $H < \frac{1}{2}$ and $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ satisfies the assumptions of Proposition* 3.1*, or $H > \frac{1}{2}$ and σ satisfies the assumptions of Proposition* 3.2*. Further, let $b_1\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a Borel function satisfying*

$$\begin{aligned}
&\forall\, N \in \mathbb{N}\ \exists\, K_N > 0\ \forall\, t \in [0,T]\ \forall\, x, y \in \mathbb{R}^n\ \ \|x\| + \|y\| \leqslant N \\
&\qquad \|b_1(t,x) - b_1(t,y)\| \leqslant K_N \|x - y\|, \\
&\exists\, K_b > 0\ \forall\, t \in [0,T]\ \forall\, x \in \mathbb{R}^n \quad \|b_1(t,x)\| \leqslant K_b(1 + \|x\|).
\end{aligned}$$

Then there exists a pathwise unique solution to the equation (3.1).

Proof. Since $\{Z_t, t \in [0,T]\}$ has continuous trajectories, standard ODE techniques may be used. □

Now we can prove the Hölder continuity of the process $\{Y_t, t \in [0,T]\}$.

Theorem 3.6. *Let $\{Y_t, t \in [0,T]\}$ be a solution to the equation* (3.1)*, where $b_1\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Borel function satisfying*

$$\exists\, K_b > 0\ \forall\, t \in [0,T]\ \forall\, x \in \mathbb{R}^n \quad \|b_1(t,x)\| \leqslant K_b(1 + \|x\|). \tag{3.7}$$

Let either $H < \frac{1}{2}$ and $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ satisfy the conditions of Proposition 3.1 *or $H > \frac{1}{2}$ and σ satisfy the conditions of Proposition* 3.2.

Then there exists a version of the process $\{Y_t, t \in [0,T]\}$ with Hölder continuous trajectories of order $0 < \gamma < H$. Moreover, for any $0 < \gamma < H$ the estimate

$$\|Y\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)} \leqslant A(1 + \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}) \quad \mathbb{P}\text{-a.s.} \tag{3.8}$$

is valid for a constant $A \equiv A(T, \tilde{x}, K_b, \gamma) > 0$.

P r o o f. Take $\gamma < H$. First we estimate $\|Y\|_{\mathcal{C}([0,T];\mathbb{R}^n)}$ in terms of $\|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)}$. We have

$$\begin{aligned}\|Y_t\| &\leqslant \|\tilde{x}\| + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)} + \int_0^t \|b_1(s, Y_s)\| \, \mathrm{d}s \\ &\leqslant \|\tilde{x}\| + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)} + K_b T + K_b \int_0^t \|Y_s\| \, \mathrm{d}s\end{aligned}$$

for all $t \in [0,T]$ $\mathbb{P}$-a.s. By Gronwall's lemma (see e.g. [15]) we get

$$\|Y_t\| \leqslant (\|\tilde{x}\| + K_b T + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)})\mathrm{e}^{K_b t}, \quad t \in [0,T] \ \mathbb{P}\text{-a.s.},$$

which implies

$$\|Y\|_{\mathcal{C}([0,T];\mathbb{R}^n)} \leqslant (\|\tilde{x}\| + K_b T + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)})\mathrm{e}^{K_b T} \quad \mathbb{P}\text{-a.s.} \tag{3.9}$$

According to Proposition 3.1 and 3.2 in the respective cases the process $\{Z_t, t \in [0,T]\}$ has trajectories in $\mathcal{C}^\gamma([0,T];\mathbb{R}^n)$, $\gamma < H$.

Select $\Omega_0 \in \mathcal{F}$, $\mathbb{P}(\Omega_0) = 1$ such that

$$\forall\, \omega \in \Omega_0 \quad Z_\cdot(\omega) \in \mathcal{C}^\gamma([0,T];\mathbb{R}^n)$$

and

$$\forall\, \omega \in \Omega_0 \ \forall\, t \in [0,T] \quad Y_t = \tilde{x} + \int_0^t b_1(s, Y_s) \, \mathrm{d}s + Z_t.$$

By (3.7) and (3.9) we obtain

$$\begin{aligned}\|Y_t - Y_s\| &= \left\| \int_s^t b_1(u, Y_u) \, \mathrm{d}u + Z_t - Z_s \right\| \leqslant \int_s^t \|b_1(u, Y_u)\| \, \mathrm{d}u + \|Z_t - Z_s\| \\ &\leqslant K_b \int_s^t (1 + \|Y_u\|) \, \mathrm{d}u + (t-s)^\gamma \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)} \\ &\leqslant K_b (t-s)[1 + (\|\tilde{x}\| + K_b T + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)})\mathrm{e}^{K_b T}] + (t-s)^\gamma \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)} \\ &\leqslant A_1 (t-s)^\gamma (1 + \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}) \quad \text{on } \Omega_0\end{aligned} \tag{3.10}$$

for $0 \leqslant s < t \leqslant T$, where A_1 is a constant depending only on T, γ, K_b, and $\tilde{x}$, hence $\{Y_t, t \in [0,T]\}$ takes values in $\mathcal{C}^\gamma([0,T];\mathbb{R}^n)$ $\mathbb{P}$-a.s. If we define

$$\overline{Y}_t(\omega) = \begin{cases} Y_t(\omega), & Y_\cdot(\omega) \in \mathcal{C}^\gamma([0,T];\mathbb{R}^n), \\ 0 & \text{otherwise}, \end{cases}$$

then $\{\overline{Y}_t, t \in [0,T]\}$ is a version of the process $\{Y_t, t \in [0,T]\}$ with trajectories in $\mathcal{C}^\gamma([0,T];\mathbb{R}^n)$. We prove the estimate (3.8). Using (3.9) and (3.10) for $\omega \in \Omega$ such that $\overline{Y}_t(\omega) = Y_t(\omega)$, $t \in [0,T]$, we obtain

$$\begin{aligned}
\|\overline{Y}\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)} &= \|\overline{Y}\|_{\mathcal{C}([0,T];\mathbb{R}^n)} + \sup_{0\leqslant v<u\leqslant T} \frac{\|\overline{Y}_u - \overline{Y}_v\|}{(u-v)^\gamma} \\
&= \|Y\|_{\mathcal{C}([0,T];\mathbb{R}^n)} + \sup_{0\leqslant v<u\leqslant T} \frac{\|Y_u - Y_v\|}{(u-v)^\gamma} \\
&\leqslant (\|\tilde{x}\| + K_b T + \|Z\|_{\mathcal{C}([0,T];\mathbb{R}^n)})\mathrm{e}^{K_b T} + A_1(1 + \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)}) \\
&\leqslant A(1 + \|Z\|_{\mathcal{C}^\gamma([0,T];\mathbb{R}^n)})),
\end{aligned}$$

where A depends only on $T, \gamma, K_b, \tilde{x}$, and (3.8) follows for the version $\{\overline{Y}_t, t \in [0,T]\}$ of the process $\{Y_t, t \in [0,T]\}$. □

4. Girsanov theorem and its application

In what follows $\{W_t, t \geqslant 0\}$ denotes an n-dimensional $\mathcal{F}_t$-Wiener process defined by (2.4).

