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MEASURING OF SECOND–ORDER STOCHASTIC DOMINANCE PORTFOLIO EFFICIENCY

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In this paper, we deal with second-order stochastic dominance (SSD) portfolio efficiency with respect to all portfolios that can be created from a considered set of assets. Assuming scenario approach for distribution of returns several SSD portfolio efficiency tests were proposed. We introduce a δ-SSD portfolio efficiency approach and we analyze the stability of SSD portfolio efficiency and δ-SSD portfolio efficiency classification with respect to changes in scenarios of returns. We propose new SSD and δ-SSD portfolio efficiency measures as measures of the stability. We derive a non-linear and mixed-integer non-linear programs for evaluating these measures. Contrary to all existing SSD portfolio inefficiency measures, these new measures allow us to compare any two δ-SSD efficient or SSD efficient portfolios. Finally, using historical US stock market data, we compute δ-SSD and SSD portfolio efficiency measures of several SSD efficient portfolios.

Keywords: stochastic dominance, stability, SSD portfolio efficiency measure

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1. INTRODUCTION

When solving portfolio selection problem several approaches can be used: mean-risk models, maximising expected utility problems, stochastic dominance criteria, etc. If the information about the risk attitude of a decision maker is not known one may adopt stochastic dominance approach to test an efficiency of a given portfolio with respect to a considered set of utility functions. If only non-satiation and risk aversion of decision maker is assumed, that is, concave utility functions are considered, second-order stochastic dominance (SSD) relation allows comparison of any two portfolios.

Stochastic dominance was introduced independently in Hadar & Russel [6], Hanoch & Levy [7], Rothschild & Stiglitz [20] and Whitmore [23].[1]

The definition of second-order stochastic dominance relation uses comparisons of either twice cumulative distribution functions, or expected utilities (see for example Levy [13]). Alternatively, one can define SSD relation using cumulative quantile functions or conditional value at risk (see Ogryczak & Ruszczyński [15] or Kopa & Chovanec [9]).

[1] For more information see Levy [13].

Similarly to well-known mean-variance criterion, second-order stochastic dominance relation can be used in portfolio efficiency analysis. A given portfolio is called SSD efficient if there exists no other portfolio preferred by all risk-averse and risk-neutral decision makers (see for example Ruszczyński & Vanderbei [21], Kuosmanen [12] or Kopa & Chovanec [9]).

To test SSD portfolio efficiency of a given portfolio relative to all portfolios created from a set of assets Post [17], Kuosmanen [12] and Kopa & Chovanec [9] proposed several linear programming algorithms. While the Post test is based on representative utility functions and strict SSD efficiency criterion, the Kuosmanen and the Kopa-Chovanec test focuses on identifying a SSD dominating portfolio. The last two tests can be formulated as optimization problems with SSD constraints. Similar types of problems were discussed in Dentcheva & Ruszczyński [2, 3, 4], Rudolf & Ruszczyński [5] and Luedtke [14]. In these papers, weak stochastic dominance relation is used, contrary to SSD portfolio tests where strict stochastic dominance relation is considered.

For SSD inefficient portfolios, several SSD portfolio inefficiency measures were introduced in Post [17], Kuosmanen [12] and Kopa & Chovanec [9]. These measures are based on a "distance" between a tested portfolio and some other portfolio identified by a SSD portfolio efficiency test.

In all SSD portfolio efficiency tests, the scenario approach is assumed, that is, the returns of assets are modeled by discrete distribution with equiprobable scenarios. Therefore, especially for SSD efficient portfolios, one can ask how the original scenarios can be changed such that a given SSD efficient portfolio remains SSD efficient for perturbed scenarios, too. To circumvent this problem, Kopa & Post [10] suggested bootstrap techniques for first-order stochastic dominance (FSD) portfolio efficiency and Kopa [11] for SSD portfolio efficiency. In both cases, the inefficiency of a US stock market portfolio was detected with more than 95% significance. Alternatively, Dentcheva, Henrion and Ruszczyński [1] used a general stability results in stochastic programming (see Rőmisch [19]) for optimization problems with weak FSD constraints.

In this paper, we introduce a δ-SSD portfolio efficiency as a new type of portfolio efficiency with respect to second-order stochastic dominance criteria.

