AUCO Czech Economic Review 4 (2010) 236–250
Acta Universitatis Carolinae Oeconomica

Rescaled Range Analysis and Detrended Fluctuation Analysis: Finite Sample Properties and Confidence Intervals

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Received 14 September 2009; Accepted 1 March 2010

Abstract We focus on finite sample properties of two mostly used methods of Hurst exponent H estimation—rescaled range analysis (R/S) and detrended fluctuation analysis (DFA). Even though both methods have been widely applied on different types of financial assets, only several papers have dealt with the finite sample properties which are crucial as the properties differ significantly from the asymptotic ones. Recently, R/S analysis has been shown to overestimate H when compared to DFA. However, we show that even though the estimates of R/S are truly significantly higher than an asymptotic limit of 0.5, for random time series with lengths from 2^9 to 2^{17}, they remain very close to the estimates proposed by Anis & Lloyd and the estimated standard deviations are lower than the ones of DFA. On the other hand, DFA estimates are very close to 0.5. The results propose that R/S still remains useful and robust method even when compared to newer method of DFA which is usually preferred in recent literature.

Keywords Rescaled range analysis, detrended fluctuation analysis, Hurst exponent, long-range dependence, confidence intervals

JEL classification G1, G10, G14, G15

1. Introduction

Long-range dependence and its presence in the financial time series has been discussed in several recent papers (Czarnecki et al. 2008; Grech and Mazur 2004; Carbone et al. 2004; Matos et al. 2008; Vandewalle et al. 1997; Alvarez-Ramirez et al. 2008; Peters 1994; Di Matteo et al. 2005; Di Matteo 2007). However, most authors interpret the results on the basis of comparison of estimated Hurst exponent H with the theoretical value for an independent process of 0.5. In more detail, Hurst exponent of 0.5 indicates two possible processes: either independent (Beran 1994) or short-range dependent process (Lillo and Farmer 2004). If $H > 0.5$, the process has significantly positive correlations at all lags and is said to be persistent (Mandelbrot and van Ness 1968). On the other hand, if $H < 0.5$, it has significantly negative correlations at all lags and the process is said to be anti-persistent (Barkoulas et al. 2000).

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However, the estimates for pure Gaussian process can strongly deviate from the limit of 0.5 (Weron 2002; Couillard and Davison 2005). Moreover, the estimates are influenced by choice of minimum and maximum scale (Weron 2002; Kristoufek 2009). There have been several papers dealing with finite sample properties of estimators of Hurst exponent (Peters 1994; Couillard and Davison 2005; Grech and Mazur 2005; Weron 2002). With the exception of Kristoufek (2009), none of the papers use the proposition for optimal scales presented elsewhere (Grech and Mazur 2004; Matos et al. 2008; Alvarez-Ramirez et al. 2005; Einstein et al. 2001). This paper attempts to fill this gap and presents results of Monte Carlo simulations for two mostly used techniques—rescaled range analysis and detrended fluctuation analysis.

In Section 2, we present and describe both techniques in detail. In Section 3, we show results of Monte Carlo simulations for time series lengths from 512 to 131,072 observations and support that R/S overestimates Hurst exponent for all examined time series lengths. The overestimation decreases significantly with growing length. In Section 4, we present results for simulations for time series of length from 256 to 131,072 observations but this time, on the same series, both procedures are applied and we comment on differences. We find out that even if R/S shows higher values of Hurst exponent than DFA, the standard deviations are lower for R/S so that the confidence intervals are narrower. Nevertheless, both methods show very similar estimates, when the bias is taken into consideration, whereas they are more correlated with growing time series length. Section 5 concludes.

2. Hurst exponent estimation methods

In this section, we briefly introduce rescaled range analysis and detrended fluctuation analysis procedures. For more detailed reviews, see Taqqu et al. (1995), Kantelhardt (2008) or references in the following subsections.

2.1 Rescaled range analysis

Rescaled range analysis (R/S) was developed by Harold E. Hurst while working as a water engineer in Egypt (Hurst 1951) and was later applied to financial time series by Mandelbrot and van Ness (1968), Mandelbrot (1970). The basic idea behind R/S analysis is that a range, which is taken as a measure of dispersion of the series, follows a scaling law. If a process is random, the measure of dispersion scales according to the square-root law so that a power in the scaling law is equal to 0.5. Such value is connected to Hurst exponent of 0.5.