Theorem 4.1 (Girsanov Theorem for fBm)**.** *Let $B^H = \{B_t^H, t \in [0,T]\}$ be an n-dimensional $\mathcal{F}_t$-fractional Brownian motion with Hurst parameter H on the interval $[0,T]$. Consider an $\mathcal{F}_t$-adapted n-dimensional process $u = \{u_t, t \in [0,T]\}$ with integrable trajectories. Set*

$$v(s) = K_H^{-1}\left(\int_0^\cdot u_r \,\mathrm{d}r\right)(s), \quad s \in [0,T],$$
$$\xi_T = \exp\left\{\int_0^T v_s^{\mathrm{T}} \,\mathrm{d}W_s - \frac{1}{2}\int_0^T \|v_s\|^2 \,\mathrm{d}s\right\},$$

and assume that

(i) $\int_0^\cdot u_s \,\mathrm{d}s \in I_{0+}^{H+\frac{1}{2}}(L^2([0,T];\mathbb{R}^n))$ $\mathbb{P}$*-a.s.,*

(ii) $\mathbb{E}(\xi_T) = 1$.

Then $\{B_t^H - \int_0^t u_s \,\mathrm{d}s, t \in [0,T]\}$ is an n-dimensional $\mathcal{F}_t$-fractional Brownian motion with Hurst parameter H on the interval $[0,T]$ under the probability $\widetilde{\mathbb{P}}$ defined by the density $\xi_T = \mathrm{d}\widetilde{\mathbb{P}}/\mathrm{d}\mathbb{P}$ with respect to $\mathbb{P}$.

Proof. Cf. [23]. □

For $\lambda \in (0,1]$ define the space

$$\mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n) := \Big\{f \in \mathcal{C}^\lambda([0,T];\mathbb{R}^n);\ \forall\,\varepsilon > 0\ \exists\,\delta \equiv \delta(\varepsilon, f) > 0$$
$$\forall\, s,t \in (0,T),\ 0 < |t-s| < \delta \Rightarrow \frac{|f(t) - f(s)|}{|t-s|^\lambda} < \varepsilon\Big\}.$$

If the space $\mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n)$ is equipped with the norm of the space $\mathcal{C}^{\lambda}([0,T];\mathbb{R}^n)$ then

1. $\mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n) \subset \mathcal{C}^{\lambda}([0,T];\mathbb{R}^n)$;
2. $\mathcal{C}^{\nu}([0,T];\mathbb{R}^n) \subset \mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n) \quad \forall\, \nu > \lambda$;
3. the space $\mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n)$ is separable (cf. [14], Theorem 1.4.11).

The separability of $\mathcal{C}^{0,\lambda,0}([0,T];\mathbb{R}^n)$ will be important when we apply Fernique's Theorem (cf. [9]).

In what follows, let $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ be a Borel mapping such that there exists an inverse $\sigma^{-1}(t)$ of $\sigma(t)$ for all $t \in [0,T]$ and let $b_2\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a Borel function. Set

$$u(s) = \sigma^{-1}(s)\, b_2(s, Y_s), \quad s \in [0,T],$$

and

$$v(s) = K_H^{-1}\Big(\int_0^{\cdot} u_r \,\mathrm{d}r\Big)(s), \quad s \in [0,T],$$

where $\{Y_s, s \in [0,T]\}$ is a solution to (3.1) with Hölder continuous trajectories of order δ, $0 < \delta < H$.

Case $H < \frac{1}{2}$.

Theorem 4.2. *Let $\{Y_t, 0 \leqslant t \leqslant T\}$ be a solution to the equation* (3.1) *whose diffusion and drift satisfy the assumptions of Theorem* 3.6. *Assume that $\sigma(t)$ is invertible for all $t \in [0,T]$. Let $b_2\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be a Borel function satisfying*

$$\exists\, K > 0\ \forall\, t \in [0,T]\ \forall\, x \in \mathbb{R}^n \quad \|\sigma^{-1}(t)\, b_2(t,x)\| \leqslant K(1 + \|x\|). \tag{4.1}$$

Then $\widetilde{B}^H = \{B_t^H - \int_0^t u_s \,\mathrm{d}s, 0 \leqslant t \leqslant T\}$ is an n-dimensional $\mathcal{F}_t^{B^H}$-fractional Brownian motion on $[0,T]$ under the probability measure defined by the density ξ_T with respect to $\mathbb{P}$, where

$$\xi_T = \exp\Big\{\int_0^T v_s^{\mathrm{T}} \,\mathrm{d}W_s - \frac{1}{2}\int_0^T \|v_s\|^2 \,\mathrm{d}s\Big\}.$$

Proof. The process $\{u_s, s \in [0,T]\}$ is $\mathcal{F}_t^{B^H}$-adapted. We show that the assumptions (i), (ii) of Theorem 4.1 are satisfied for $\{u_s, s \in [0,T]\}$. The condition (i) of Theorem 4.1 is equivalent to

$$v \in L^2([0,T]) \quad \mathbb{P}\text{-a.s.} \tag{4.2}$$

To check (ii) it suffices to show that there exists $\Delta > 0$ and a partition $\{0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T\}$ such that $|t_{i+1} - t_i| < \Delta$ and

$$\mathbb{E}\,\exp\Big\{\int_{t_i}^{t_{i+1}} \|v_s\|^2 \,\mathrm{d}s\Big\} < +\infty \tag{4.3}$$

holds for all $i = 0, \dots, N(\Delta) - 1$, since this implies

$$\mathbb{E}\left[\exp\left\{\int_{t_i}^{t_{i+1}} v_s^{\mathrm{T}}\,\mathrm{d}W_s - \frac{1}{2}\int_{t_i}^{t_{i+1}} \|v_s\|^2\,\mathrm{d}s\right\}\Big|\mathcal{F}_{t_i}^{B^H}\right] = 1 \quad \mathbb{P}\text{-a.s.} \tag{4.4}$$

(cf. [10], Lemma 7.1.3). Using repeatedly (4.4) we obtain

$$\mathbb{E}\left[\exp\left\{\int_0^T v_s^{\mathrm{T}}\,\mathrm{d}W_s - \frac{1}{2}\int_0^T \|v_s\|^2\,\mathrm{d}s\right\}\right] = 1,$$

which is (ii) from Theorem 4.1. Thus it is sufficient to prove (4.2) and (4.3).