Fixing the number of equiprobable scenarios, we identify the maximal perturbation of original scenarios satisfying δ-SSD portfolio efficiency condition for a given portfolio. The magnitude of this maximal perturbation, expressed in terms of a distance between original and perturbed scenarios, can be considered as a measure of δ-SSD efficiency and the limiting case for $\delta \to 0$ leads to a new SSD efficiency measure. We consider only special perturbations where all scenarios are equiprobable and the number of scenarios is fixed. The more general approach can not be used because all SSD portfolio efficiency tests were developed only for equiprobable scenarios.

Contrary to the SSD inefficiency measures discussed above, δ-SSD and SSD portfolio efficiency measures are defined as measures of stability. In comparison with bootstrap techniques suggested by Kopa & Post [10] and Kopa [11], this new stability approach is more robust because it is not based only on a subsampling of given scenarios. The results reached in Dentcheva, Henrion and Ruszczyński [1] for optimization problems with weak FSD constraints can probably be extended for weak SSD constraints. However, this extension would be too technically and computationally demanding for SSD portfolio efficiency testing based on scenario approach and strict SSD relation. Moreover, the general stability results do not deal with any measure of stability.

We apply our stability analysis to the historical US stock market data (six Famma and French portfolios and a riskless asset) in order to compute the values of our δ-SSD and SSD portfolio efficiency measures for two SSD efficient portfolios. As the first portfolio, we choose the portfolio with the highest mean return. Since CVaR is consistent with SSD relation we find the second portfolio by solving mean-CVaR problem. For more details about the consistency see Ogryczak & Ruszczyński [15]. Another way of identifying a SSD efficient portfolio satisfying some required properties was presented in Roman, Darby-Dowman, and Mitra [18].

The remainder of the paper is organized as follows. The Preliminaries section starts with notation, assumptions and definitions for the SSD relation and SSD portfolio efficiency. We introduce a δ-SSD relation and δ-SSD portfolio efficiency as a new type of SSD relation and SSD portfolio efficiency. It is followed by a section dealing with SSD portfolio efficiency test derived in Kuosmanen [12] and it's modification for δ-SSD portfolio efficiency. In Section 4, we state our main stability ideas and we introduce new measures of SSD portfolio efficiency and δ-SSD portfolio efficiency as measures of stability. Using US stock market data, the final section presents a numerical illustration where we compute the δ-SSD and SSD portfolio efficiency measures for two SSD efficient portfolios .

2. PRELIMINARIES

We consider a random vector $\mathbf{r} = (r_1, r_2, \ldots, r_N)'$ of returns of N assets with a discrete probability distribution described by T equiprobable scenarios. The returns of the assets for the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \ldots, x_N^t)$ is the tth row of matrix X. We will use $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)'$ for a vector of portfolio weights and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in \mathbb{R}^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \ \lambda_n \geq 0, \ \ n = 1, 2, \ldots, N\}.$$

Alternatively, one can consider any bounded polytope:

$$\Lambda' = \{\boldsymbol{\lambda} \in \mathbb{R}^N | A\boldsymbol{\lambda} \geq \mathbf{b}\}.$$

The tested portfolio is denoted by $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_N)'$. Following Ruszczyński and Vanderbei [21], Kuosmanen [12], Kopa and Chovanec [9], we define second-order stochastic dominance relation in the strict form in the context of SSD portfolio efficiency. Let $F_{\mathbf{r}'\boldsymbol{\lambda}}(x)$ denote the cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$. The twice cumulative probability distribution function of returns of portfolio $\boldsymbol{\lambda}$ is defined as:

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(t) \quad = \quad \int_{-\infty}^{t} F_{\mathbf{r}'\boldsymbol{\lambda}}(x)\,\mathrm{d}x. \tag{1}$$

Definition 2.1. Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by second-order stochastic dominance ($\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$) if and only if

$$F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}(t) \leq F^{(2)}_{\mathbf{r}'\boldsymbol{\tau}}(t) \qquad \forall\, t \in \mathbb{R}$$

with strict inequality[2] for at least one $t \in \mathbb{R}$.

The following SSD criteria can be used as alternative definitions of the SSD relation:

(i) $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if and only if $\mathrm{E}u(\mathbf{r}'\boldsymbol{\lambda}) \geq \mathrm{E}u(\mathbf{r}'\boldsymbol{\tau})$ for all concave utility functions u provided the expected values above are finite and strict inequality is fulfilled for at least some concave utility function, see for example Levy [13].

