Pexeso ("Concentration game") as an arbiter of bounded-rationality models

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Abstract. Among board games, Pexeso ("Concentration game") for two players is a game almost purely output-oriented, i.e. the optimal play is not given by strategic plans with long-term horizon (e.g. no short-term tactical sacrifice is observed). So, the optimal strategy and game dynamics may be calculated almost analytically, assuming a given rationality restrictions of the players.

In the paper, the optimal strategy for two players is solved using dynamic programming. Further, it is proved that for rational players, the game would end with "stalemate" (the game never ends) with a probability close to 1.

Further, the game dynamics is described if a rationality restriction is given such that the players perform a random move instead of the optimal one with probabilities $P, Q > 0$. In this case, the probability of a stalemate is equal to zero.

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1 Introduction

Concentration game, (also known as "Memory", "Pelmanism", "Pexeso", "Shinkei-suijaku" or "Pairs") is a card board game.

At the beginning, there are $2n$ cards on the board face down, consisting of n pairs, where each pair of cards has the same symbol. The aim of the players is to collect pairs of cards.

In the sense of Game theory it is zero-sum sequential finite stochastic game with perfect information. This is way to solve stochastic-optimal strategy (and evaluate P(finishing game) too) with using Dynamic programming methods ([3]). For real players, Concentration game has sense if one assumes restriction of a perfect rationality of the players, in particular as a consequence of non perfect medium-time memory.

In this article, we describe first the dynamics of a game of two completely rational players (with perfect memory). The result is presented using a recurrent formula, obtained by means of dynamic programming for each of the positions $[n, k]$ ($2n$ cards on the board, k of those known).

Further we show that the optimal play of two rational players leads with a high probability to a position, where the player whose turn it is is disadvantaged. The simplest such position is $[2, 0]$ (two pairs of card are on the table, none of those is known yet).

The player on the turn is forced by the rules of the game to accept this disadvantage – he is forced to turn two cards face up. In the majority of similar positions (e.g. $[4, 3]$), the player on turn may get rid of this disadvantage by "pretending restricted rationality": he turns over two known cards. If both players continue this way, the game ends with a draw. The probability that a game starting with $[n, 0]$ of two perfect rational players will be really played out, is only $O(e^{-n})$.

From observing real players, we see that a draw almost never happens. Being an empirical proof of the rationality restriction of real players, it may further be used as a first test of the relevance of a model using players with restricted rationality. We assume such a model to be more realistic, if it implies that a draw happens rarely or never.

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2 The rules of Concentration game

On the board, there are $2n$ cards face down. The player on the turn turns a card face up and then another card. He and all other players see the symbols on the two cards. If the symbols are equal, he collects the pair of cards and continues with another pair of cards. If not, other players continue. Real rules add the condition that cards with nonequal symbols must be again turned face down so that the players cannot see them; in case of perfect rational players, this requirement is superfluous.

The game continues until any cards remain on the board. The payoff of a player is the number of acquired cards. In case of two players, we will measure the payoff by the difference of the number of cards of a player and his opponent.

3 Dynamic programming results

4 Rational players

4.1 Payoffs for 2 rational players

We may assume that a player on the turn collects first all pairs of cards that are known and were not collected by the opponent. The rules admit the following follow-up possibilities:

0) Turn over two known cards

1) Turn over first a known card, then an unknown card

2) Turn over first an unknown card, then

a) in case it builds a pair with a known card

a1) turn over an unknown card

a2) turn over the corresponding known card

b) if it does not build a pair with any known card

b0) turn over another unknown card

b1) turn over any known card

Some positions exclude some of the mentioned strategies, in particular in case $[n, 0]$, the strategies 0), 1), 2a1) and 2b1) are excluded and the case $[n, 1]$ excludes 0).

Substrategies a1, a2 are dominated by a0 and may be omitted.

Let as denote for the position $[n, k]$, $k \leq n$ $A(n, k) =$ E(payoff of the player on turn minus the payoff of the opponent). We first derive a general recurrent formula for $k \geq 2$ based on strategy 2:

Turning over the first card leads with a probability $p = \frac{k}{2n-k}$ into the state 2a) and with $1-p = \frac{2n-2k}{2n-k}$ into the state 2b)

In case of 2a) the player scores a point and continues further from the position $[n-1, k-1]$, so his payoff is $1 + A(n-1, k-1)$.

In case of 2b), he further chooses between b0) and b1).

b0) avoids risk and leads to the payoff $-A(n, k+1)$

b1) gives

– with probability $\frac{1}{2n-k-1}$ one point plus continuing in the position $[n-1, k]$, i.e. $1 + A(n-1, k)$

– with probability $\frac{2n-2k-2}{2n-k-1}$ loss of being on turn and the opponents comes to a position $[n, k+2]$,

– with probability $\frac{k}{2n-k-1}$ loss of being on turn and a point to the opponent and a position $[n-1, k+1]$ to the opponent.