In the procedure, one takes returns of the time series of length T and divides them into N adjacent sub-periods of length υ while $N\upsilon = T$. Each sub-period is labeled as I_n with $n = 1, 2, \ldots, N$. Moreover, each element in I_n is labeled $r_{k,n}$ with $k = 1, 2, \ldots, \upsilon$. For each sub-period, one calculates an average value and constructs new series of accumulated deviations from the arithmetic mean values (a profile).

The procedure follows in calculation of the range, which is defined as a difference between a maximum and a minimum value of the profile $X_{k,n}$, and a standard deviation of the original returns series for each sub-period I_n. Each range R_{I_n} is standardized by the corresponding standard deviation S_{I_n} and forms a rescaled range as

$$(R/S)_{I_n} = \frac{R_{I_n}}{S_{I_n}}. \tag{1}$$

The process is repeated for each sub-period of length υ. We get average rescaled ranges $(R/S)_\upsilon$ for each sub-interval of length υ.

The length υ is increased and the whole process is repeated. We use the procedure used in recent papers so that we use the length υ equal to the power of a set integer value. Thus, we set a basis b, a minimum power *pmin* and a maximum power *pmax* so that we get $\upsilon = b^{pmin}, b^{pmin+1}, \ldots, b^{pmax}$ where $b^{pmax} \leq T$ (Weron 2002).

Rescaled range then scales as

$$(R/S)_\upsilon \sim c\upsilon^H \tag{2}$$

where c is a finite constant independent of υ (Taqqu et al. 1995; Di Matteo 2007). A linear relationship in double-logarithmic scale indicates a power scaling (Weron 2002). To uncover the scaling law, we use an ordinary least squares regression on logarithms of each side of (2). We suggest using logarithm with basis equal to b. Thus, we get

$$\log_b (R/S)_\upsilon \sim \log_b c + H \log_b \upsilon, \tag{3}$$

where H is Hurst exponent.

2.2 Detrended fluctuation analysis

Detrended fluctuation analysis (DFA) was firstly proposed by Peng et al. (1994) while examining series of DNA nucleotides. Compared to the R/S analysis examined above, DFA uses different measure of dispersion—squared fluctuations around trend of the signal. As DFA is based on detrending of the sub-periods, it can be used for non-stationary time series contrary to R/S.

Starting steps of the procedure are the same as the ones of R/S analysis as the whole series is divided into non-overlapping periods of length υ which is again set on the same basis as in the mentioned procedure and the series profile is constructed. The following steps are based on Grech and Mazur (2005). Polynomial fit $X_{\upsilon,l}$ of the profile is estimated for each sub-period I_n. The choice of order l of the polynomial is rather a rule of thumb but is mostly set as the first or the second order polynomial trend as higher orders do not add any significant information (Vandewalle et al. 1997). The procedure is then labeled as DFA-0, DFA-1 and DFA-2 according to an order of the filtering trend (Hu et al. 2001). We stick to the linear trend filtering and thus use DFA-1 in the paper. A detrended signal $Y_{\upsilon,l}$ is then constructed as

$$Y_{\upsilon,l}(t) = X(t) - X_{\upsilon,l}. \tag{4}$$

Fluctuation $F^2_{DFA}(\upsilon,l)$, which is defined as

$$F^2_{DFA}(\upsilon,l) = \frac{1}{\upsilon}\sum_{t=1}^{\upsilon} Y^2_{\upsilon,l}(t), \tag{5}$$

scales as

$$F_{DFA}^2(\upsilon,l) \sim c\upsilon^{2H(l)}, \tag{6}$$

where again c is a constant independent of υ (Weron 2002).

We again run an ordinary least squares regression on logarithms of (6) and estimate Hurst exponent $H(l)$ for set l-degree of polynomial trend in same way as for R/S as

$$\log_b F_{DFA}(\upsilon,l) \sim \log_b c + H(l)\log_b \upsilon. \tag{7}$$

DFA can be adjusted and various filtering functions $X_{\upsilon,l}$ can be used. For a detailed review of DFA, see Kantelhardt (2008).

3. Finite sample properties of R/S and DFA

3.1 R/S analysis

R/S analysis has one significant advantage compared to the other methods—as it has been known and tested for over 50 years, the methods for testing have been well developed and applied.