(ii) $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if and only if $F^{(-2)}_{\mathbf{r}'\boldsymbol{\lambda}}(\frac{p}{T}) \geq F^{(-2)}_{\mathbf{r}'\boldsymbol{\tau}}(\frac{p}{T})$ for all $p = 1, 2, \ldots, T$ with strict inequality for at least some p where the second quantile function $F^{(-2)}_{\mathbf{r}'\boldsymbol{\lambda}}$ is the convex conjugate function of $F^{(2)}_{\mathbf{r}'\boldsymbol{\lambda}}$ in the sense of Fenchel duality, see Ogryczak & Ruszczyński [15]. Let $k = T - p$. Since

$$\mathrm{CVaR}_{1-\frac{p}{T}}(-\mathbf{r}'\boldsymbol{\lambda}) = -\frac{F^{(-2)}_{\mathbf{r}'\boldsymbol{\tau}}(\frac{p}{T})}{\frac{p}{T}}$$

for all $p = 1, 2, \ldots, T$, where conditional value at risk (CVaR) can be defined via the optimization problem:

$$\begin{aligned} \mathrm{CVaR}_{\frac{k}{T}}(Y) \quad &= \quad \min_{a, w_t} \quad a + \frac{1}{(1-\frac{k}{T})T} \sum_{t=1}^{T} w_t \\ \text{s.t.} \quad w_t \quad &\geq \quad y_t - a \\ w_t \quad &\geq \quad 0, \end{aligned} \tag{2}$$

we can alternatively formulate the criterion in the following way: $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if and only if $\mathrm{CVaR}_{\frac{k}{T}}(-\mathbf{r}'\boldsymbol{\lambda}) \leq \mathrm{CVaR}_{\frac{k}{T}}(-\mathbf{r}'\boldsymbol{\tau})$ for all $k = 0, 1, \ldots, T-1$ with strict inequality for at least some k. See Kopa and Chovanec [9], Uryasev & Rockafellar [22] and Pflug [16] for details.

[2] This type of SSD relation is sometimes referred to as the strict second-order stochastic dominance. If no strict inequality is required then the relation can be called the weak second-order stochastic dominance.

(iii) $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$ if and only if there exists a double stochastic matrix $W = \{w\}_{ij}$ such that $(WX\boldsymbol{\tau} \leq X\boldsymbol{\lambda}$ and $\mathbf{1}'WX\boldsymbol{\tau} < \mathbf{1}'X\boldsymbol{\lambda})$ or $(WX\boldsymbol{\tau} = X\boldsymbol{\lambda}$ and $\sum_{i=1}^{T} w_{ii} < T)$ where $\mathbf{1}' = (1, 1, \ldots, 1)$. See Kuosmanen [12] and Hardy, Littlewood & Polya [8] (Theorem 46) for details.

Since $\mathbf{1}'W = \mathbf{1}'$ for all double stochastic matrices W, using criterion (iii) we define a new type of SSD relation.

Definition 2.2. Let $\delta > 0$. Portfolio $\boldsymbol{\lambda} \in \Lambda$ dominates portfolio $\boldsymbol{\tau} \in \Lambda$ by the δ-second-order stochastic dominance $(\mathbf{r}'\boldsymbol{\lambda} \succ_{\delta-SSD} \mathbf{r}'\boldsymbol{\tau})$ if there exists a double stochastic matrix $W = \{w\}_{ij}$ such that $X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau}$ and $\mathbf{1}'X\boldsymbol{\lambda} - \mathbf{1}'X\boldsymbol{\tau} \geq \delta$.

The strictly positive parameter δ in Definition 2 is chosen sufficiently small, that is, such that if $X\boldsymbol{\lambda} \geq WX\boldsymbol{\tau}$ and $\mathbf{1}'X\boldsymbol{\lambda} - \mathbf{1}'X\boldsymbol{\tau} < \delta$ then vectors $X\boldsymbol{\lambda}$ and $WX\boldsymbol{\tau}$ are empirically indistinguishable.[3] It is easily seen that if portfolio $\boldsymbol{\lambda}$ δ-SSD dominates portfolio $\boldsymbol{\tau}$ for some $\delta > 0$ then $\boldsymbol{\lambda}$ SSD dominates $\boldsymbol{\tau}$. On the other hand, SSD relation need not imply δ-SSD relation for any $\delta > 0$. Hence, δ-SSD relation for some $\delta > 0$ is only sufficient condition of SSD relation.

Definition 2.3. A given portfolio $\boldsymbol{\tau} \in \Lambda$ is *SSD inefficient* if and only if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\mathbf{r}'\boldsymbol{\lambda} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is *SSD efficient*.

