A Note on Pole Assignment in Linear Singular Systems.

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Abstract. This paper is devoted to some aspects of pole assignment by state feedback to the non-square linear systems. Under the condition of weak regularizability necessary and sufficient conditions for pole assignment are established.

I. Introduction

Consider a linear, time-invariant system in the form

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t),\; t \geq 0, x(0) = x_0, \tag{1}$$

where $E, A \in \mathbb{R}^{q\times n}$, $B \in \mathbb{R}^{q\times m}$ with B of full column rank. The system (1) will be referred to as the triple (E, A, B). Applying the state feedback

$$u(t) \;=\; Fx(t) + v(t), \tag{2}$$

$F \in \mathbb{R}^{m\times n}$, to the system (1) gives the closed-loop system

$$E\dot{x}(t) \;=\; (A + BF)x(t) + Bv(t) \tag{3}$$

where $v(t)$ is a new external control input. The system (3) describes how we can change the behavior of the system (1) by the state feedback (2).

Taking the Laplace transform of (1) the system can be written in the form (see the details on matrix pencil in [1]) :

$$(sE - A)X(s) \;=\; BU(s) + Ex_0, \tag{4}$$

where $X(s), U(s)$ denote the Laplace transforms of $x(t), u(t)$, respectively.

The pole structure of the system (1) is defined by the zero structure of the pencil $sE-A$. Particularly, the finite zero structure of $sE - A$ is given by the invariant polynomials of $sE - A$, while the infinite zero structure is defined by the negative powers of s occurring in the Smith-McMillan from at infinity of $sE - A$ [3]. The problems of pole structure assignment (PSA) and pole assignment (PA), which is just a special case of PSA, are defined (see [3]) as follows:

Given a system (1), monic polynomials $\psi_1(s) \rhd \psi_2(s) \rhd \ldots \rhd \psi_r(s)$, ($\psi_i \rhd \psi_{i+1}$ means that ψ_{i+1} divides ψ_i) and integers $d_1 \geq d_2 \geq \ldots \geq d_{k_d}$, under what conditions there exists a matrix F in (2) such that the polynomials $\psi_i(s)$ and integers d_i will be the invariant polynomials and infinite zero orders of $sE-A-BF$.

The PA and PSA problems have been widely studied by many authors in the case of square systems. The seminal work in this direction belongs to Rosenbrock. In the work [6] he gives necessary and sufficient conditions for the PSA in the case of the explicit (E is square and invertible) and controllable systems. This result is generalized [7] to the explicit and uncontrollable system. A generalization to the square, implicit (E is singular) and controllable systems is given in [8]. Necessary and sufficient conditions for the PA and necessary conditions for the PSA in the case of square (regularizable) system are given in [3].

II. Background

As far as notation is concerned, mostly standard symbols are used, see [2, 3] for instance. If not, they are defined just before used for the first time. A square system (1) is called regularizable if there exists an F in (2) such that $\mathrm{rank}\,(sE - A - BF)$ is full.

1. *Feedback Canonical Form*

Under the action of the feedback group, which consists of quadruples (P, Q, G, F), where $P,\ Q,\ G$ are invertible matrices and F is an $m \times n$ matrix, each system (E, A, B) can be brought into the *feedback canonical form* (FCF) described by the below introduced types of blocks (see details on FCF in [4]), i.e.

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P(A + BF)Q, PBG) =: (E_C,\ A_C,\ B_C)$$

These operations bring the pencil $[sE - A \quad -B]$ into the form in which

$$\begin{aligned}(sE_C - A_C) := \mathrm{blockdiag} \quad & (\{sE_{\epsilon_i} - A_{\epsilon_i}\}_{i=1}^{k_\epsilon}\,; \{sI_{\sigma_i} - N_{\sigma_i}\}_{i=1}^{k_\sigma}\,; \{sE_{q_i} - A_{q_i}\}_{i=1}^{k_q}\,; \\ & \{sN_{p_i+1} - I_{p_i+1}\}_{i=1}^{k_p}\,; \{sI_{l_i} - A_{l_i}\}_{i=1}^{k_l}\,; \{sE_{\eta_i} - A_{\eta_i}\}_{i=1}^{k_\eta})\end{aligned}$$

