NONLINEAR FUNCTIONALS IN STOCHASTIC PROGRAMMING; A NOTE ON STABILITY AND EMPIRICAL ESTIMATES

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Abstract: Economic processes are very often influenced simultaneously by a decision parameter (that can be chosen according to conditions) and a random factor. Since mostly it is necessary to determine the decision parameter without knowledge of a random element realization, a deterministic optimization problem has to be defined. This deterministic problem can usually depend on an "underlying" probability measure corresponding to the random element. The investigation of such types problems often belong to the stochastic programming field. The great attention has been focus on the problems in which objective functions depend "linearly" on the probability measure. This note is focus on the cases when the above mentioned assumption is not fulfilled; see e.g. Markowitz functionals or some risk measures. We try to cover static (one stage problems) as well as dynamic approaches (multistage stochastic programming case).
Keywords: Optimization problems with random element, one stage stochastic programming problems, multistage stochastic programming problems, linear and nonlinear functionals, risk measures.

1 Introduction

Optimization problems depending on a probability measure correspond to many applications. They can be often investigated in the framework of the stochastic programming theory; in one–stage as well as in multistage settings. Objective functions are there mostly a linear "functional" of the "underlying" probability measure. However, it happens relatively often that this assumption is not fulfilled (see e.g. [**?**], [6], [9]). In this note, we focus on this nonlinear case. First, we recall some corresponding one–stage problems, furthermore we try to generalize the definition and corresponding results to the multistage case.

2 One–Stage Stochastic Programming Problems

We start with a "classical" one–stage problem. To this end let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi(:= \xi(\omega) = [\xi_1(\omega), \ldots, \xi_s(\omega)])$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F(z), z \in R^s)$ the distribution function of ξ; $P_F, Z(:= Z_F)$ the probability measure and support corresponding to F. Let, moreover, $g_0(:= g_0(x, z))$ be a real–valued (say continuous) function defined on $R^n \times R^s$; $X \subset R^n$ be a nonempty "deterministic" set. If the symbol E_F denotes the operator of mathematical expectation corresponding to F, then many economic applications (considering with respect only to one time point) can be introduced as the problem:

Find

$$\varphi(F) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X\}. \tag{1}$$

Evidently, the objective function in (1) depends linearly on the probability measure P_F. However, some applications correspond to optimization problems in which this assumption is not fulfilled. Let us consider the following very simple portfolio problem.:

Find

$$\max \sum_{k=1}^{n} \xi_k x_k \quad \text{s.t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n, \quad s = n, \tag{2}$$

where x_k is a fraction of the unit wealth invested in the asset k, ξ_k denotes the return of the asset $k \in \{1, 2, \ldots n\}$. If $\xi_k, k = 1, \ldots, n$ are known, then (2) is a linear programming problem. However, $\xi_k, k = 1, \ldots, n$ are mostly random variables with unknown realizations in a time decision. If we denote

$$\mu_k = \mathsf{E}_F \xi_k, \quad c_{k,j} = \mathsf{E}_F(\xi_k - \mu_k)(\xi_j - \mu_j), \quad k, j = 1, \ldots n, \tag{3}$$

then it is reasonable to set to the portfolio selection two–objective optimization problem:

Find

$$\max \sum_{k=1}^{n} \mu_k x_k, \quad \min \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,j} x_j \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n, \tag{4}$$

where $\sum\limits_{k=1}^{n} \sum\limits_{j=1}^{n} x_k c_{k,j} x_j$ can be considered as a risk measure.

Evidently, there exists only rarely a possibility to find an optimal solution simultaneously with respect to the both criteria. Markowitz suggested (see e.g. [2]) to replace the problem (4) by one–criterion optimization problem of the form:

Find

$$\varphi^M(F) = \max\{\sum_{k=1}^{n} \mu_k x_k - K \sum_{k=1}^{n} \sum_{j=1}^{n} x_k c_{k,j} x_j\} \quad \text{s. t.} \quad \sum_{k=1}^{n} x_k \le 1, \quad x_k \ge 0, \quad k = 1, \ldots, n, \tag{5}$$

where $K \ge 0$ is a constant.