This definition classifies portfolio $\boldsymbol{\tau} \in \Lambda$ as SSD efficient if and only if no other portfolio is better (in the sense of the SSD relation) for all risk averse and risk neutral decision makers. Another definition of SSD efficiency was presented in Post [17]. Based on Definition 2, we can similarly define δ-SSD portfolio efficiency.

Definition 2.4. A given portfolio $\boldsymbol{\tau} \in \Lambda$ is *δ-SSD inefficient* if and only if there exists portfolio $\boldsymbol{\lambda} \in \Lambda$ such that $\mathbf{r}'\boldsymbol{\lambda} \succ_{\delta-SSD} \mathbf{r}'\boldsymbol{\tau}$. Otherwise, portfolio $\boldsymbol{\tau}$ is δ-SSD efficient.

Since δ-SSD relation implies SSD relation, δ-SSD portfolio efficiency is a necessary condition of SSD portfolio efficiency, that is, every SSD efficient portfolio is δ-SSD efficient for all strictly positive δ.

3. SSD AND δ–SSD PORTFOLIO EFFICIENCY TEST

In this section we present the linear programming test of SSD portfolio efficiency in the form of necessary and sufficient condition derived in Kuosmanen [12]. From the three SSD efficiency tests: the Post test [17], the Kopa–Chovanec test [9] and the Kuosmanen test [12], we choose the last one, because the Kuosmanen test can be easily modify to a new δ-SSD efficiency test. The Kuosmanen test is based on criterion (iii) and it tries to identify a portfolio $\boldsymbol{\lambda} \in \Lambda$ that SSD dominates the given portfolio $\boldsymbol{\tau}$.

[3] This kind of approximation is sometimes used in empirical finance.

Lemma 3.1. (The Kuosmanen test) Let

$$\theta^* = \max_{W,\boldsymbol{\lambda}} \sum_{t=1}^{T} \left(\mathbf{x}^t\boldsymbol{\lambda} - \mathbf{x}^t\boldsymbol{\tau}\right) \tag{3}$$

$$\begin{aligned} \text{s.t.} \quad X\boldsymbol{\lambda} &\geq WX\boldsymbol{\tau} \\ \sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} &= 1, \quad w_{ij} \geq 0 \quad i,j = 1,2,\ldots,T \\ \boldsymbol{\lambda} &\in \Lambda \end{aligned}$$

and

$$\theta^{**} = \min_{W,\boldsymbol{\lambda},S^+,S^-} \sum_{j=1}^{T}\sum_{i=1}^{T}(s_{ij}^+ + s_{ij}^-) \tag{4}$$

$$\begin{aligned} \text{s.t.} \quad X\boldsymbol{\lambda} &= WX\boldsymbol{\tau} \\ s_{ij}^+ - s_{ij}^- &= w_{ij} - \frac{1}{2} \quad i,j = 1,2,\ldots,T \\ s_{ij}^+, s_{ij}^-, w_{ij} &\geq 0 \quad i,j = 1,2,\ldots,T \\ \sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} &= 1 \quad i,j = 1,2,\ldots,T \\ \boldsymbol{\lambda} &\in \Lambda \end{aligned}$$

where $S^+ = \{s_{ij}^+\}_{i,j=1}^T$, $S^- = \{s_{ij}^-\}_{i,j=1}^T$ and $W = \{w_{ij}\}_{i,j=1}^T$. Let ϵ_k denote the number of k-way ties in $X\boldsymbol{\tau}$.[4] Then portfolio $\boldsymbol{\tau}$ is SSD efficient if and only if

$$\theta^* = 0 \quad \wedge \; \theta^{**} = \frac{T^2}{2} - \sum_{k=1}^{T} k\epsilon_k.$$

Let $\boldsymbol{\lambda}^*$ and $\boldsymbol{\lambda}^{**}$ be the optimal solution of (3) and (4), respectively. If $\theta^* > 0$ then $\mathbf{r}'\boldsymbol{\lambda}^* \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$. If $\theta^* = 0$ and $\theta^{**} < \frac{T^2}{2} - \sum_{k=1}^{T} k\epsilon_k$ then $\mathbf{r}'\boldsymbol{\lambda}^{**} \succ_{SSD} \mathbf{r}'\boldsymbol{\tau}$.

If $\theta^* > 0$ then problem (4) need not to be solved, because portfolio $\boldsymbol{\tau}$ is SSD inefficient and the optimal solution $\boldsymbol{\lambda}^*$ is a SSD dominating portfolio, see Kuosmanen [12] for more details.