where $E_{\epsilon_i} := [I_{\epsilon_i}\ 0],\ A_{\epsilon_i} := [0\ I_{\epsilon_i}] \in \mathbb{R}^{\epsilon_i \times (\epsilon_i+1)}, E_{q_i} := [I_{q_i}\ 0]^T,\ A_{q_i} := [0\ I_{q_i}]^T \in \mathbb{R}^{(q_i+1)\times q_i},\ E_{\eta_i} := [I_{\eta_i}\ 0]^T,\ A_{\eta_i} := [0\ I_{\eta_i}]^T \in \mathbb{R}^{(\eta_i+1)\times \eta_i}$, N_j is nilpotent matrix with the index of nilpotency j. The quantities describing the corresponding blocks are called nonproper indices, $\epsilon_1 \geq \ldots \geq \epsilon_{k_\epsilon} \geq 0$; proper indices, $\sigma_1 \geq \ldots \geq \sigma_{k_\sigma} > 0$; almost proper indices, $q_1 \geq \ldots \geq\ q_{k_q} \geq 0$; almost nonproper indices, $p_1 \geq \ldots \geq p_{k_p} > 0$; the fixed invariant polynomials $\alpha_i(s) = s^{l_i} + a_{il_i}s^{l_i-1} + \cdots + a_{i1}s + a_{i0}$, (they represent the hidden finite modes of the system) $\alpha_1 \triangleright \alpha_2 \triangleright \cdots \triangleright \alpha_{k_l}$; row minimal indices of $[sE_C - A_C \ \ - B_C]$, $\eta_1 \geq \ldots \geq \eta_{k_\eta} \geq 0$.

The matrix B_C is of the form $B_C \ := \ \mathrm{blockdiag}\,\{0, B_\sigma, B_q, 0, 0, 0\}$, where $B_\sigma \ := \mathrm{blockdiag}$ $\left\{[0\cdots01]^T \in \mathbb{R}^{\sigma_i}\right\}$, $B_q := \mathrm{blockdiag}\left\{[0\cdots01]^T \in \mathbb{R}^{q_i+1}\right\}$.

2. *Normal External Description*

Definition 1. *The matrices* $N(s), D(s)$ *are said to form NED of the system (1) if they satisfy the following conditions:*

- $[sE{-}A \ \ -B]\begin{bmatrix} N(s) \\ D(s)\end{bmatrix} = 0$ *where* $\begin{bmatrix} N(s) \\ D(s)\end{bmatrix}$ *forms a minimal polynomial basis for* $\mathrm{Ker}[sE{-}A \ \ -B]$.
- $N(s)$ *forms a minimal polynomial basis for* $\mathrm{Ker}\Pi[sE{-}A]$ *where* Π *is a maximal annihilator of* B

It is assumed that the column degrees of $\left[N^T(s)\ D^T(s)\right]^T$ are non-increasingly ordered.

For the square and regularizable systems the column degrees $c_i := \deg_{ci}\left[N^T(s)\ D^T(s)\right]^T$ are the controllability indices of the system. When $\deg_{\mathrm{ci}}N(s) > \deg_{ci}D(s)$ ($\deg_{ci}N(s) \leq \deg_{ci}D(s)$) such degrees are called by nonproper (proper) controllability indices, $i = 1, 2, \ldots, k_\epsilon + k_\sigma$. A square regularizable system is called controllable iff $\sum_i c_i = \mathrm{rank}E$ [5].