The condition for a time series to reject long-term dependence is that $H = 0.5$. However, it holds only for infinite samples and therefore is an asymptotic limit. The correction for finite samples is thoroughly tested in Couillard and Davison (2005). Anis and Lloyd (1976), which we note AL76, states the expected value of rescaled range as

$$E(R/S)_\upsilon = \frac{\Gamma(\frac{\upsilon-1}{2})}{\sqrt{\pi}\Gamma(\frac{\upsilon}{2})}\sum_{i=1}^{\upsilon-1}\sqrt{\frac{\upsilon-1}{i}}. \tag{8}$$

We performed original tests for time series lengths from $T = 512 = 2^9$ up to $T = 131{,}072 = 2^{17}$. All steps of R/S analysis on 10,000 time series drawn from standardized normal distribution $N(0,1)$ were performed. Hurst exponent was estimated by log-log regression according to the presented procedure. Averaged rescaled ranges applied in the regression were the ones for $2^4 \le \upsilon \le 2^{T-2}$. The logic behind this step is rather intuitive—very small scales can bias the estimate as standard deviations are based on just few observations; on the other hand, large scales can bias the estimate as

Table 1. Monte Carlo simulations descriptive statistics (R/S)

	512	1024	2048	4096	8192	16384	32768	65536	131072
Mean	0.5763	0.5647	0.5570	0.5494	0.5430	0.5380	0.5338	0.5296	0.5267
AL76	0.5657	0.5572	0.5500	0.5438	0.5386	0.5342	0.5304	0.5272	0.5132
SD	0.0551	0.0404	0.0310	0.0246	0.0199	0.0162	0.0138	0.0118	0.0102
Skewness	0.0104	0.0003	−0.0231	−0.0316	−0.0223	−0.0331	−0.0329	0.0068	−0.0762
Kurtosis	−0.1316	0.0730	−0.0595	−0.0567	0.0220	−0.0271	0.0136	−0.1108	0.0237
Jarque-Bera	7.4569	2.1800	2.3895	3.0314	1.0196	2.1440	1.8737	5.2405	9.9080
P-value	0.0240	0.3362	0.3028	0.2197	0.6006	0.3423	0.3919	0.0728	0.0071

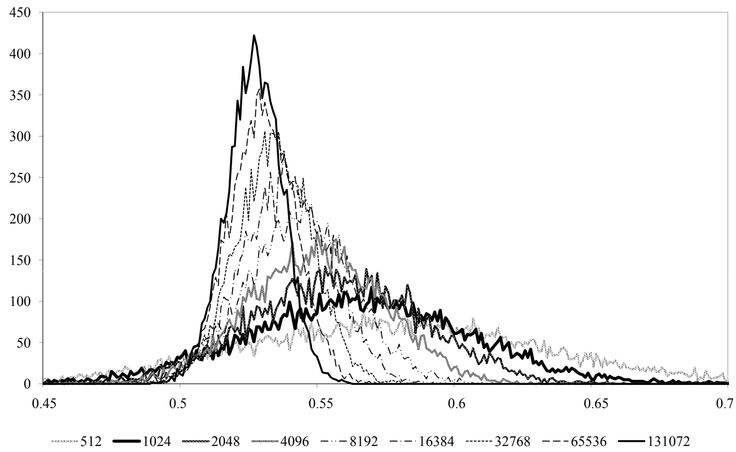

Figure 1. Histogram of Monte Carlo simulations (R/S)

outliers or simply extreme values are not averaged out (Peters 1994; Grech and Mazur 2004; Matos et al. 2008; Alvarez-Ramirez et al. 2005; Einstein et al. 2001). The same procedure is applied for DFA-1 later.

The expected values of Hurst exponent and corresponding descriptive statistics together with Jarque-Bera test (Jarque and Bera 1981) for normality are summarized in Table 1 and histograms are showed in Figure 1.

The estimates of Hurst exponent are not equal to 0.5 as predicted by asymptotic theory. Therefore, one must be careful when accepting or rejecting hypotheses about long-term dependence present in time series solely on its divergence from 0.5. This statement is most valid for short time series. However, the Jarque-Bera test rejected normality of Hurst exponent estimates for time series lengths of 512, 65,536 and 131,072 and therefore, we should use percentiles rather than standard deviations for the estimation of confidence intervals (Weron 2002). Nevertheless, the differences for mentioned estimates not normally distributed are only of the order of the tenths of the thousandth and therefore, we present confidence intervals based on standard deviations for R/S. Standard deviation can be estimated as

$$\hat{\sigma}(H) \approx \frac{1}{\pi T^{0.3}} \tag{9}$$

with R^2 of 98.55% so that the estimates are very reliable (Figure 2). Therefore, we propose (9) for other time series lengths but for the same minimum and maximum scales only as the estimates can vary for different scales choice (Peters 1994; Weron 2002; Couillard and Davison 2005; Kristoufek 2009).