By Proposition 3.1 $\{Z_t, t \in [0,T]\}$ is a Gaussian process with Hölder continuous trajectories on $[0,T]$ of order γ, $0 < \gamma < H$, and

$$\mathcal{C}^{\gamma}([0,T];\mathbb{R}^n) \subset \mathcal{C}^{0,\nu,0}([0,T];\mathbb{R}^n;\|\cdot\|_{\mathcal{C}^{\nu}([0,T];\mathbb{R}^n)}) \subset \mathcal{C}^{\nu}([0,T];\mathbb{R}^n)$$

for any ν, $H > \gamma > \nu > 0$. Due to the separability of the space $\mathcal{C}^{0,\nu,0}([0,T];\mathbb{R}^n;\|\cdot\|_{\mathcal{C}^{\nu}([0,T];\mathbb{R}^n)})$, $Z\colon \Omega \to \mathcal{C}^{0,\nu,0}([0,T];\mathbb{R}^n;\ \|\cdot\|_{\mathcal{C}^{\nu}([0,T];\mathbb{R}^n)})$ is a Gaussian random variable. Thus, by Fernique's Theorem (cf. [9]) we get

$$\mathbb{E}\,\exp\{\zeta\|Z\|^2_{\mathcal{C}^{\nu}([0,T];\mathbb{R}^n)}\} < +\infty \tag{4.5}$$

for some $\zeta > 0$. Denote

$$\varphi(s) = s^{\frac{1}{2}-H}\sigma^{-1}(s)\,b_2(s,Y_s).$$

For any $t_1, t_2 \in [0,T]$, $t_1 < t_2$, we have

$$\begin{aligned}\int_{t_1}^{t_2}\|v_s\|^2\,\mathrm{d}s &= \int_{t_1}^{t_2}\left\|K_H^{-1}\Big(\int_0^{\cdot} u_r\,\mathrm{d}r\Big)\right\|^2\mathrm{d}s = \int_{t_1}^{t_2}\|s^{H-\frac{1}{2}}(I_{0+}^{\frac{1}{2}-H}\varphi)(s)\|^2\,\mathrm{d}s\\ &= \frac{1}{\Gamma^2(\frac{1}{2}-H)}\int_{t_1}^{t_2}\left\|s^{H-\frac{1}{2}}\int_0^s (s-r)^{-H-\frac{1}{2}}r^{\frac{1}{2}-H}\sigma^{-1}(r)\,b_2(r,Y_r)\,\mathrm{d}r\right\|^2\mathrm{d}s.\end{aligned}$$

In view of (4.1) we get

$$\begin{aligned}\int_{t_1}^{t_2}\|v_s\|^2\,\mathrm{d}s &\leqslant \frac{1}{\Gamma^2(\frac{1}{2}-H)}\int_{t_1}^{t_2} s^{2H-1}\left[\int_0^s (s-r)^{-H-\frac{1}{2}}r^{\frac{1}{2}-H}K(1+\|Y_r\|)\,\mathrm{d}r\right]^2\mathrm{d}s\\ &\leqslant \frac{K^2(1+\|Y\|_{L^{\infty}([0,T];\mathbb{R}^n)})^2}{\Gamma^2(\frac{1}{2}-H)}B^2\Big(\frac{3}{2}-H,\frac{1}{2}-H\Big)\int_{t_1}^{t_2} s^{1-2H}\,\mathrm{d}s =: I_1.\end{aligned}$$

Obviously,

$$\int_{t_1}^{t_2} s^{1-2H}\,\mathrm{d}s = \int_0^{t_2-t_1}(s+t_1)^{-\frac{1}{2}}(s+t_1)^{\frac{3}{2}-2H}\,\mathrm{d}s \leqslant 2T^{\frac{3}{2}-2H}(t_2-t_1)^{\frac{1}{2}}, \tag{4.6}$$

and using the estimate (3.8) from Theorem 3.6 we have

$$(4.7)\qquad (1+\|Y\|_{L^\infty([0,T];\mathbb{R}^n)})^2 \leqslant [1+A(1+\|Z\|_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)})]^2 \\ \leqslant 2(1+2A^2+2A^2\|Z\|^2_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)}) \quad \mathbb{P}\text{-a.s.}$$

From (4.6) and (4.7) we obtain

$$(4.8)\qquad I_1 = \frac{K^2(1+\|Y\|_{L^\infty([0,T];\mathbb{R}^n)})^2}{\Gamma^2(\frac{1}{2}-H)} B^2\Big(\frac{3}{2}-H,\frac{1}{2}-H\Big)\int_{t_1}^{t_2} s^{1-2H}\,\mathrm{d}s \\ \leqslant \widetilde{B}[1+(t_2-t_1)^{\frac{1}{2}}\|Z\|^2_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)}]$$

for any $t_1,t_2\in[0,T]$, $t_1<t_2$, $\mathbb{P}$-a.s., where $\widetilde{B}$ is a constant depending only on K, T, H, A.

For $t_1=0$, $t_2=T$, (4.8) reads

$$\int_0^T \|v_s\|^2\,\mathrm{d}s \leqslant \widetilde{B}[1+T^{\frac{1}{2}}\|Z\|^2_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)}] < +\infty \quad \mathbb{P}\text{-a.s.},$$

which implies (4.2).

Take $\Delta>0$ such that $\widetilde{B}\Delta^{\frac{1}{2}}<\zeta$ and consider a partition $\{0=t_0<t_1<\ldots<t_{N(\Delta)}=T\}$ such that $|t_{i+1}-t_i|<\Delta$, $i=0,\ldots,N(\Delta)-1$. By (4.5) and (4.8) we obtain

$$\mathbb{E}\exp\Big\{\int_{t_i}^{t_{i+1}}\|v_s\|^2\,\mathrm{d}s\Big\} \leqslant \mathbb{E}\exp\{\widetilde{B}[1+(t_{i+1}-t_i)^{\frac{1}{2}}\|Z\|^2_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)}]\} \\ \leqslant \mathrm{e}^{\widetilde{B}}\,\mathbb{E}\exp\{\zeta\|Z\|^2_{\mathcal{C}^\nu([0,T];\mathbb{R}^n)}\} < +\infty,$$

which verifies (4.3) and the proof is complete. □

Case $H>\frac{1}{2}$.

Theorem 4.3. *Let $\{Y_t, 0\leqslant t\leqslant T\}$ be a solution to the equation* (3.1) *whose diffusion and drift satisfy the assumptions of Theorem* 3.6. *Assume that $\sigma(t)$ is invertible for all $t\in[0,T]$. Let $b_2\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ be a Borel function satisfying*

$$(4.9)\qquad \exists\,\alpha\in(1-\frac{1}{2H},1)\ \exists\,\beta\in(H-\frac{1}{2},1)\ \exists\,C>0\ \forall\,s,t\in[0,T]\ \forall\,x,y\in\mathbb{R}^n \\ \|\sigma^{-1}(t)\,b_2(t,x)-\sigma^{-1}(s)\,b_2(s,y)\| \leqslant C(\|x-y\|^\alpha+|t-s|^\beta).$$