Konno and Yamazaki introduced in [7] another risk measure $w(x)$ by

$$w(x) = \mathsf{E}_F | \sum_{k=1}^{n} \xi_k x_k - \mathsf{E}_F[\sum_{k=1}^{n} \xi_k x_k]|. \tag{6}$$

Some other suitable risk measures can be found e.g. in [9].

Evidently, $w(x)$ is a Lipschitz function of $\mathsf{E}_F[\sum\limits_{k=1}^{n} \xi_k x_k]$ and, consequently, the problem

Find

$$\max\{\lambda \sum_{k=1}^{n} \mu_k x_k - (1-\lambda) \mathsf{E}_F | \sum_{k=1}^{n} \xi_k x_k - \mathsf{E}_F[\sum_{k=1}^{n} \xi_k x_k]|, \quad \lambda \in \langle 0, 1 \rangle \tag{7}$$

can be covered by the more general problem:

Find

$$\varphi(F) := \bar{\varphi}(F) = \inf\{\mathsf{E}_F g_0^1(x, \xi, \mathsf{E}_F h(x, \xi)) | x \in X\}, \tag{8}$$

where $h(x, z) = (h_1(x, z), \ldots, h_{m_1}(x, z))$ is m_1–dimensional vector function defined on $R^n \times R^s$, $g_0^1(x, z, y)$ is a real–valued (say uniformly continuous) function defined on $R^n \times R^s \times R^{m_1}$.

3 Multistage Stochastic Programming Problems

Many real–life problems with a random factor, those are developing over time, can be treated by multistage stochastic techniques. To this end let random factors ξ^k and decisions x^k $k = 0, 1, \ldots$ follow the scheme:

$$\begin{aligned} & x^0 \longrightarrow \xi^0 \longrightarrow x^1(:= x^1(\xi^0, x^0)) \longrightarrow \xi^1 \longrightarrow x^2(:= x^2(\bar{\xi}^1, \bar{x}^1)) \longrightarrow \xi^2 \longrightarrow \ldots \quad \ldots \longrightarrow \\ & x^{M-1}(:= X^{M-1}(\bar{\xi}^{M-2}, \bar{x}^{M-2})) \longrightarrow \xi^{M-1} \longrightarrow x^M(:= x^M(\bar{\xi}^{M-1}, \bar{x}^{M-1})) \longrightarrow \xi^M \ldots, \end{aligned} \tag{9}$$

where $\bar{x}^k = [x^0, x^1, \ldots, x^k], \quad \bar{\xi}^k = [\xi^0, \xi^1, \ldots, \xi^k], \quad k = 0, 1, \ldots$

Evidently, it follows from the relation (9) that for every $k = 0, 1, \ldots$ the decision x^k can depend on $x^0, \ldots, x^{k-1}$ and $\xi^0, \ldots \xi^{k-1}$, however it can not depend on $x^{k+1}, \ldots$ and $\xi^k, \ldots$ We say that the decision has to be nonanticipative.

Considering the above mentioned situation with respect to a discrete time interval $\langle 0, M\rangle$ and supposing that the decision parameter can be determined with respect to the average of a corresponding objective function, we can set usually to the relation (9) a "classical" multistage ($M + 1$–stage) stochastic programming problem (for more details see e.g. [1] or [10]):

Find

$$\varphi_{\mathcal{F}}(M) = \inf\{\mathsf{E}_{F^{\xi^0}} g^0_{\mathcal{F}}(x^0, \xi^0) | x^0 \in \mathcal{K}^0\}, \tag{10}$$

where the function $g^0_{\mathcal{F}}(x^0, z^0)$ is defined recursively

$$\begin{aligned} g^k_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) \quad & = \quad \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}} g^{k+1}_{\mathcal{F}}(\bar{x}^{k+1}, \bar{\xi}^{k+1}) | x^{k+1} \in \mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k)\}, \\ k = 0, 1, \ldots, M-1, & \\ g^M_{\mathcal{F}}(\bar{x}^M, \bar{z}^M) \quad & := \quad g^M_0(\bar{x}^M, \bar{z}^M), \quad \mathcal{K}_0 := X^0. \end{aligned} \tag{11}$$