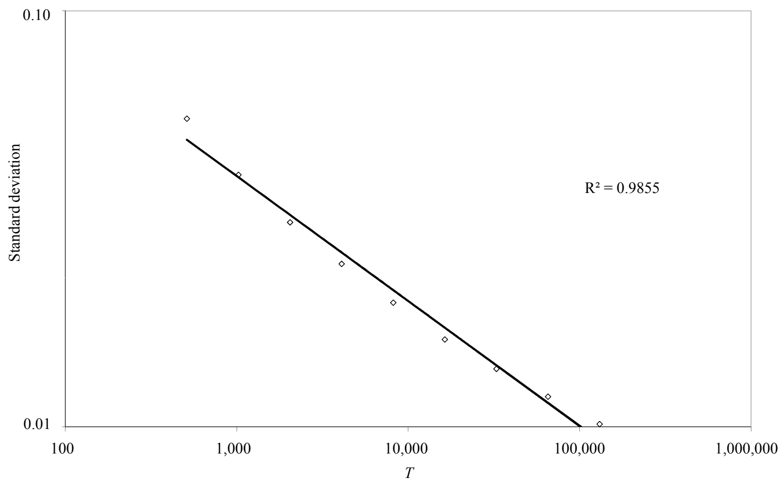

Figure 2. Standard deviations based on Monte Carlo simulations (R/S)

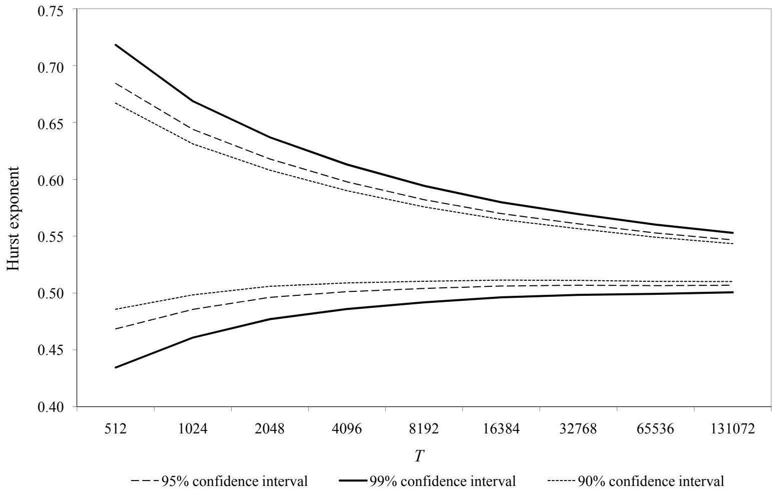

Figure 3. Confidence intervals for R/S

In Figure 3, we present the estimated confidence intervals for 90%, 95% and 99% two-tailed significance level. From the chart, we can see that all shown confidence intervals are quite wide for short time series. Even if time series of 512 observations yields H equal to 0.65, we cannot reject the hypothesis of no long-term dependence in the process even at 90% significance level.

3.2 DFA

DFA-1 was already shown to estimate Hurst exponent with expected value close to 0.5 for random normal series (Weron 2002; Grech and Mazur 2005) so that there is no need for similar procedure as for rescaled range presented before. We present the results of simulations for DFA-1 with minimum scale of 16 observations and maximum scale of one quarter of the time series length as was the case for R/S.

Table 2. Monte Carlo simulations descriptive statistics (DFA)

	512	1024	2048	4096	8192	16384	32768	65536	131072
Mean	0.5079	0.5062	0.504	0.5031	0.5025	0.5022	0.502	0.5015	0.5013
SD	0.0687	0.0500	0.0386	0.0304	0.0247	0.0202	0.0173	0.0149	0.0126
Skewness	0.1189	0.0630	0.0430	−0.0069	0.0053	−0.0258	−0.0398	−0.0227	−0.0323
Kurtosis	−0.0205	−0.0512	−0.0796	−0.0711	−0.0795	−0.0739	−0.0051	0.0109	−0.0919
Jarque-Bera	23.741	7.7276	5.7584	2.2171	2.7205	3.4246	2.658	0.899	5.3017
P-value	0.0000	0.0210	0.0562	0.3300	0.2566	0.1804	0.2647	0.6379	0.0706