The action of the state feedback upon the system (E_C, A_C, B_C) leads to

$$[sE_C{-}A_C{-}B_CF \quad -B_C]\begin{bmatrix} N_C(s) \\ D_C(s){-}FN_C(s)\end{bmatrix} = 0$$

It should be noted that the NED of the system reflects only the information on the $\epsilon_i,\ \sigma_i$- blocks and does not depend on the quantities $q_i,\ p_i,\ \eta_i$ and polynomials $\alpha_i(s)$. These quantities represent the hidden part of the system. More particularly, the zeros of $\alpha_i(s)$ are the finite uncontrollable modes of (1) while $q_i,\ p_i$ give the orders of uncontrollable mode at infinity. To remedy this situation the matrix B_C is extended in such a way that the hidden part of the original system will appear in the NED. The system $(E_C,\ A_C,\ [B_C\ \overline{B}_C])$ modified by this trick is called the extended system of (1) (see for details [3]). The NED of the extended system is

$$\left[N_E^T(s),\ D_E^T(s)\right]^T := \left[\text{diag}\,\{N_C^T(s), \overline{N}_C^T(s)\},\ \text{diag}\,\{D_C^T(s), \overline{D}_C^T(s)\}\right]^T$$

where $\overline{N}_C(s),\ \overline{D}_C(s)$ form an NED of the hidden part of the system.

The matrix describing the modification of the system which can be done by a state feedback is in the form

$$D_{EF}(s) := \begin{bmatrix} D_C(s) - FN_C(s) & -F\overline{N}_C(s) \\ 0 & \overline{D}_C(s) \end{bmatrix} = \begin{bmatrix} D_{11}(s) & S_\sigma(s)+D_{12}(s) & D_{13}(s) & D_{14}(s) & D_{15}(s) & D_{16}(s) \\ D_{21}(s) & D_{22}(s) & S_q+D_{23}(s) & D_{24}(s) & D_{25}(s) & D_{26}(s) \\ 0 & 0 & -I_{k_q} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{k_p} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_\alpha(s) & 0 \\ 0 & 0 & 0 & 0 & 0 & S_\eta(s) \end{bmatrix}$$

where

$S_\sigma := \text{diag}\,\{s^{\sigma_i}\}_{i=1}^{k_\sigma},\ S_q := \text{diag}\,\{s^{q_i}\}_{i=1}^{k_q},\ S_\alpha := \text{diag}\,\{\alpha_i(s)\}_{i=1}^{k_l},\ S_\eta := \text{blockdiag}\left\{\begin{bmatrix} s^{\eta_i} \\ -1 \end{bmatrix}\right\}_{i=1}^{k_\eta}$

and $D_{ij}(s)$ are arbitrary matrices satisfying the conditions

$$\deg_{ci}\begin{bmatrix} D_{1j}(s) \\ D_{2j}(s) \end{bmatrix} \leq \deg_{ci} N_j(s), \quad i = 1, 2, \ldots, j = 1, 2, \ldots, 6$$

To deal with finite and infinite zeros of the pencil in a unified way, the conformal mapping $s = (1 + aw)/w$, where $a \in \mathbb{R},\ a \neq 0$ and is not a pole of (1), is used. Then the finite zero structure of the pencil $w\tilde{E}_C - \tilde{A}_C$ at the point $w = 0$ determines the infinite zero structure of $sE_C - A_C$, where $\tilde{E}_C$ and $\tilde{A}_C$ are the w-analogues of E_C and A_C that are derived by applying the conformal mapping upon the system (1) (see [3] for detail). It should be noted that many properties of the system can be stated in the terms of the matrix $D_{EF}(s)$ - see [2, 3].

Proposition 1. *The following holds:*

- *The system (1) is regularizable by state feedback F in (2) if and only if $k_\epsilon = k_q$ and $k_\eta = 0$.*
- *The non unit invariant factors of both $sE_C - A_C - B_CF$ and $D_{EF}(s)$ coincide for any F.*
- *The infinite zero orders of $sE_C - A_C - B_CF$ and $D_{EF}(s)\text{diag}\{s^{-c_i}\}$ are the same.*

This proposition allows us to reformulate the problems of studying the structure of $sE_C - A_C - B_CF$ in terms of the structure $D_{EF}(s)$.