Then the conclusion of Theorem 4.2 *holds true.*

Proof. We verify (4.2) and (4.3) which appear in the proof of Theorem 4.2. First we prove that for any $\delta \in (0, H)$ there exists a constant $\widetilde{B} > 0$ such that

$$\int_{t_1}^{t_2} \|v_s\|^2 \, \mathrm{d}s \leqslant \widetilde{B}[1 + (t_2 - t_1)^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}] \tag{4.10}$$

holds $\mathbb{P}$-a.s. whenever $t_1, t_2 \in [0, T]$, $t_1 < t_2$. Setting again

$$\varphi(s) = s^{\frac{1}{2}-H} \sigma^{-1}(s)\, b_2(s, Y_s),$$

we may estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} \|v_s\|^2 \, \mathrm{d}s \qquad (4.11)\\
&= \int_{t_1}^{t_2} \Big\|K_H^{-1}\Big(\int_0^{\cdot} u_r \, \mathrm{d}r\Big)(s)\Big\|^2 \mathrm{d}s = \int_{t_1}^{t_2} \|s^{H-\frac{1}{2}} (D_{0+}^{H-\frac{1}{2}} \varphi)(s)\|^2 \, \mathrm{d}s\\
&= \int_{t_1}^{t_2} \frac{1}{\Gamma^2(\frac{3}{2} - H)} \Big\| s^{H-\frac{1}{2}} \Big[\frac{\varphi(s)}{s^{H-\frac{1}{2}}} + \Big(H - \frac{1}{2}\Big) \int_0^s \frac{\varphi(s) - \varphi(r)}{(s-r)^{H+\frac{1}{2}}} \, \mathrm{d}r\Big]\Big\|^2 \mathrm{d}s\\
&\leqslant \frac{2}{\Gamma^2(\frac{3}{2} - H)} \Big[\int_0^{t_2 - t_1} \|\varphi(v + t_1)\|^2 \, \mathrm{d}v\\
&\quad + \int_0^{t_2-t_1} \Big\|\Big(H - \frac{1}{2}\Big)(v + t_1)^{H-\frac{1}{2}} \int_0^{v+t_1} \frac{\varphi(v + t_1) - \varphi(r)}{(v + t_1 - r)^{H+\frac{1}{2}}} \, \mathrm{d}r\Big\|^2 \mathrm{d}v\Big]\\
&:= \frac{2}{\Gamma^2(\frac{3}{2} - H)} (I_1 + I_2)
\end{aligned}$$

for $t_1, t_2 \in [0, T]$, $t_1 < t_2$. Without loss of generality we suppose

$$\|\sigma^{-1}(0)\, b_2(0, \tilde{x})\| \leqslant C, \tag{4.12}$$

where C is the constant from (4.9). Fix any δ, $0 < \delta < H$. By (4.9), (4.12) and the Hölder continuity of order δ of the trajectories of the process $\{Y_t, 0 \leqslant t \leqslant T\}$ we obtain

$$\begin{aligned}
I_1 &= \int_0^{t_2-t_1} \|\varphi(v + t_1)\|^2 \, \mathrm{d}v\\
&= \int_0^{t_2-t_1} (v + t_1)^{1-2H} \|\sigma^{-1}(v + t_1)\, b_2(v + t_1, Y_{v+t_1}) \pm \sigma^{-1}(0)\, b_2(0, Y_0)\|^2 \, \mathrm{d}v\\
&\leqslant 2C^2 \int_0^{t_2-t_1} (v + t_1)^{1-2H} [(\|Y_{v+t_1} - Y_0\|^\alpha + (v + t_1)^\beta)^2 + 1] \, \mathrm{d}v\\
&\leqslant 2C^2 \int_0^{t_2-t_1} (v + t_1)^{1-2H} [2((v + t_1)^{2\alpha\delta} \|Y\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} + (v + t_1)^{2\beta}) + 1] \, \mathrm{d}v.
\end{aligned}$$

Using (3.8) to estimate Y, we get

$$\begin{aligned}
(4.13)\quad I_1 &\leqslant 2C^2 \int_0^{t_2-t_1} (v+t_1)^{1-2H} \\
&\quad \times [2((v+t_1)^{2\alpha\delta} A^{2\alpha}(1+\|Z\|_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)})^{2\alpha} + (v+t_1)^{2\beta}) + 1]\,\mathrm{d}v \\
&\leqslant B_1[1+(t_2-t_1)^{2(1-H)}\|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}] \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where B_1 depends only on H, C, T, α, δ, β, A. Setting $\psi(s) = \sigma^{-1}(s)\,b_2(s, Y_s)$, $s \in [0,T]$, we have

$$\begin{aligned}
I_2 &= \int_0^{t_2-t_1} \left\|(H-\frac{1}{2})(v+t_1)^{H-\frac{1}{2}} \int_0^{v+t_1} \frac{\varphi(v+t_1)-\varphi(r)}{(v+t_1-r)^{H+\frac{1}{2}}}\,\mathrm{d}r\right\|^2 \mathrm{d}v \\
&\leqslant 2\Big(H-\frac{1}{2}\Big)^2 \int_0^{t_2-t_1} (v+t_1)^{2H-1}\Bigg[\Bigg\|\int_0^{v+t_1} \frac{\varphi(v+t_1)-r^{\frac{1}{2}-H}\psi(v+t_1)}{(v+t_1-r)^{H+\frac{1}{2}}}\,\mathrm{d}r\Bigg\|^2 \\
&\quad + \Bigg\|\int_0^{v+t_1} \frac{r^{\frac{1}{2}-H}\psi(v+t_1)-\varphi(r)}{(v+t_1-r)^{H+\frac{1}{2}}}\,\mathrm{d}r\Bigg\|^2\Bigg]\,\mathrm{d}v \\
&\leqslant 2\Big(H-\frac{1}{2}\Big)^2 \int_0^{t_2-t_1} (v+t_1)^{2H-1}\Bigg\{\Bigg[\int_0^{v+t_1} \frac{|(v+t_1)^{\frac{1}{2}-H}-r^{\frac{1}{2}-H}|}{(v+t_1-r)^{H+\frac{1}{2}}}\,\mathrm{d}r\|\psi(v+t_1)\|\Bigg]^2 \\
&\quad + \Bigg[\int_0^{v+t_1} \frac{r^{\frac{1}{2}-H}}{(v+t_1-r)^{H+\frac{1}{2}}}\|\psi(v+t_1)-\psi(r)\|\,\mathrm{d}r\Bigg]^2\Bigg\}\,\mathrm{d}v.
\end{aligned}$$

By (2.10) from Lemma 2.2, (4.9), (4.12), the Hölder continuity of order δ of the trajectories of the process $\{Y_t, 0 \leqslant t \leqslant T\}$ and the estimate (3.8) we obtain

$$\begin{aligned}
(4.14)\qquad I_2 &\leqslant 2\Big(H-\frac{1}{2}\Big)^2 C^2 \\
&\quad \times \int_0^{t_2-t_1} (v+t_1)^{2H-1}\Big\{2C^2(H)(v+t_1)^{2-4H}[2\|Y_{v+t_1}-Y_0\|^{2\alpha} + 2(v+t_1)^{2\beta}+1] \\
&\quad + \Bigg[\int_0^{v+t_1} \frac{r^{\frac{1}{2}-H}}{(v+t_1-r)^{H+\frac{1}{2}}}\{(v+t_1-r)^{\alpha\delta}\|Y\|^{\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} + (v+t_1-r)^{\beta}\}\,\mathrm{d}r\Bigg]^2\Bigg\}\,\mathrm{d}v \\
&\leqslant 2\Big(H-\frac{1}{2}\Big)^2 C^2 \int_0^{t_2-t_1} (v+t_1)^{2H-1} \\
&\quad \times \Big\{2C^2(H)(v+t_1)^{2-4H}[4(v+t_1)^{2\alpha\delta}A^{2\alpha}(1+\|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}) + 2(v+t_1)^{2\beta}+1] \\
&\quad\quad + \Big[A^\alpha(1+\|Z\|^{\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)})\int_0^1 [u^{\frac{1}{2}-H}(1-u)^{\alpha\delta-H-\frac{1}{2}} \\
&\quad\quad\quad \times (v+t_1)^{\alpha\delta-2H+1} + u^{\frac{1}{2}-H}(1-u)^{\beta-H-\frac{1}{2}}(v+t_1)^{\beta-2H+1}]\,\mathrm{d}u\Big]^2\Big\}\,\mathrm{d}v \\
&\leqslant B_2[1+(t_2-t_1)^{2(1-H)}\|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}] \quad \mathbb{P}\text{-a.s.},
\end{aligned}$$