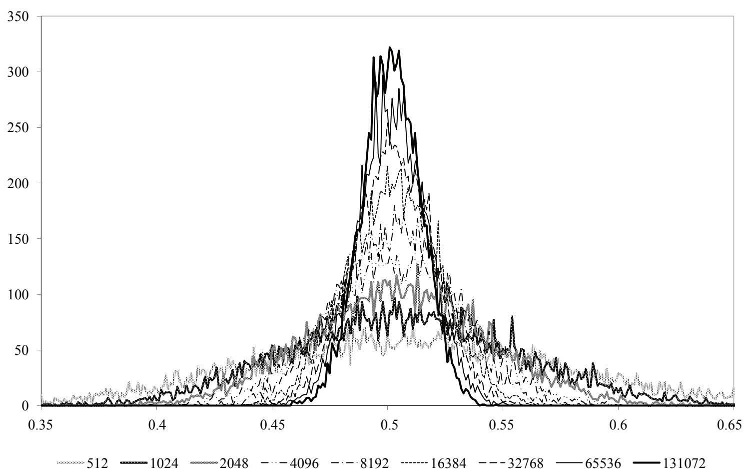

Figure 4. Histogram of Monte Carlo simulations (DFA)

Figure 4 and Table 2 show that expected values for DFA-1 are very close to the asymptotic limit of 0.5 even for short time series. Normal distribution of the simulated Hurst exponents cannot be rejected with exception for two lowest scales. Therefore, we stick to the use of standard deviations for estimation of confidence intervals. The standard deviation can be modeled as

$$\hat{\sigma}(H) \approx \frac{0.3912}{T^{0.3}}. \tag{10}$$

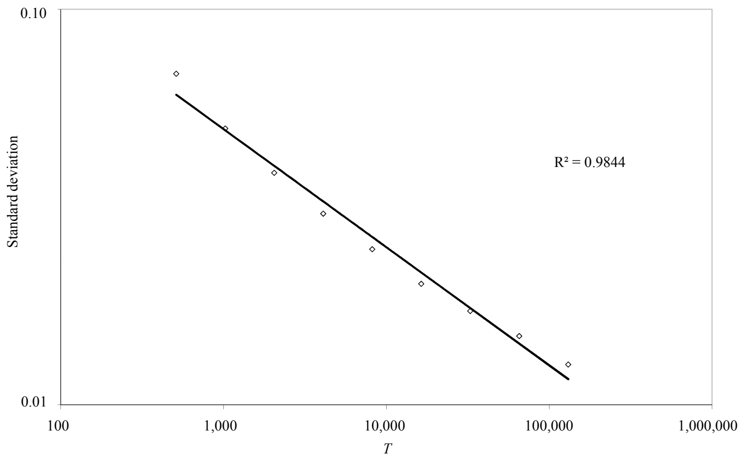

Figure 5. Standard deviations based on Monte Carlo simulations (DFA)

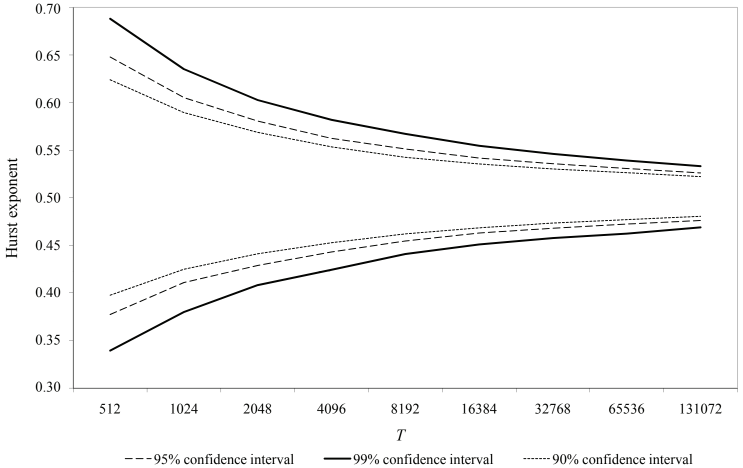

Figure 6. Confidence intervals for DFA

The evolution of standard deviation for different time series lengths together with the fit are shown in Figure 5. The fit is again reliable with R^2 equal to 98.44%. Note that power values in both (9) and (10) are equal to 0.3 which might be the case of future research. The estimates for the expected value of Hurst exponent are close to 0.5 so that we do not present any approximation for different time series lengths. Therefore, we propose to use 0.5 as the expected values and our approximation of standard deviation for construction of confidence intervals for different time series lengths than the ones we present.