If a given portfolio $\boldsymbol{\tau}$ is SSD inefficient then, from the entire set of SSD dominating portfolios, the Kuosmanen test identifies that with the highest mean return. That is, (3) and (4) can be reformulated in the following way:

$$\max_{\boldsymbol{\lambda}\in\Lambda} \quad f(\boldsymbol{\lambda},\boldsymbol{\tau}) \tag{5}$$

$$\text{s.t.} \quad \mathbf{r}'\boldsymbol{\lambda} \quad \succ_{SSD} \quad \mathbf{r}'\boldsymbol{\tau}, \tag{6}$$

where $f(\boldsymbol{\lambda},\boldsymbol{\tau}) = T\left(\mathrm{E}(\mathbf{r}'\boldsymbol{\lambda}) - \mathrm{E}(\mathbf{r}'\boldsymbol{\tau})\right) = \sum_{t=1}^{T} \mathbf{x}^t\boldsymbol{\lambda} - \mathbf{x}^t\boldsymbol{\tau}$.

[4]We say that a k-way tie occurs if k elements of $X\boldsymbol{\tau}$ are equal to each other.

Problem (5) – (6) is an optimization problem with a stochastic dominance constraint. Contrary to problems discussed in Dentcheva & Ruszczyński [2, 3, 4], Rudolf & Ruszczyński [5] and Luedtke [14], the stochastic dominance constraint (6) is in the strict form.

The optimal value θ^* of (3) can be considered as a measure of SSD portfolio inefficiency. It gives us information about the maximal possible difference, expressed in mean return (or sum of returns), between the tested portfolio and a SSD dominating portfolio. The alternative SSD portfolio inefficiency measures arise from the Post test and the Kopa–Chovanec test. All these three measures allow comparison of two SSD inefficient portfolios. Unfortunately, these measures are not suitable for SSD efficiency measuring, because all these measures are equal to zero for all SSD efficient portfolios. Therefore, for SSD portfolio efficiency measuring, we suggest another approach, based on the δ-SSD portfolio efficiency and stability of δ-SSD portfolio efficiency classification. Firstly, we modify the Kuosmanen test to δ-SSD portfolio efficiency test.

Lemma 3.2. (The δ-SSD portfolio efficiency test) Let

$$\theta^*_\delta = \max_{W,\boldsymbol{\lambda}} \sum_{t=1}^{T} \left(\mathbf{x}^t\boldsymbol{\lambda} - \mathbf{x}^t\boldsymbol{\tau}\right) \tag{7}$$

$$\begin{aligned} \text{s.t.} \qquad X\boldsymbol{\lambda} &\geq WX\boldsymbol{\tau} \\ \sum_{t=1}^{T} \left(\mathbf{x}^t\boldsymbol{\lambda} - \mathbf{x}^t\boldsymbol{\tau}\right) &\geq \delta \\ \sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} &= 1, \quad w_{ij} \geq 0 \qquad i,j = 1,2,\ldots,T \\ \boldsymbol{\lambda} &\in \Lambda. \end{aligned}$$

If an optimal solution of (7) exists then portfolio $\boldsymbol{\tau}$ is δ-SSD inefficient and $\mathbf{r}'\boldsymbol{\lambda}^* \succ_{\delta-SSD} \mathbf{r}'\boldsymbol{\tau}$. Otherwise, $\boldsymbol{\tau}$ is δ-SSD efficient portfolio.

The proof of Lemma 3.2 directly follows from Lemma 3.1, criterion (iii), Definition 2.2 and Definition 2.4.

4. STABILITY OF SSD AND δ-SSD PORTFOLIO EFFICIENCY CLASSIFICATION

In previous sections a fixed scenario matrix was considered and all portfolio efficiency tests were done for this scenario matrix. Unfortunately, usually we do not have perfect information about the distribution of returns. Therefore, the stability of SSD portfolio efficiency and δ-SSD portfolio efficiency with respect to changes in the scenario matrix is investigated in this section.

Since the SSD portfolio efficiency tests and the δ-SSD portfolio efficiency test are derived under the assumption of equiprobable scenarios collected in matrix X we will consider only perturbation matrices X_p of the original matrix X which have exactly T rows, that is, we admit only approximations with T equiprobable scenarios. Let $\mathcal{X}_p$ be the set of all such perturbation matrices. In this section we analyze how the results of the SSD and δ-SSD portfolio efficiency test for a given portfolio depend on the original scenario matrix X and which other matrices X_p from a neighbourhood[5] of X guarantee the SSD or δ-SSD portfolio efficiency of the given portfolio.