where B_2 is a constant depending on $H, C, T, \alpha, \delta, \beta, A, C(H)$. Plugging (4.13) and (4.14) into (4.11) we get that $\mathbb{P}$-a.s.

$$\int_{t_1}^{t_2} \|v_s\|^2 \,\mathrm{d}s \leqslant \widetilde{B}[1 + (t_2 - t_1)^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}]$$

holds for all $t_1, t_2 \in [0, T],\ t_1 < t_2$, where

$$\widetilde{B} = \frac{2}{\Gamma^2(\frac{3}{2} - H)}(B_1 + B_2),$$

which completes the proof of (4.10).

Using (4.10) for $t_1 = 0, t_2 = T$ we get

$$\int_0^T \|v_s\|^2 \,\mathrm{d}s \leqslant \widetilde{B}[1 + T^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}] < +\infty \quad \mathbb{P}\text{-a.s.},$$

hence (4.2) holds.

In order to verify (4.3), choose $\Delta > 0$ such that $\widetilde{B}\Delta^{2(1-H)} < \zeta$ and a partition $\{0 = t_0 < t_1 < \ldots < t_{N(\Delta)} = T\}$, $|t_{i+1} - t_i| < \Delta,\ i = 1, \ldots, N(\Delta) - 1$. We have

$$\begin{aligned}
\mathbb{E} \exp\left\{ \int_{t_i}^{t_{i+1}} \|v_s\|^2 \,\mathrm{d}s \right\} &\leqslant \mathbb{E} \exp\{\widetilde{B}[1 + (t_{i+1} - t_i)^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}]\} \\
&= \mathrm{e}^{\widetilde{B}} \Bigg[\int_{0 \leqslant \|Z\|_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} \leqslant 1} \exp\{\widetilde{B}\Delta^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}\} \,\mathrm{d}\mathbb{P} \\
&\quad + \int_{\|Z\|_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} > 1} \exp\{\widetilde{B}(\Delta^{2(1-H)} \|Z\|^{2\alpha}_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}\} \,\mathrm{d}\mathbb{P} \Bigg] \\
&\leqslant \mathrm{e}^{\widetilde{B}} \Bigg[\mathrm{e}^{\zeta} \mathbb{P} \{0 \leqslant \|Z\|_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} \leqslant 1\} \\
&\quad + \int_{\|Z\|_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)} > 1} \exp\{\zeta \|Z\|^2_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}\} \,\mathrm{d}\mathbb{P} \Bigg] \\
&\leqslant \mathrm{e}^{\widetilde{B}} [\mathrm{e}^{\zeta} + \mathbb{E} \exp\{\zeta \|Z\|^2_{\mathcal{C}^\delta([0,T];\mathbb{R}^n)}\}] < +\infty,
\end{aligned}$$

which shows (4.3) and the proof may be completed as in the proof of Theorem 4.3. □

5. Existence of a weak solution

Consider the equation

$$X_t = \tilde{x} + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H \tag{5.1}$$

where $b\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ and $\{B_t^H, t \in [0,T]\}$ is an n-dimensional $\mathcal{F}_t^{B^H}$-fractional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with parameter H on the interval $[0,T]$.

Proposition 5.1. *Suppose that trajectories of the process $\{u_t, t \in [0,T]\}$ are $\mathbb{P}$-a.s. in $L^\infty([0,T];\mathbb{R}^n)$. Then for any $\varphi \in \mathcal{H}$ (case $H < \frac{1}{2}$) or $\varphi \in L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))$ (case $H > \frac{1}{2}$)*

$$\int_0^t \varphi(s)\,\mathrm{d}B_s^H = \int_0^t \varphi(s)\,\mathrm{d}\widetilde{B}_s^H + \int_0^t \varphi(s)u(s)\,\mathrm{d}s \quad \widetilde{\mathbb{P}}\text{-a.s.},\ t \in [0,T], \tag{5.2}$$

where $\{\widetilde{B}_t^H, t \in [0,T]\}$ is the n-dimensional $\mathcal{F}_t^{B^H}$-fBm on $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$ defined in Theorems 4.2 and 4.3 in the respective cases $H < \frac{1}{2}$ and $H > \frac{1}{2}$.

Remark 5.2. Note that the stochastic integrals on the left-hand and the right-hand sides of (5.2) are defined on different probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, respectively. However, these spaces differ only by the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, which are mutually absolutely continuous. Therefore, (5.2) makes sense.

P r o o f. Take any $t \in [0,T]$, $a, b \in [0,t]$. If $\varphi = I_{[a,b)}$ then the left-hand side of (5.2) is equal to

$$\int_0^t \varphi(s)\,\mathrm{d}B_s^H = B_b^H - B_a^H. \tag{5.3}$$

Further, with this choice of φ,

$$\begin{aligned}\int_0^t \varphi(s)\,\mathrm{d}\widetilde{B}_s^H + \int_0^t \varphi(s)u(s)\,\mathrm{d}s &= \widetilde{B}_b^H - \widetilde{B}_a^H + \int_a^b u(s)\,\mathrm{d}s \\ &= B_b^H - \int_0^b u(s)\,\mathrm{d}s + B_a^H - \int_0^a u(s)\,\mathrm{d}s + \int_a^b u(s)\,\mathrm{d}s = B_b^H - B_a^H,\end{aligned}$$

so (5.2) has been shown for $\varphi = I_{[a,b)}$, $a, b \in [0,t]$. By linearity we can extend (5.2) to all $\varphi \in \mathcal{E}$. The rest of the proof is divided into two parts.