$\xi^j := \xi^j(\omega)$, $j = 0, 1, \ldots, M$ denotes an s–dimensional random vector defined on a probability space $(\Omega, \mathcal{S}, P)$; $F^{\xi^j}(z^j)$, $z^j \in R^s, j = 0, 1 \ldots, M$ the distribution function of the ξ^j and $F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})$, $z^k \in R^s, \bar{z}^{k-1} \in R^{(k-1)s}, k = 1, \ldots, M$ the conditional distribution function (ξ^k conditioned by $\bar{\xi}^{k-1}$); $P_{F^{\xi^j}},, P_{F^{\xi^{k+1}|\bar{\xi}^k}}$, $j = 0, 1, \ldots, M, k = 0, 1, \ldots,, M-1$ the corresponding probability measures; $Z^j := Z_{F^{\xi^j}} \subset R^s, j = 0, 1, \ldots, M$ the support of the probability measure $P_{F^{\xi^j}}$. Furthermore, the symbol $g^M_0(\bar{x}^M, \bar{z}^M)$ denotes a uniformly continuous function defined on $R^{n(M+1)} \times R^{s(M+1)}$; $X^0 \subset R^n$ is a nonempty compact set; the symbol $\mathcal{K}^{k+1}_{\mathcal{F}}(\bar{x}^k, \bar{z}^k) := \mathcal{K}^{k+1}_{F^{\xi^{k+1}|\bar{\xi}^k}}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$ denotes a multifunction mapping $R^{n(k+1)} \times R^{(k+1)}$ into the space of subsets of R^n. $\bar{\xi}^k(:= \bar{\xi}^k(\omega)) = [\xi^0, \ldots, \xi^k]$; $\bar{z}^k = [z^0, \ldots, z^k], z^j \in R^s$; $\bar{x}^k = [x^0, \ldots, x^k]$, $x^j \in R^n$; $\bar{Z}^k := \bar{Z}^k_{\mathcal{F}} = Z_{F^{\xi^0}} \times Z_{F^{\xi^1}} \ldots \times Z_{F^{\xi^k}}$, $j = 0, 1, \ldots, k$, $k = 0, 1, \ldots, M$. Symbols $\mathsf{E}_{F^{\xi^0}}$, $\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}}, k = 0, 1, \ldots, M-1$ denote the operators of mathematical expectation corresponding to $F^{\xi^0}, F^{\xi^{k+1}|\bar{\xi}^k = \bar{z}^k}, k = 0, \ldots, M-1, \quad \mathcal{F} = \{F^{\xi^0(z^0, F^{\xi^k|\bar{\xi}^{k-1}}(z^k|\bar{z}^{k-1})}, k = 1, \ldots, M\}$.

The problem (10) is a "classical" one–stage stochastic programming problem depending on the probability measure P_{ξ^0}, the problems (11) are for $k = 0, 1, \ldots, M$ parametric one–stage stochastic programming problems depending on the conditional probability measures $P_{F^{\xi^{k+1}|\bar{\xi}^k}}$. Simultaneously, the objective functions depend "linearly" on the above mentioned measures. However, this assumption is not fulfilled every time; see the former section for one–stage case. Now we try to generalize one–stage case to the multistage approach. To this end we assume:

i.1 There exist m_1–dimensional vector functions $\bar{h}^j(x^j, z^j) = (h_1^j(x^j, z^j), \ldots, h_{m_1}^j(x^j, z^j))$ defined on $R^n \times R^s$ and real–valued (say uniformly continuous) functions $\bar{g}_0^j(x^j, z^j, y^j)$ defined on $R^n \times R^s \times R^{m_1}, j = 0, 1, \ldots, M$ such that

$$g_0^M(\bar{x}^M, \bar{z}^M) := \sum_{j=0}^{M} \bar{g}_0^j(x^j, z^j, \mathsf{E}_{F^{\xi^j|\bar{\xi}^{j-1}=\bar{z}^{j-1}}} \bar{h}^j(x^j, \xi^j)). \tag{12}$$