Let matrix $\Upsilon = \{\upsilon_{ij}\}_{i,j=1}^{T}$ be defined as $\Upsilon = X_p - X$. Let $D(X, X_p) = \max_{i,j} |\upsilon_{ij}|$ denote a distance between matrices X and X_p on $\mathcal{X}_p$. We introduce a new measure of δ-SSD portfolio efficiency as a measure of stability.

Definition 4.1. The δ-SSD portfolio efficiency measure γ_δ of δ-SSD efficient portfolio $\boldsymbol{\tau} \in \Lambda$ is defined as the optimal value of the following optimization problem:

$$\begin{aligned} \gamma_\delta(\boldsymbol{\tau}) = \max \quad & \varepsilon \\ \text{s.t.} \quad \boldsymbol{\tau} \text{ is } \delta - SSD & \text{ efficient for all } \quad X_p \in \mathcal{X}_p \quad \text{such that} \quad D(X, X_p) \le \varepsilon. \end{aligned} \tag{8}$$

This measure gives us information how large is the neighborhood of X such that for all matrices from this neighborhood the portfolio $\boldsymbol{\tau}$ is classified as δ-SSD efficient. The problem (8) consists of infinitely many δ-SSD efficiency constraints. Moreover, according to the Lemma 3.2, each constraint involves a maximization problem what makes problem (8) practically unsolvable. Therefore we reinterpret the δ-SSD portfolio efficiency measure for a given δ-SSD efficient portfolio $\boldsymbol{\tau} \in \Lambda$ as the minimal distance between the original matrix X and any other matrix X_p that makes portfolio $\boldsymbol{\tau}$ δ-SSD inefficient, that is,

$$\begin{aligned} \gamma_\delta(\boldsymbol{\tau}) = \min_{X_p \in \mathcal{X}_p} \quad & D(X, X_p) \\ \text{s.t.} \quad \boldsymbol{\tau} \text{ is } \delta - SSD & \text{ inefficient for } \quad X_p. \end{aligned} \tag{9}$$

Using Lemma 3.2, the SSD portfolio efficiency measure $\gamma_\delta(\boldsymbol{\tau})$ can be computed in a much less computationally demanding way:

$$\gamma_\delta(\boldsymbol{\tau}) = \min_{\boldsymbol{\lambda} \in \Lambda, X_p \in \mathcal{X}_p} \quad D(X, X_p) \tag{10}$$

$$\begin{aligned} \text{s.t.} \quad X\boldsymbol{\lambda} - WX\boldsymbol{\tau} \quad & \ge \quad 0 \\ \sum_{t=1}^{T} \left(\mathbf{x}^t \boldsymbol{\lambda} - \mathbf{x}^t \boldsymbol{\tau} \right) \quad & \ge \quad \delta \\ \sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} = 1, \quad w_{ij} \quad & \ge \quad 0 \quad i, j = 1, 2, \ldots, T. \end{aligned}$$

Since $\Upsilon = X_p - X$ and $D(X, X_p) = \max_{i,j} |\upsilon_{ij}|$ the measure $\gamma_\delta(\boldsymbol{\tau})$ can be computed using the following non-linear program.

[5]for a given metric on $\mathcal{X}_p$

$$\gamma_\delta(\boldsymbol{\tau}) = \min_{\boldsymbol{\lambda}\in\Lambda,\Upsilon,\varepsilon} \varepsilon \tag{11}$$

$$\begin{array}{rcll}
\text{s.t.} \quad (X+\Upsilon)\boldsymbol{\lambda} - W(X+\Upsilon)\boldsymbol{\tau} & \geq & 0 & \\
\sum_{t=1}^{T} \left((\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\lambda} - (\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\tau}\right) & \geq & \delta & \\
\sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} = 1, \quad w_{ij} & \geq & 0 & i,j = 1,2,\ldots,T \\
-\varepsilon \ \leq \ \upsilon_{ij} & \leq & \varepsilon & i,j = 1,2,\ldots,T,
\end{array}$$

where $\boldsymbol{\upsilon}^t = (\upsilon_{t1}, \upsilon_{t2}, \ldots, \upsilon_{tT})$ is the tth row of matrix Υ. For a given portfolio $\boldsymbol{\tau}$ we have $\gamma_\delta(\boldsymbol{\tau}) \geq 0$ for all $\delta > 0$. Moreover, if $\delta_1 < \delta_2$ then the set of feasible solutions of (11) is larger for δ_1 than for δ_2 and consequently $\gamma_{\delta_1}(\boldsymbol{\tau}) \leq \gamma_{\delta_2}(\boldsymbol{\tau})$. Therefore, we can define a measure of SSD efficiency in the following way.