(i) $H < \frac{1}{2}$. By the definition of $\mathcal{H}$

$$\forall\, \varphi \in \mathcal{H}\ \exists\, \varphi_n \in \mathcal{E} \quad \|\varphi_n - \varphi\|_{\mathcal{H}} \longrightarrow 0,\ n \to +\infty.$$

Using the isometry (2.3)

$$\int_\Omega \left\| \int_0^t \varphi_n(s)\,\mathrm{d}\widetilde{B}^H_s - \int_0^t \varphi(s)\,\mathrm{d}\widetilde{B}^H_s \right\|^2 \mathrm{d}\widetilde{\mathbb{P}} = \|\mathcal{K}^*_H(\varphi_n - \varphi)(s)\|^2_{L^2([0,t];\mathcal{L}(\mathbb{R}^n))}$$
$$\leqslant \|\mathcal{K}^*_H(\varphi_n - \varphi)(s)\|^2_{L^2([0,T];\mathcal{L}(\mathbb{R}^n))} = \|\varphi_n - \varphi\|^2_{\mathcal{H}} \longrightarrow 0,\ n \to +\infty,$$

thus

$$\int_0^t \varphi_n(s)\,\mathrm{d}\widetilde{B}^H_s \longrightarrow \int_0^t \varphi(s)\,\mathrm{d}\widetilde{B}^H_s,\ n \to +\infty \quad \text{in } L^2(\Omega, \widetilde{\mathbb{P}}; \mathbb{R}^n).$$

Hence there exists a subsequence of (φ_n) (denoted again by (φ_n)) such that

$$\int_0^t \varphi_n(s)\,\mathrm{d}\widetilde{B}^H_s \longrightarrow \int_0^t \varphi(s)\,\mathrm{d}\widetilde{B}^H_s,\ n \to +\infty \quad \widetilde{\mathbb{P}}\text{-a.s.} \tag{5.4}$$

Repeating the same procedure for $\{B^H_t, t \in [0,T]\}$ we get

$$\int_0^t \varphi_n(s)\,\mathrm{d}B^H_s \longrightarrow \int_0^t \varphi(s)\,\mathrm{d}B^H_s,\ n \to +\infty \quad \mathbb{P}\text{-a.s.}$$

and since the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent we also have

$$\int_0^t \varphi_n(s)\,\mathrm{d}B^H_s \longrightarrow \int_0^t \varphi(s)\,\mathrm{d}B^H_s,\ n \to +\infty \quad \widetilde{\mathbb{P}}\text{-a.s.} \tag{5.5}$$

As $\mathcal{H} \hookrightarrow L^1([0,T];\mathcal{L}(\mathbb{R}^n))$ ([3], Remark 2.1) we have

$$\|\varphi_n - \varphi\|_{L^1([0,T];\mathcal{L}(\mathbb{R}^n))} \longrightarrow 0,\ n \to +\infty.$$

It follows that

$$\left\| \int_0^t \varphi_n(s)u(s)\,\mathrm{d}s - \int_0^t \varphi(s)u(s)\,\mathrm{d}s \right\| = \left\| \int_0^t (\varphi_n(s) - \varphi(s))u(s)\,\mathrm{d}s \right\| \tag{5.6}$$
$$\leqslant \|u\|_{L^\infty([0,T];\mathbb{R}^n)} \|\varphi_n - \varphi\|_{L^1([0,T];\mathcal{L}(\mathbb{R}^n))} \longrightarrow 0,\ n \to +\infty \quad \widetilde{\mathbb{P}}\text{-a.s.}$$

Our statement follows from

$$\int_0^t \varphi_n(s)\,\mathrm{d}B^H_s = \int_0^t \varphi_n(s)\,\mathrm{d}\widetilde{B}^H_s + \int_0^t \varphi_n(s)u(s)\,\mathrm{d}s \quad \widetilde{\mathbb{P}}\text{-a.s.},\ t \in [0,T],$$

by letting $n \to +\infty$ and using (5.4), (5.5) and (5.6).

(ii) $H > \frac{1}{2}$. Take $\varphi \in L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))$, then there exist $\varphi_n \in \mathcal{E}$ such that

$$\|\varphi_n - \varphi\|_{L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n))} \longrightarrow 0,\ n \to +\infty.$$

However, $L^{1/H}([0,T];\mathcal{L}(\mathbb{R}^n)) \hookrightarrow \mathcal{H}$ according to (2.5) and hence

$$\|\varphi_n - \varphi\|_{\mathcal{H}} \longrightarrow 0,\ n \to +\infty.$$

Repeating the above arguments we find that (5.4), (5.5) and (5.6) are valid in the case $H > \frac{1}{2}$ as well and the proof is complete. □

Definition 5.3. Let $(\overline{\mathcal{F}}_t)_{t\geqslant 0}$ be an augmented filtration on a complete probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. By a *weak solution* to the equation

$$\overline{X}_t = \tilde{x} + \int_0^t b(s, \overline{X}_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\overline{B}_s^H \tag{5.7}$$

we mean a couple of $\overline{\mathcal{F}}_t$-adapted processes $(\overline{B}^H, \overline{X})$ with continuous trajectories on a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ such that $\overline{B}^H$ is an $\overline{\mathcal{F}}_t$-fractional Brownian motion and $\overline{X}$ and $\overline{B}^H$ satisfy the equation (5.7) for all $t \in [0,T]$ $\overline{\mathbb{P}}$-a.s.

We prove the existence of a weak solution to the equation (5.1).

Theorem 5.4. *Let $b_1, b_2\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $\sigma\colon [0,T] \to \mathcal{L}(\mathbb{R}^n)$ be Borel mappings such that $b = b_1 + b_2$ on $[0,T]\times\mathbb{R}^n$ and assume that $\sigma(t)$ is regular for all $t\in[0,T]$. Suppose that*

$$\exists K_b > 0\ \forall t\in[0,T]\ \forall x\in\mathbb{R}^n \quad \|b_1(t,x)\| \leqslant K_b(1+\|x\|),$$

and let there exist a solution Y to the equation (3.1)*. Set $u(t) = \sigma^{-1}(t) b_2(t, Y_t)$, $t \in [0,T]$. Assume that $u \in L^\infty([0,T];\mathbb{R}^n)$ $\mathbb{P}$-almost surely and either $H < \frac{1}{2}$, $\sigma \in \mathcal{C}^{\delta^*}([0,T];\mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$ and*

$$\exists K > 0\ \forall t\in[0,T]\ \forall x\in\mathbb{R}^n \quad \|\sigma^{-1}(t)\, b_2(t,x)\| \leqslant K(1+\|x\|),$$

or $H > \frac{1}{2}$, $\sigma \in L^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$ and

$$\exists \alpha \in \Big(1 - \frac{1}{2H}, 1\Big)\ \exists\beta\in\Big(H - \frac{1}{2}, 1\Big)\ \exists C > 0\ \forall s,t\in[0,T]\ \forall x,y\in\mathbb{R}^n$$
$$\|\sigma^{-1}(t)\, b_2(t,x) - \sigma^{-1}(s)\, b_2(s,y)\| \leqslant C(\|x-y\|^\alpha + |t-s|^\beta).$$

Then there exists a weak solution $(\overline{B}^H, \overline{X})$ to the equation (5.1)*. Moreover, the probability laws of $\overline{X}$ and Y are equivalent.*