Proposition 2. *Given a regularizable system (1) ($k_\epsilon = k_q$ and $k_\eta = 0$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$, then there exists an F in (2) such that* $\det(sE - A - BF) = \psi(s)$ *and the sum of the infinite zero orders of $sE - A - BF$ equals d if and only if the conditions (5) - (7) (and (8) if $k_\epsilon = 0$) are satisfied.*

$$\deg \psi(s) + d = \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i \tag{5}$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)...\alpha_{k_l}(s) \tag{6}$$

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i \ . \tag{7}$$

$$\deg \psi(s) = \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_l} l_i \tag{8}$$

III. Main results

A natural question arising here is under which conditions there exists a state feedback (2) yielding the full rank pencil $sE- A-BF$.

Proposition 3. *There exists an F in (2) such that the pencil $sE- A-BF$ is of (a) full row rank if and only if $k_\epsilon \geq k_q$ and $k_\eta = 0$, or (b) full column rank if and only if $k_q \geq k_\epsilon$.*

Proof. For pencils having more columns than rows it easily follows, from the form of $D_{EF}(s)$, that $sE - A - BF$ is of full row rank if and only if (a) holds. In the case there are more rows than columns and $\operatorname{rank} sE - A - BF$ is full, say n,

$$\operatorname{rank}(sE - A - BF) \leq \operatorname{rank}\Pi(sE - A) + \operatorname{rank} B$$

where Π is a maximal annihilator of B. This condition is equivalent to

$$n - \operatorname{rank}(sE - A - BF) \geq n - \operatorname{rank}\Pi(sE - A) - \operatorname{rank} B \tag{9}$$

Then, as $n - \operatorname{rank}\Pi(s - A)$ is the number of the column minimal indices of $[sE - A - B]$, that is to say $k_\epsilon + k_\sigma$, and $\operatorname{rank} B = k_\sigma + k_q$, it follows from (9) that $0 \geq k_\epsilon + k_\sigma - k_\sigma - k_q$ and consequently follows (b).

Conversely, if (a), or (b), holds for a pencil $[sE–A \ \ –B]$, then it is always possible to find an F such that $sE - A - BF$ will be of full row or column rank. A construction of such an F is illustrated in the example below.

Examle 1. *Let*

$$[sE - A \quad - B] := \begin{bmatrix} s & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & s & -1 \end{bmatrix}$$

Defining $F = [1, 0, 0, 0]$, the pencil

$$sE - A - BF = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & s \end{bmatrix}$$

is clearly of full row rank.

Notice that if the conditions (a) and (b) of Proposition 3 are satisfied simultaneously then system is of full row and column rank, that is the condition on regularizability of the system follows.

The systems satisfying either (a) or (b) of Proposition 3 might be called *weakly (row or column) regularizable* since we cannot speak of the characteristic polynomial assignment but just of the full rank assignment, which implies that (at least) one of the principle (of order $\min\{q, n\}$) minors of $sE - A - BF$ will be a prescribed polynomial and the corresponding submatrix will have a prescribed sum of infinite zero orders.

If the systems are not weakly regularizable, we cannot speak about the pole assignment as it is not well defined. The following example shows this point.

Examle 2. *Let* $\epsilon_1 = \eta_1 = 1$ *and* $\sigma_1 = 3$*. Then the matrix* $D_{EF}(s)$ *is of the form*

$$D_{EF}(s) = \begin{bmatrix} \alpha_0 + \alpha_1 s & s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 & \gamma \\ 0 & 0 & s \\ 0 & 0 & -1 \end{bmatrix},$$

which shows that there is no F *resulting in* $D_{EF}(s)$ *(and hence* $sE - A - BF$*) nonsingular.*

Let $\psi(s)$ denote a principal minor of $D_{EF}(s)$ (or equivalently a principal minor of $sE-A-BF$ due to Proposition 1) and $\psi'(s)$ be a principal minor of $D_{\epsilon\sigma}(s) := \begin{bmatrix} D_{11}(s) & S_\sigma(s) + D_{12}(s) \\ D_{21}(s) & D_{22}(s) \end{bmatrix}$. Consider a weakly (row) regularizable system, i.e. $k_\epsilon > k_q$ and $k_\eta = 0$. Then the matrix $D_{EF}(s)$ can be made row reduced with the matrix $D_{\epsilon\sigma}(s)$ column reduced , see the procedure in the proof of Corollary 1 in [3]. Then the maximal sum of the degrees of $D_{\epsilon\sigma}(s)$ satisfy

$$\deg \psi'(s) \ \leq \ \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i \tag{10}$$