The multistage problem (10), (11) then can be (according to (9)) replaced by the following problem with nonlinear objective functions:

Find

$$\bar{\varphi}_{\mathcal{F}}(M) = \inf\{\mathsf{E}_{F^{\xi^0}}[\bar{g}_0^0(x^0, \xi^0, \mathsf{E}_{F^{\xi^0}} \bar{h}^0(x^j, \xi^0)) + \bar{g}_{\mathcal{F}}^0(x^0, \xi^0)] | x^0 \in \mathcal{K}^0\}, \tag{13}$$

where the function $g_{\mathcal{F}}^0(x^0, z^0)$ is defined recursively

$$\begin{aligned}
\bar{g}_{\mathcal{F}}^k(\bar{x}^k, \bar{z}^k) &= \inf\{\mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}}[\bar{g}_0^{k+1}(x^{k+1}, \xi^{k+1}, \mathsf{E}_{F^{\xi^{k+1}|\bar{\xi}^k=\bar{z}^k}} \bar{h}^{k+1}(x^{k+1}, \xi^{k+1})) + \\
&\quad \bar{g}_{\mathcal{F}}^{k+1}(x^{k+1}, \xi^{k+1})] | x^{k+1} \in \mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k)\}, \\
k &= 0, 1, \ldots, M-2, \\
\bar{g}_{\mathcal{F}}^{M-1}(\bar{x}^{M-1}, \bar{z}^{M-1}) &:= \inf\{\mathsf{E}_{F^{\xi^M|\bar{\xi}^{M-1}}} \bar{g}_0^M(x^M, \xi^M, \mathsf{E}_{F^{\xi^M|\bar{\xi}^{M-1}=\bar{z}^{M-1}}}, \bar{h}^M(x^M, \xi^M)) | \\
&\quad x^M \in \mathcal{K}_{\mathcal{F}}^M(\bar{x}^{M-1}, \bar{z}^{M-1})\}, \quad \mathcal{K}^0 := X^0, \\
\bar{g}_{\mathcal{F}}^M(\bar{x}^M, \bar{z}^M) &:= \bar{g}_0^M(x^M, z^M, \mathsf{E}_{F^{\xi^M|\bar{\xi}^{M-1}=\bar{z}^{M-1}}} \bar{h}^M(x^M, \xi^M)).
\end{aligned} \tag{14}$$

4 Problem Analysis

Of course the investigation of the problems (10), (11) or (13), (14) is very complicated. The stability (w.r.t. probability measure space) and empirical estimates (of the problem (10), (11)) have been investigated e.g. in [3], [5]. To investigate the problems (13), (14) we recall corresponding results for one–stage case. To this end let $\mathcal{P}(R^s)$ denote the set of Borel probability measures on $R^s, s \geq 1$ and let $\mathcal{M}_1(R^s) = \{P \in \mathcal{P}(R^s) : \int_{R^s} \|z\|_s^1 P(dz) < \infty\}$, $\|\cdot\|_s^1$ denotes $\mathcal{L}_1$ norm in R^s. We introduce the assertion proven in [6], based on the approach employed in [4].