P r o o f. Let $\widetilde{B}^H$ be the process defined in Theorem 4.2 (case $H < \frac{1}{2}$) or 4.3 (case $H > \frac{1}{2}$) and $Y = \{Y_t, t\in[0,T]\}$ be a version of solution to (3.1) with Hölder continuous trajectories of order δ, $0 < \delta < H$. We show that the couple of $\mathcal{F}_t^{B^H}$-adapted processes $(\widetilde{B}^H, Y)$ is a weak solution to (5.1) on the probability space $(\Omega, \mathcal{F}, \widetilde{\mathbb{P}})$, where $\widetilde{\mathbb{P}}$ is the probability measure defined by the density

$$\xi_T = \exp\left\{\int_0^T v_s^{\mathrm{T}}\,\mathrm{d}W_s - \frac{1}{2}\int_0^T \|v_s\|^2\,\mathrm{d}s\right\}$$

with respect to $\mathbb{P}$. The process $\widetilde{B}^H$ is a $\mathcal{F}_t^{B^H}$-fractional Brownian motion so it is sufficient to show that the process $(\widetilde{B}^H, Y)$ satisfies the equation (5.1) for all $t \in [0, T]$ $\widetilde{\mathbb{P}}$-a.s. Fix $t \in [0, T]$. Using Propositon 5.1 and the fact that Y is a solution to the equation (3.1) we have

$$\begin{aligned} Y_t = \tilde{x} + \int_0^t b_1(s, Y_s)\,\mathrm{d}s + Z_t &= \tilde{x} + \int_0^t b_1(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H \\ &= \tilde{x} + \int_0^t b_1(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\widetilde{B}_s^H + \int_0^t \sigma(s)\,\sigma^{-1}(s)\,b_2(s, Y_s)\,\mathrm{d}s \\ &= \tilde{x} + \int_0^t b_1(s, Y_s)\,\mathrm{d}s + \int_0^t b_2(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\widetilde{B}_s^H \\ &= \tilde{x} + \int_0^t b(s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}\widetilde{B}_s^H \quad \widetilde{\mathbb{P}}\text{-a.s.,} \end{aligned}$$

therefore $(\widetilde{B}^H, Y)$ satisfies the equation (5.1) for all $t \in [0, T]$ $\widetilde{\mathbb{P}}$-a.s. So $(\overline{B}^H, \overline{X}) := (\widetilde{B}^H, Y)$ is a weak solution to (5.1). The equivalence of the laws of $\overline{X}$ and Y may be proved in a similar way as in the Wiener case $H = \frac{1}{2}$. □

6. Equation of the stochastic oscillator

The last section is devoted to an example of a stochastic nonlinear oscillator driven by a fractional Brownian motion. Consider the n-dimensional stochastic differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x_t + F\Big(t, x_t, \frac{\mathrm{d}}{\mathrm{d}t} x_t\Big) = \overline{\sigma}(t) \frac{\mathrm{d}}{\mathrm{d}t} \overline{B}_t^H,$$

which can be rewritten as

$$\begin{aligned} x_t &= x_0 + \int_0^t v_s\,\mathrm{d}s, \\ v_t &= v_0 - \int_0^t F(t, x_s, v_s)\,\mathrm{d}s + \int_0^t \overline{\sigma}(s)\,\mathrm{d}\overline{B}_s^H. \end{aligned} \tag{6.1}$$

Moreover, consider the linear equations

$$\begin{aligned} y_t &= x_0 + \int_0^t w_s\,\mathrm{d}s \\ w_t &= v_0 + \int_0^t \overline{\sigma}(s)\,\mathrm{d}\overline{B}_s^H. \end{aligned} \tag{6.2}$$

For $t \in [0,T]$ and $y = (x,v)^T \in \mathbb{R}^{2n}$ denote

$$X_t := \begin{pmatrix} x_t \\ v_t \end{pmatrix}, \quad Y_t := \begin{pmatrix} y_t \\ w_t \end{pmatrix},$$
$$b_1(y) := \begin{pmatrix} v \\ 0 \end{pmatrix},$$
$$b_2(t,y) := \begin{pmatrix} 0 \\ -F(t,y) \end{pmatrix},$$
$$y_0 := \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

and

$$\sigma(t) := \begin{pmatrix} 0 & 0 \\ 0 & \overline{\sigma}(t) \end{pmatrix},$$

$\sigma(t)$ being a $2n \times 2n$ matrix.

Then the equations (6.1) and (6.2) can be rewritten in the matrix form

$$X_t = y_0 + \int_0^t (b_1(X_s) + b_2(s,X_s))\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H \tag{6.3}$$

and

$$Y_t = y_0 + \int_0^t b_1(Y_s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}B_s^H, \tag{6.4}$$

where $\{B_t^H, 0 \leqslant t \leqslant T\}$ is a $2n$-dimensional $\mathcal{F}_t^{B^H}$-fractional Brownian motion whose second n components are the components of $\overline{B}^H$. Suppose that the matrix $\overline{\sigma}(t)$ is regular for all $t \in [0,T]$. Let

$$\Sigma(t) = \begin{pmatrix} 0 & 0 \\ 0 & \overline{\sigma}^{-1}(t) \end{pmatrix}, \quad t \in [0,T],$$

denote the $2n \times 2n$ matrix with the property

$$\sigma(t)\Sigma(t) = \Sigma(t)\sigma(t) = \begin{pmatrix} 0 & 0 \\ 0 & I_{n\times n} \end{pmatrix}, \quad t \in [0,T],$$

where $I_{n\times n}$ is the $n \times n$ identity matrix.

If we replace the inverse $\sigma^{-1}(t)$ by the matrix $\Sigma(t)$ all statements in the previous sections hold true.

Suppose that $\overline{\sigma}$ is a Borel function and

- either $H < \frac{1}{2}$ and $\overline{\sigma} \in \mathcal{C}^{\delta^*}([0,T]; \mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$,

- or $H > \frac{1}{2}$ and $\overline{\sigma} \in L^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$.

The function $b_1\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$; $y = (x,v)^{\mathrm{T}} \mapsto (v,0)^{\mathrm{T}}$ is Lipschitz (consequently b_1 satisfies the condition (3.7)). Thus there exists a unique solution to the equation (6.4) (cf. Proposition 3.5) and it has a Hölder continuous version of order γ, $0 < \gamma < H$ (Theorem 3.6).

Assume that the trajectories of the process $\{\overline{\sigma}^{-1}(t)F(t,Y_t), t \in [0,T]\}$ are in $L^\infty([0,T];\mathbb{R}^n)$ and suppose moreover that

- either $H < \frac{1}{2}$ and $\exists\, K > 0\ \forall\, t \in [0,T]\ \forall\, y \in \mathbb{R}^{2n}\ \ \|\overline{\sigma}^{-1}(t)F(t,y)\| \leqslant K(1 + \|y\|)$,
- or $H > \frac{1}{2}$ and $\exists\, \alpha \in (1 - \frac{1}{2}H^{-1}, 1)\ \exists\, \beta \in (H - \frac{1}{2}, 1)\ \exists\, C > 0\ \forall\, s,t \in [0,T]\ \forall\, y_1, y_2 \in \mathbb{R}^{2n}$

$$\|\overline{\sigma}^{-1}(t)F(t,y_1) - \overline{\sigma}^{-1}(s)F(t,y_2)\| \leqslant C(\|y_1 - y_2\|^\alpha + |t-s|^\beta).$$

Then the assumptions of Theorem 5.4 on the map $(t,y) \mapsto \Sigma(t)\, b_2(t,y)$, $t \in [0,T]$, $y \in \mathbb{R}^{2n}$ are satisfied because

$$\Sigma(t)\, b_2(t,y) = \begin{pmatrix} 0 \\ -\overline{\sigma}(t)F(t,y) \end{pmatrix},$$

hence the equation (6.3) has a weak solution and so the equation (6.1) has a weak solution.