The infinite zero structure is given by the $q_i,\ p_i$- blocks as the rank deficiency of $\tilde{D}_{EF}(w)$ at $w = 0$ equals to $k_q + k_p$. So, the the following condition must also be satisfied

$$d \geq \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i \,. \tag{11}$$

The condition (6) is still necessary and if the conditions (10) and (11) are taken into account, we obtain

$$\deg \psi(s) + d \leq \sum_{i=1}^{k_q} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i \tag{12}$$

The equality need not be satisfied at this case. When the system is weakly (column) regularizable ($k_q \geq k_\epsilon$) a similar condition to (12) can be derived and completed by the following inequalities.

$$\deg \psi'(s) \ \leq \ \sum_{i=1}^{k_\epsilon} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i \tag{13}$$

$$d \geq \sum_{i=k_q-k_\epsilon+1}^{k_q} q_i + \sum_{i=1}^{k_p} p_i \tag{14}$$

The conditions (10) - (14) give us just necessary conditions as far as the degree of principal minors are concerned. The following example shows that they are not sufficient in general.

Examle 3. *Let* $\epsilon_1 = 0$ *and* $\sigma_1 = 3$*. Then the matrix* $D_{EF}(s)$ *is of the form*

$$D_{EF}(s) = \begin{bmatrix} \alpha_0 & s^3 + \beta_2 s^2 + \beta_1 s + \beta_0 \end{bmatrix}$$

Then the degrees of principal minors are either 0 *or* 3*, but never* 1 *or* 2 *even if they satisfy (12).*

It can noticed that the maximal number of zeros assignable by state feedback (2), when (E, A, B) is a weakly regularizable, is not increased by additional blocks of nonproper or almost proper indices and row minimal indices of $[sE - A \ \ -B]$. There always exists an F such that

$$\deg \psi(s) + d = \sum_{i=1}^{k_r} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i$$

where $r := \min\{k_\epsilon, k_q\}$. All the above observations are summarized in the following theorem.

Theorem 1. *Given a weakly regularizable system (1) (i.e. $k_\epsilon \geq k_q$ and $k_\eta = 0$ or $k_q \geq k_\epsilon$), a monic polynomial $\psi(s)$, and an integer $d \geq 0$, then there exists an F in (2) such that a principal minor of $sE - A - BF$ is equal to $\psi(s)$ and the sum of the infinite zero orders of the corresponding submatrix equals d if and only if the following conditions are satisfied:*

$$\deg \psi(s) + d = \sum_{i=1}^{k_r} \epsilon_i + \sum_{i=1}^{k_\sigma} \sigma_i + \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i + \sum_{i=1}^{k_l} l_i \tag{15}$$

$$\psi(s) \triangleright \alpha_1(s)\alpha_2(s)...\alpha_{k_l}(s) \tag{16}$$

$$d \geq \sum_{i=1}^{k_r} q_i + \sum_{i=1}^{k_p} p_i \ . \tag{17}$$

where $r := \min\{k_\epsilon, k_q\}$.

IV. Conclusions

The problem of pole (structure) assignment for the non-square systems (1) is considered. If the conditions (a) and (b) of Proposition 3 are satisfied then the problem is well defined. For such systems, which we call weakly regularizable, necessary and sufficient conditions are established in Theorem 1. The conditions are stated in terms of principal minors rather than in terms of invariant factors that are used in [3], where necessary conditions (which are also sufficient only in various particular cases) are established. Theorem 1 gives additional insight in the picture of modifying the poles of a singular system by state feedback.

Acknowledgement

This work is co-financed from the student grant SGS 2010/7821 Interactive Mechatronics Systems Using the Cybernetics Principles.

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