Definition 4.2. The SSD portfolio efficiency measure γ of SSD efficient portfolio $\boldsymbol{\tau} \in \Lambda$ is defined as: $\gamma(\boldsymbol{\tau}) = \lim_{\delta\to 0^+} \gamma_\delta(\boldsymbol{\tau}) = \inf_{\delta>0} \gamma_\delta(\boldsymbol{\tau})$.

4.1. One scenarion perturbation – A given scenario

Assume that only the tth scenario can be changed, that is $\upsilon_{ij} = 0$ for all $i \neq t$. Then $D(X, X_p) = \max_j |\upsilon_{tj}|$ and the corresponding δ-SSD efficiency measure γ_δ^t is defined as

$$\gamma_\delta^t(\boldsymbol{\tau}) = \min_{\boldsymbol{\lambda}\in\Lambda,\Upsilon,\varepsilon} \varepsilon \tag{12}$$

$$\begin{array}{rcll}
\text{s.t.} \quad (X+\Upsilon)\boldsymbol{\lambda} - W(X+\Upsilon)\boldsymbol{\tau} & \geq & 0 & \\
\sum_{t=1}^{T} \left((\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\lambda} - (\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\tau}\right) & \geq & \delta & \\
\sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} = 1, \quad w_{ij} & \geq & 0 & i,j = 1,2,\ldots,T \\
-\varepsilon \ \leq \ \upsilon_{tj} & \leq & \varepsilon & j = 1,2,\ldots,T \\
\upsilon_{ij} & = & 0 & i \neq t \quad j = 1,2,\ldots,T.
\end{array}$$

Similarly to the complete scenario perturbation case, the SSD efficiency measure for one scenario perturbation is: $\gamma^t(\boldsymbol{\tau}) = \lim_{\delta\to 0^+} \gamma_\delta(\boldsymbol{\tau}) = \inf_{\delta>0} \gamma_\delta(\boldsymbol{\tau})$.

4.2. One scenarion perturbation – An arbitrary scenario

In this section we still assume that only one scenario can be changed. Contrary to the previous case, now we do not prescribe which scenario it is. Therefore we again consider $D(X, X_p) = \max_{i,j} |\upsilon_{ij}|$ as in the general case and the δ-SSD portfolio efficiency measure for this situation is defined as:

$$\overline{\gamma}_\delta(\boldsymbol{\tau}) = \min_{\boldsymbol{\lambda}\in\Lambda,\Upsilon,\varepsilon} \varepsilon \tag{13}$$

$$\begin{array}{rcl}
\text{s.t.} \qquad (X+\Upsilon)\boldsymbol{\lambda} - W(X+\Upsilon)\boldsymbol{\tau} & \geq & 0 \\
\sum_{t=1}^{T} \left((\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\lambda} - (\mathbf{x}^t + \boldsymbol{\upsilon}^t)\boldsymbol{\tau} \right) & \geq & \delta \\
\sum_{j=1}^{T} w_{ij} = 1, \quad \sum_{i=1}^{T} w_{ij} = 1, \quad w_{ij} & \geq & 0 \quad i,j = 1,2,\ldots,T \\
-\varepsilon \;\leq\; \upsilon_{ij} & \leq & \varepsilon \quad i,j = 1,2,\ldots,T \\
\upsilon_{ij} & \leq & My_i \;\; j = 1,2,\ldots,T \\
\sum_{i=1}^{T} y_i = 1, \quad y_i & \in & \{0,1\},
\end{array}$$

where M is a sufficiently large constant, for example $M = 2\sum_{i,j=1}^{T} |x_{ij}|$. Problem (13) is more computationally demanding than (12) because T binary variables are added. The corresponding SSD efficiency measure is again defined as the limiting case: $\overline{\gamma}(\boldsymbol{\tau}) = \lim_{\delta\to 0^+} \gamma_\delta(\boldsymbol{\tau}) = \inf_{\delta>0} \gamma_\delta(\boldsymbol{\tau})$.