In the remaining part of this section the equivalence of the laws of weak solutions is studied for $\overline{\sigma}(t) \equiv \overline{\sigma}$ independent of t. Under the above assumptions we prove that the laws of the solutions $(x_t, v_t)^{\mathrm{T}}$ are equivalent to the Lebesgue measure on $\mathbb{R}^{2n}$ for each $t > 0$, $H \in (0,1) \setminus \{\frac{1}{2}\}$ and $(x_0, v_0)^{\mathrm{T}} \in \mathbb{R}^{2n}$. We show that the covariance matrix Q_T of the random variable Y_T ($\{Y_t, t \in [0,T]\}$ is a solution to the equation (6.4) which is Gaussian) is positive definite therefore the law of Y_T has a (positive) density with respect to the Lebesgue measure on $\mathbb{R}^{2n}$. Hence, obviously, the probability law of Y_t for $t \in [0,T]$ is equivalent to the Lebesgue measure on $\mathbb{R}^{2n}$.

Using the above notation the equation (6.4) can be rewritten as

$$Y_t = y_0 + \int_0^t AY_s \,\mathrm{d}s + \int_0^t \sigma \,\mathrm{d}B_s^H, \tag{6.5}$$

where

$$A = \begin{pmatrix} 0 & I_{n\times n} \\ 0 & 0 \end{pmatrix}$$

is a $2n \times 2n$ matrix. Then the solution is given by the formula (cf. [8], [7])

$$Y_t = \exp\{At\}y_0 + \int_0^t \exp\{A(t-s)\}\sigma \,\mathrm{d}B_s^H = \exp\{At\}y_0 + \int_0^t \begin{pmatrix} 0 & (t-s)\overline{\sigma} \\ 0 & \overline{\sigma} \end{pmatrix} \mathrm{d}B_s^H,$$

since

$$\exp\{At\} = \begin{pmatrix} I_{n\times n} & tI_{n\times n} \\ 0 & I_{n\times n} \end{pmatrix}.$$

So the law of Y_T is Gaussian $N(\exp\{AT\}y_0, Q_T)$. The computation of Q_T is divided into two cases.

Case $H > \frac{1}{2}$. The covariance matrix Q_T has the form (cf. [7])

$$Q_T = H(2H-1)\int_0^T\int_0^T \exp\{As\}\sigma\sigma^{\mathrm{T}}\exp\{A^{\mathrm{T}}r\}|r-s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s.$$

Using the identities

$$\int_0^T\int_0^T rs|r-s|^{2H-2}\,\mathrm{d}r,\ \mathrm{d}s = T^{2H+2}\frac{1}{2H(2H-1)(H+1)},$$
$$\int_0^T\int_0^T r|r-s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s = T^{2H+1}\frac{1}{2H(2H-1)},$$
$$\int_0^T\int_0^T |r-s|^{2H-2}\,\mathrm{d}r\,\mathrm{d}s = T^{2H}\frac{1}{H(2H-1)}$$

we get

$$Q_T = T^{2H}\begin{pmatrix} \dfrac{T^2}{2(H+1)}\overline{\sigma\sigma}^{\mathrm{T}} & \dfrac{T}{2}\overline{\sigma\sigma}^{\mathrm{T}} \\ \dfrac{T}{2}\overline{\sigma\sigma}^{\mathrm{T}} & \overline{\sigma\sigma}^{\mathrm{T}} \end{pmatrix}. \tag{6.6}$$

Case $H < \frac{1}{2}$. It is sufficient to compute $\mathbb{E}\,[\int_0^T (T-s)\,\mathrm{d}(B_s^H)^i]^2$ and $\mathbb{E}\,[(B_T^H)^i \times \int_0^T (T-s)\,\mathrm{d}(B_s^H)^j]$ for $i,j = 1,\dots,2n$. In the norm of the space $\mathcal{H}$ the function $f(s) = T-s,\ s\in[0,T]$, may be approximated by

$$f_l(s) = \sum_{k=0}^{l-1}\frac{l-k}{l}TI_{[\frac{k}{l}T,\frac{k+1}{l}T)}(s),\ s\in[0,T],\ l\in\mathbb{N},$$

so we have

$$\mathbb{E}\left[\int_0^T f_l(s)\,\mathrm{d}(B_i^H)_s\right]^2 = \left(\frac{T}{l}\right)^{2H+2}\sum_{k=1}^{l}k^{2H+1}$$
$$\longrightarrow T^{2H+2}\frac{1}{2H+2},\ l\to+\infty,\ i=1,\dots,2n,$$

and

$$\mathbb{E}\left[(B_T^H)^i\int_0^T f_l(s)\,\mathrm{d}(B_s^H)^j\right] = \frac{1}{2}T^{2H+1}\Big[1+\frac{1}{l}\Big]$$
$$\longrightarrow \frac{1}{2}T^{2H+1},\ l\to+\infty,\ i,j=1,\dots,2n,$$

therefore

$$\mathbb{E}\left[\int_0^T (T-s)\,\mathrm{d}(B_s^H)^i\right]^2 = \frac{T^{2H+2}}{2H+2}, \quad \mathbb{E}\left[(B_T^H)^i\int_0^T (T-s)\,\mathrm{d}(B_s^H)^j\right] = \frac{1}{2}T^{2H+1},$$

for $i,j=1,\dots,2n$. It is easy to see that Q_T has the same form (6.6) as in the case $H>\frac{1}{2}$.

Now we prove that Q_T is positive definite. For $x=(x_1,x_2)^{\mathrm{T}}\in\mathbb{R}^{2n}$ we obtain

$$\langle Q_T x, x\rangle = T^{2H}[(ay_1+by_2)^{\mathrm{T}}(ay_1+by_2)+cy_2^{\mathrm{T}}y_2],$$

where

$$y_1=\overline{\sigma}^{\mathrm{T}}x_1,\ y_2=\overline{\sigma}^{\mathrm{T}}x_2$$

and

$$a=\frac{T}{\sqrt{2(H+1)}}>0,\ b=\sqrt{\frac{H+1}{2}}>0,\ c=\frac{1-H}{2}>0.$$

By the regularity of $\overline{\sigma}$, $(y_1,y_2)^{\mathrm{T}}\neq 0$ for $x\neq 0$, and therefore $\langle Q_T x,x\rangle>0$ for all $x\neq 0$.

The equivalence of the probability laws of the solutions to the equations (6.1) and (6.2) follows from Theorem 5.4, so it follows that the laws of the solutions $(x_t,v_t)^{\mathrm{T}}$ to (6.1) are equivalent to the Lebesgue measure on $\mathbb{R}^{2n}$ for each $t>0,\ H\in(0,1)\setminus\{\frac{1}{2}\}$ and $(x_0,v_0)^{\mathrm{T}}$.

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