With using the DP Algorithm (combine forward & backward methods) [1], the expected profit coming from the position $[n, k]$ for $k \geq 2$ is then

$$A(n,k) = \frac{k}{2n-k}(1 + A(n-1,k-1)) + \frac{2n-2k}{2n-k} max[-A(n,k+1), \frac{1}{2n-k-1}(1 + A(n-1,k))-$$

$$-\frac{k}{2n-k-1}(1 + A(n-1,k+1) - \frac{2n-2k-2}{2n-k-1}A(n,k+2)]$$

The special cases $A(n,0)$ will be derived by excluding some of the possibilities in the general case:

$$A(n,0) = \frac{1}{2n-1}(1 + A(n-1,0)) - \frac{2n-2}{2n-1}A(n,2)$$

$$A(1,1) = 1, \quad A(n,k) = n \; if \; k \geq n$$

[4] uses the similar, but a bit different formula for derivation the simple asymptotic approximation of the best strategy.

Solving the equation for all $[n,k]$, $k \leq n$, $n \leq 64$ results in 754 positions such that $A(n,k) < 0$; from those, 672 (exactly one third) fulfills the condition $k \geq 2$, so the perfect rationality of both players leads to a never ending repetition of the strategy 0.

4.2 Probability of finishing a game

Based on similar considerations, we derive with using DP Algorithmus (combine backward & forward methods) [1], a formula for the probability $P(n,k)$ that a game starting in the position $[n,k]$ will be finished, i.e. that the state $[0,0]$ will be achieved. We assume that the player in the position $[n,k]$ chooses the strategy maximizing the expected value $A(n',k')$ of positions achiavable from $A(n,k)$. In particular, for $n-2 \geq k \geq 2$, this is one of the positions $[n-1,k-1]$, $[n,k+1]$, $[n-1,k]$, $[n,k+2]$.

Further: in case $A(n,k) < 0$, he chooses for $k \geq 2$ a strategy that leads again to $[n,k]$ with probability 1.

For $k \leq 2$ we have

$$P(n,k) = 0 \; if \; A(n,k) < 0$$

and for $k \geq 1$

$$P(n,k) = \frac{k}{2n-k}P(n-1,k-1) + \frac{2n-2k}{2n-k}Q(n,k)$$

where

$$Q(n,k) = \begin{cases} P(n,k+1) \\ \frac{1}{2n-k-1}(P(n-1,k)) + \frac{k}{2n-k-1}P(n-1,k+1) + \frac{2n-2k-2}{2n-k-1}P(n,k+2) \end{cases}$$

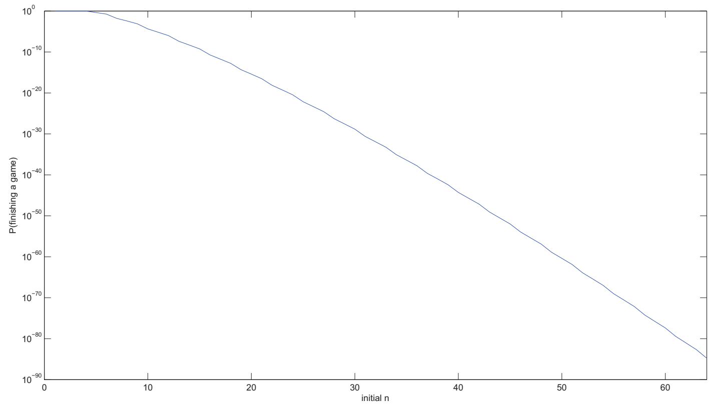
if

$-A(n,k+1) > \frac{1}{2n-k-1}(1 + A(n-1,k)) - \frac{k}{2n-k-1}(1 + A(n-1,k+1) - \frac{2n-2k-2}{2n-k-1}A(n,k+2)$.

Further

$$P(n,0) = \frac{1}{2n-1}P(n-1,0) + \frac{2n-2}{2n-1}P(n,2)$$

Maximal n s.t. $P(n,0) = 1$ is $n = 3$. $P(4,0) = 0.43$, $P(64,0) = 4.05 \cdot 10^{-87}$ (see plot).



5 Bounded rationality - erroneous players

Let as assume that both players think rationally, but instead of the chosen card they turn over a random card laying on the board with probabilities p and q respectively. The values p, q are known to the players before the game starts. We further assume that the error does not happen during the first part of one's move in which he just collects all the known pairs. We may equivalently consider an additional rule that if the position of both cards of a pair is known, the moderator turnes both cards face up.

We denote by $A(n, k)$ resp. $B(n, k)$ the expected values of the difference of the profit of a player and his opponent for the first, resp. the second player and consider the boundary condition $A(n, k) = B(n, k) = n$ for $k \geq n$.

For $p, q > 0$ the game achieves the state $[0, 0]$ with probability 1, because in each position $[n, k]$, the lowering of n after a turn may always happen by accident, with a probability at least p^2, resp. q^2.

6 Conclusion

The model of two completely rational players with perfect memory leads almost always to an infinite progress of the game (a draw); this may be an argument against perfect rationality. On the other hand, erroneous players finish the game with probability 1. This is however only a weak argument for realness of this particular model of restricted rationality.

Acknowledgements

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