Even though the expected values are in hand with asymptotic limit, the constructed confidence intervals are still rather wide (Figure 6) and rejection of hypothesis for short time series might be again quite problematic. Nevertheless, the confidence intervals are quite narrow for long time series. However, the most interesting results come if, for a single time series, we estimate Hurst exponent with both R/S and DFA-1 and compare the results. We present the results in detail in the following section.

4. Simultaneous finite sample properties

We again simulated 10,000 random standardized normally distributed $N(0,1)$ time series for each set length. This time, we estimated Hurst exponent based on both R/S and DFA-1 on each time series while estimating the results for the lengths from 256 to 131,072 observations. Descriptive statistics for differences between estimates of R/S and DFA-1 are summed in Table 3. The results show that R/S on average overestimates

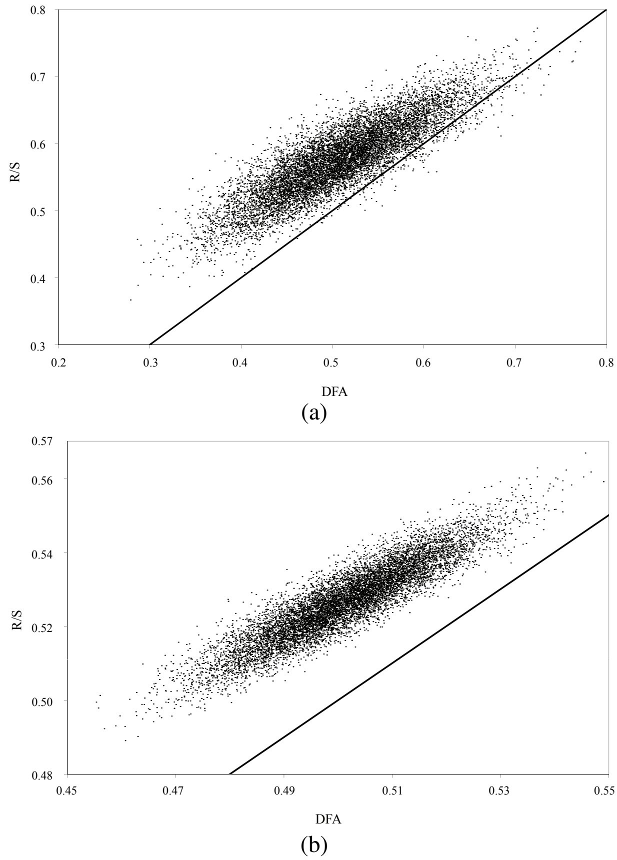

Figure 7. Comparison of R/S and DFA-1 estimates

Table 3. Descriptive statistics of differences between R/S and DFA estimates

	256	512	1024	2048	4096	8192	16384	32768	65536	131072
Mean	0.0783	0.0687	0.0598	0.0525	0.0458	0.0406	0.0358	0.0321	0.0285	0.0256
SD	0.0573	0.0351	0.0239	0.0174	0.0136	0.0110	0.0089	0.0075	0.0063	0.0054
Max	0.3159	0.2130	0.152	0.1130	0.0989	0.0861	0.0750	0.0624	0.0600	0.0477
Min	−0.1143	−0.0726	−0.032	−0.0073	−0.0057	−0.0059	0.0035	0.0081	0.0059	0.0052
$P_{97.5}$	0.1933	0.1394	0.1074	0.087	0.0734	0.0626	0.0541	0.0472	0.0410	0.0366
$P_{2.5}$	−0.0320	0.0012	0.014	0.0193	0.0202	0.0195	0.0189	0.0177	0.0167	0.0151
Skew.	0.1114	0.0832	0.0962	0.0944	0.1539	0.0849	0.1523	0.1217	0.1263	0.1177
Kurt.	0.1187	0.0829	0.0192	−0.0332	0.0992	0.0947	0.1030	0.0252	0.0417	0.1214
J.-B.	26.565	14.394	15.584	15.319	43.585	15.737	43.064	24.955	27.292	29.221
P-value	0.0000	0.0007	0.0004	0.0005	0.0000	0.0004	0.0000	0.0000	0.0000	0.0000