5. EMPIRICAL APPLICATION

To illustrate our portfolio efficiency measuring, we apply it to the US stock market data in order to compute the δ-SSD portfolio efficiency measure γ_δ, and SSD portfolio efficiency measure γ of two SSD efficient portfolios. The investment universe of stocks is proxied by the well-known six value-weighted Fama and French portfolios. The last considered asset is the riskless asset that is proxied by the one-year US government bond index from Ibbotson Associates. We consider yearly excess returns from 1963 to 2002 (40 annual observations). Excess returns are computed by subtracting the riskless rate from the nominal returns, that is, the riskless asset always has a return of zero. Table 1 shows descriptive statistics for our data set.

We start with identifying two SSD efficient portfolios. Since short sales are not allowed and no two assets have the same mean, the portfolio consisting only of the asset with the highest mean $\boldsymbol{\tau}_1 = (0,0,1,0,0,0,0)$ is obviously SSD efficient. Ogryczak & Ruszczyński [15] proved that several mean-risk models are consistent with SSD relation, e. g., for CVaR as a measure of risk. Therefore, if mean-CVaR model has an unique optimal solution then it is a SSD efficient portfolio. Solving mean-CVaR model with $\alpha = 0.95$ we identified the second SSD efficient portfolio $\boldsymbol{\tau}_2 = (0,0,0.385,0.016,0,0.013,0.586)$ were the minimal required mean was equal to the mean of market portfolio proxied by the CRSP all-share index. We solve problems (11) for both SSD efficient portfolios and five levels $\delta = 1, 0.1, 0.01, 0.001, 0.0001$ using GAMS system (solver COINIPOPT). The results are presented in Table 2.

Table 1. Descriptive statistics for 6 Famma and French portfolios formed on market capitalization of equity and book-to-market equity ratio (SG = small growth, SN = small neutral, SV = small value, BG = big growth, BN = big neutral and BV = big value).

	Mean	St.dev.	Skew.	Kurt.	Min.	Max.
SG	5.309	28.520	0.323	0.175	–49.28	83.68
SN	11.301	22.728	–0.308	0.062	–37.38	65.48
SV	13.861	23.158	–0.373	–0.222	–33.86	61.14
BG	5.303	18.820	–0.317	–0.537	–40.49	34.67
BN	6.340	16.120	–0.241	–0.090	–34.13	34.73
BV	8.946	17.723	–0.690	–0.026	–34.24	40.34

Table 2. δ-SSD efficiency measures for portfolio $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$.

	$\delta = 1$	$\delta = 0.1$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.0001$
$\gamma_\delta(\boldsymbol{\tau}_1)$	1.369	1.369	1.369	1.369	1.369
$\gamma_\delta(\boldsymbol{\tau}_2)$	0.937	0.412	0.393	0.388	0.388

From Table 2 we can see that $\gamma_\delta(\boldsymbol{\tau}_1) = 1.369$ for all $\delta \in \langle 1, 0.0001\rangle$ and therefore we can expect that $\gamma(\boldsymbol{\tau}_1) = \inf_{\delta>0} \gamma_\delta(\boldsymbol{\tau}_1) = 1.369$. To prove it, we apply the modified Kousmanen test where we use $X + \Upsilon$ instead of X and we include the additional constraints:[6]

$$-(1.369 - \xi) \quad \leq \quad v_{ij} \quad \leq \quad (1.369 - \xi) \qquad i, j = 1, 2, \ldots, T, \tag{14}$$

where ξ is a sufficiently small number,[7] in our case we choose $\xi = 0.0005$. This modified test tries to identify a SSD dominating portfolio for any feasible perturbed scenario matrix. We can find that the test fails to identify a SSD dominating portfolio for completely perturbed scenario matrices X_p with $D(X, X_p) \leq 1.3685$. Therefore, we can conclude that the SSD portfolio efficiency measure of portfolio $\boldsymbol{\tau}_1$ is equal to 1.369. By analogy, we can easily check that $\gamma(\boldsymbol{\tau}_2) = 0.388$ where we use $\xi = 0.0005$ and

$$-(0.388 - \xi) \quad \leq \quad v_{ij} \quad \leq \quad (0.388 - \xi) \qquad i, j = 1, 2, \ldots, T.$$

instead of (14).

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[6]Note that including matrix of variables Υ makes the test non-linear.

[7]The choice of ξ depends on a prescribed accuracy level. In our example we round all values to three decimal digits accuracy.

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