Proposition 1. [6] Let X be a compact set, G be an arbitrary s–dimensional distribution function. Let, moreover, P_F, $P_G \in \mathcal{M}_1(R^s)$. If

1. $g_0^1(x, z, y)$ is for $x \in X$, $z \in R^s$ a Lipschitz function of $y \in Y$ with a Lipschitz constant L^y; $Y = \{y \in R^{m_1} : y = h(x, z) \quad \text{for some} \quad x \in X, z \in R^s\}$,
2. for every $x \in X, y \in Y$ there exist finite mathematical expectations $\mathsf{E}_F g_0^1(x, \xi, \mathsf{E}_F h(x, \xi))$, $\mathsf{E}_F g_0^1(x, \xi, \mathsf{E}_G h(x, \xi))$, $\mathsf{E}_G g_0^1(x, \xi, \mathsf{G}_F h(x, \xi))$,
3. $h_i(x, z), i = 1, \ldots, m_1$ are for every $x \in X$ Lipschitz functions of z with the Lipschitz constants L_h^i (corresponding to $\mathcal{L}_1$ norm),
4. $g_0^1(x, z, y)$ is for every $x \in X, y \in R^{m_1}$ a Lipschitz function of $z \in R^s$ with the Lipschitz constant L^z (corresponding to $\mathcal{L}_1$ norm),

then there exist $\hat{C} > 0$ such that

$$|\bar{\varphi}(F) - \bar{\varphi}(G)| \leq \hat{C} \sum_{i=1}^{s} \int_{-\infty}^{\infty} |F_i(z_i) - G_i(z_i)| dz_i. \tag{15}$$

Evidently, the assertion of Proposition 1 can be employed for the investigation of empirical estimates of the problem (8) (for more details see [6]). There has been proven that convergence rates of the problems (1), (8) are (under the corresponding assumptions) the same. They can depend on the tails of one–dimensional marginals distribution functions.

To investigate the problem (13), (14) we introduce a system of the next assumptions:

i.2 There exists a random vector $\varepsilon^k := \varepsilon^k(\omega)), k = \ldots, -1, 0, 1, \ldots$ defined on $(\Omega, \mathcal{S}, P)$

- ξ^0, ε^k (defined on $(\Omega, \mathcal{S}, P), k = 1, 2, \ldots$ are stochastically independent,
- $\varepsilon^k, k = 0, 1 \ldots$ are identically distributed. (We denote the distribution function corresponding to ε^1 by the symbol F^{ε}),

i.3 there exists a Lipschitz vector (s–dimensional) function $H(z)$ defined on R^s such that (for sequence of s–dimensional random vectors $\{\xi^k\}_{k=-\infty}^{\infty}$ one of the following conditions is valid

- $\xi^k = \varepsilon^k H(\xi^{k-1}), k = \ldots - 1, 0, 1, \ldots,$
- ξ^k follows random sequence such that $\xi^k = \varepsilon^k + H(\xi^{k-1}), k = \ldots - 1, 0, 1, \ldots$

i.4 The multifunction $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k), k = 0, 1, \ldots, M-1$ do not depend on the system $\mathcal{F}$.

A similar system of the assumptions have been already employed in [5], [8].

Employing the proofs technique of the paper [6] we can (under some additional assumptions) obtained (for problems (13), (14)) very similar results to them for one–stage case. Evidently, to this end it is necessary to find out assumptions under which the functions $\bar{g}_{\mathcal{F}}^k(\bar{x}^k, \bar{z}^k), k = 0, \ldots, M$ are uniformly continuous and Lipschitz functions of z^k with the Lipschits constant not depending on $\bar{x}^k, \bar{z}^{k-1}$. Furthermore the constraint sets have to be compact sets $\mathcal{K}_{\mathcal{F}}^{k+1}(\bar{x}^k, \bar{z}^k)$, $k = 0, 1, \ldots, M-1$ have to be compact. To this end the approach of the papers [3], [5] can be employed. However, more detailed investigation is over the possibility of this note.

5 Conclusions

In the note we have tried to introduce some types of optimization problems in which objective functions are not linear "functionals" of the "underlying" probability measures. Furthermore, we tried to give a brief sketch of their stability and empirical estimates investigation. According to this, it is possible to see that the results in the case of linear dependence and some nonlinear case are the same for corresponding one–stage problems. Moreover, if the random element follows autoregressive random sequences (in the multistage case) we can obtain also very similar results for the multistage case.

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