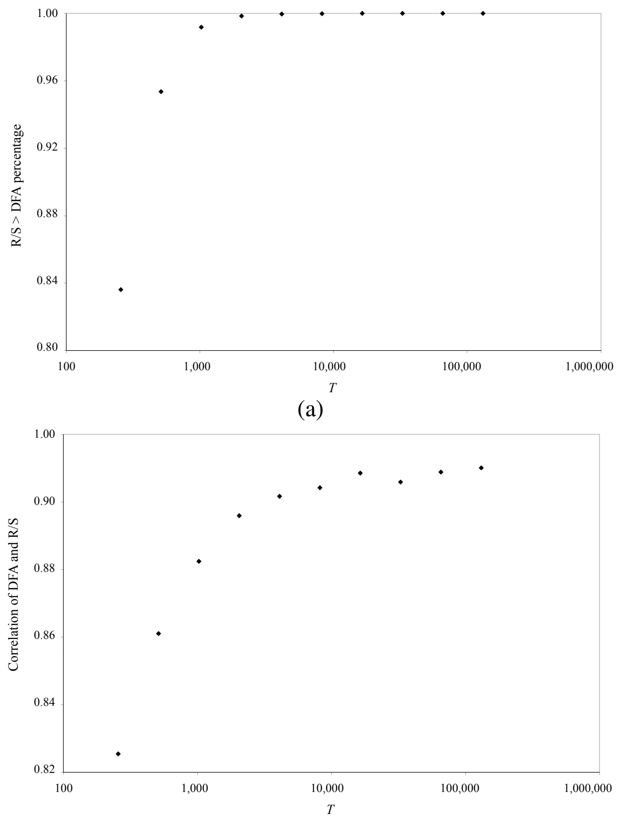

Figure 8. Comparison of R/S and DFA-1 estimates and corresponding correlations

Hurst exponent when compared to DFA-1 while the overestimation decreases with growing time series length. For illustration, we present Figure 7 which shows the estimates for both techniques for the time series lengths of 512 and 131,072.

From the figure, we can see that estimates are both strongly correlated and also that the relationship between both estimates is rather linear and not related in more complicated way. Moreover, the overestimation of Hurst exponent by R/S is evidently decreasing with the time series length. The proportion of estimates which are higher for R/S than for DFA-1 is illustrated in Figure 8a. From the time series of length 4,096 onwards, all of the estimates are higher for R/S. Figure 8b shows the evolution of correlations between the estimates of the used methods for different time series lengths. We can see that the correlations are quite high even for short time series and convergence above the value of 0.9 for the time series with more than 2,048 observations.

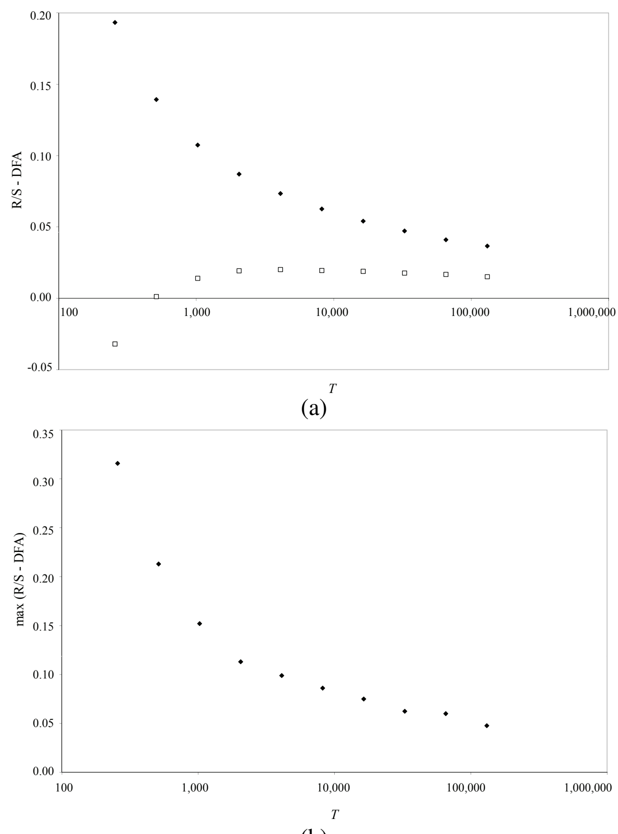

Figure 9. Comparison of R/S and DFA-1 percentiles and maximum differences

Different aspects are shown in Figure 9. Percentiles (97.5% and 2.5%) show that the estimates can differ significantly for low scales. The difference can be as high as 0.32 for time series length of 256 observations. Nevertheless, the difference narrows significantly for longer time series. The statistics are summed in Table 4.

Table 4. Further statistics

	256	512	1024	2048	4096	8192	16384	32768	65536	131072
Correlation	0.8255	0.8611	0.8825	0.8960	0.9017	0.9043	0.9086	0.9059	0.9089	0.9101
R/S $>$ DFA (%)	0.8362	0.9536	0.9918	0.9984	0.9996	0.9998	1.0000	1.0000	1.0000	1.0000
Max. difference	0.3159	0.2130	0.1520	0.1130	0.0989	0.0861	0.0750	0.0624	0.0600	0.0477

However, the most important findings, which contradict results in Weron (2002), are based on results of estimated standard deviations of Hurst exponents. R/S is generally considered as the less efficient method and is replaced by DFA in majority of recent applied papers (Grech and Mazur 2004; Czarnecki et al. 2008; Alvarez-Ramirez et al. 2008). Reasons for such replacement are usually stated as bias for non-stationary data and general overestimation of Hurst exponent of R/S. However, we have already shown that the overestimation is built in the procedure for finite samples (as was already shown in Weron 2002, Couillard and Davison 2005, Peters 1994). Moreover, non-stationarity is usually not the case for the financial time series while the statement is more valid for daily data which are mostly examined (Cont 2001). Further, as we show in Figure 10, the standard deviations are lower for R/S than for DFA-1 for all examined time series lengths. Therefore, also confidence intervals are narrower for R/S which makes the long-term dependence better testable by this procedure. The values of the standard deviations are more important than expected values of the Hurst exponent for the hypothesis testing. Nevertheless, we need to keep in mind that expected values for Hurst exponent based on R/S for finite samples are far from the asymptotic limit.

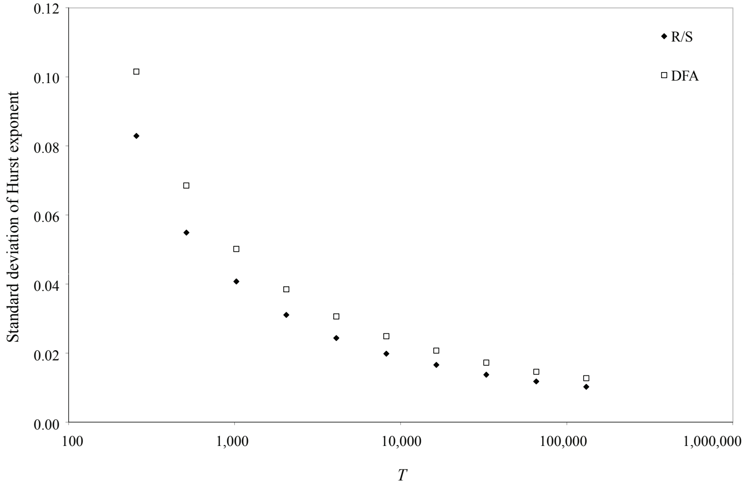

Figure 10. Comparison of standard deviations of R/S and DFA-1

5. Conclusions and discussion

We have shown that rescaled range analysis can still stand the test against new methods. Our comparison with detrended fluctuation analysis has supported the known fact that R/S overestimates Hurst exponent. However, the overestimation is in hand with estimates of Anis and Lloyd (1976) and thus is not unexpected. Importantly, the standard deviations of R/S are lower than those of DFA-1 which is crucial for the construction of confidence intervals for hypothesis testing. The results are different from the ones of Weron (2002) who asserts that DFA-1 is a clear winner when compared to R/S. Such difference is caused by different choice of minimum and maximum scales for Hurst exponent estimation. Our results are based on recommendations of several other authors (Peters 1994; Grech and Mazur 2004; Matos et al. 2008; Alvarez-Ramirez et al. 2005; Einstein et al. 2001) so that we use minimum scale of 16 observations with maximum scale equal to a quarter of time series length. The choice of scales is thus crucial for final results and its research should be of future interest.

Nevertheless, we show that both methods show similar results which become closer as the time series becomes longer. We show that testing the hypothesis of no long-range dependence for short time series, especially with 256 and 512 observations, can be complicated as the confidence intervals are very broad.

Acknowledgment The support from the Czech Science Foundation under Grants 402/09/H045, 402/09/0965, the Grant Agency of Charles University (GAUK) under project 118310, Ministry of Education MSMT 0021620841 and project SVV 261 501 are gratefully acknowledged.

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