Vaguely Motivated Cooperation

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Abstract The transferable utility (TU) cooperative games are used as an effective mathematical representation of cooperation and coalitions forming. This contribution deals with a modified form of such games in which the expected pay-offs of coalitions are known only vaguely, where the vagueness is modelled by means of fuzzy quantities and some other fuzzy set theoretical concepts. Such games were investigated in [8] and in some other papers. Their cores and Shapley values were analyzed and some of their basic properties were shown. This contribution is to extend that analysis, namely from the point of view of the motivation of players to cooperate in coalitions, as well as the relation between the willingness to cooperate and the ability to find the conditions under that the cooperation can be percepted as fair.

Key words: Cooperative game, TU-game, Fuzzy characteristic function, Fuzzy Shapley value, Willingness for cooperation.

1 Introduction

The concepts of coalition and bargaining, introducing the cooperative behaviour of players into games of strategy, appear in the game theory since its very beginning [14] and they form its significant component in many fundamental works (see, e. g., [7] or [15]). The coalition forming is, essentially, based on the expectations of further development of the game. It regards both – the structure of realized coalitions, as well as their presumed incomes. The expectations are mostly rather vague than stochastic, where the vagueness follows mostly from the subjectivity existing in the estimations and evaluations of the acceptability of particular potential results of bargaining. The theory of fuzzy set (where the seminal work is [17]) offered the game theory effective tools for mathematical processing of vagueness, mentioned above.

First, the attention was focused on the coalitions forming and its modifications influenced by the vagueness. Fuzzy coalitions defined as fuzzy subsets of the all-players set, allow the parallel participation of some player in several coalitions (see, e. g. [1, 2, 3]) and this model is investigated till now (e. g., [5, 11, 12, 13]). Other demonstration of vagueness in the cooperative games, i. e., the uncertainty regarding the expected incomes of coalitions and its distribution among their members, was investigated rather later. It was briefly mentioned in [8], and more thoroughly analyzed in [9], and recently this model is developed in several other papers.

The aim of this paper is to contribute to the understanding of methodical tools used in [9], and to interpret the conclusions derived there. Our main attention is oriented to the phenomenon of forming cooperation from the point of view of the players' motivation under uncertainty on the expected pay-offs.

The process of negotiation on the eventual cooperation includes two principal periods. Each of them is connected with specific level of motivation, and also its aim reflects different tightness of the accepted agreement.

— The first period covers the stage of a non-cooperative game of strategy, in which the players recognize the advantages of the cooperative actions coordinated with other players. The external attribute of this period is the forming of coalitions based on the knowledge of their expected gains. The classical deterministic game theory has developed the concept of core whose non-emptiness indicates the convenience of the universal coalition of all players.
— The second period includes rather more empathy among all players. The core, following from (in some sense given) expected incomes of coalitions, is very rarely a one-element set. If it is not empty then it offers many potential distributions of the total income of coalition among its members. The choice of the very distribution is the second period of the cooperative negotiations. In distinction from the first period, this choice does not immediately follow from the formal properties of the coalitional pay-offs. It is necessary to combine their values with some, more or less subjective, idea of rightful rates of particular players on the total profit of the coalition. As every player has, in too many cases, his own idea of such justice, there has to be an external authority, either some "judge" not belonging among the players, or some general, commonly acceptable rule, re-distributing the profit. The classical theory of cooperative games with transferable utility offers the Shapley value (see [16, 15, 7]) as such rule possessing acceptable and rational properties of the individual pay-offs.

The periods of negotiation mentioned above are, in the deterministic game model, solved for a long time, already. The theory of fuzzy cooperative games in the form with fuzzy characteristic function analyses these concepts and methods especially in [9]. Nevertheless, some problems regarding the mutual relation between fuzzy core and fuzzy Shapley value were passed or, at most, only registered without more thorough discussion. In this paper, we aim to contribute to their analysis by methodological comments and several general results dealing with the two periods of negotiation, mentioned above.

The following sections are organized as follows. The next Section 2 summarizes the basic concepts which are dealt in the rest of this contribution, and including the notions of fuzzy core and fuzzy Shapley value. The general conclusions following from this analysis and regarding the motivation of player to the cooperation are presented in Section 3. The last Section 4 includes a conclusive remark.

2 The Models

The basic elements of both models of cooperative game with transferable utility analyzed in this paper – the deterministic one and its fuzzy extension – are briefly recollected in this section. The deterministic case is well known from the classical literature (see, e. g., [7] or [15]), meanwhile its fuzzification was suggested in [9]. The subsection dealing with the deterministic model is completed by the concepts of core and Shapley value. Their fuzzified counterparts are analyzed in the last subsection of this section.

2.1 TU Cooperative Game – Deterministic Case

Let us recollect, first, the fundamental definitions of the model of *cooperative game with transferable utility* (TU-game).

In the whole paper, we denote by R the set of all real numbers.

If M is a set, then we denote in the following sections by $\mathscr{P}(M)$ the set of all subsets of M (the potential set of M).

The TU-game is defined as a pair (I, v), where $I = \{1, 2, \ldots, n\}$ is a non-empty and finite *set of players* and $v : \mathscr{P}(I) \to R$ such that $v(\emptyset) = 0$ is called *characteristic function* of the game.

Every real-valued vector $(x_i)_{i \in I} \in R^n$ such that $x_1 + x_2 + \cdots + x_n \leq v(I)$ is called an *imputation* in the TU-game (I, v). The basic solution concept in such game is the set of imputations $C \subset R^n$ called the *core* of the game and such that

$$C = \left\{ x \in R^n : \sum_{i \in I} x_i \leq v(I) \text{ and for all } K \in \mathscr{P}(I), \sum_{i \in K} x_i \geq v(K) \right\}. \quad (2.1)$$

It is evident that the coalition I of all players can be effectively formed in a TU-game, only if its core C is non-empty. Moreover, the non-emptiness of core is the single information which the players need to recognize that the coalitional cooperation over complete set I is desirable.

Comment 1. The players need not any exogenous authority to conclude if the cooperation covering complete set I is possible, i. e., if it can be useful for all of them. The non-emptiness of the core follows from the definitoric elements of the game, I and v, and it does demand any other assumption or rule.

Anyhow, the construction of the core C itself does not mean that the negotiations are finished. If $C \neq \emptyset$ then it usually contains more than one imputation and each of the players can have his own idea of the core imputations which is the most righteous one. The structure of the given game itself, i. e., the pair (I, v) and its knowledge, does not guarantee the objective choice of a distribution of the value $v(I)$ among the players, respected and accepted by all of them.

Comment 2. The critical moment of the negotiation is the step from the retrieval of the core and its reduction on one single imputation. In other words, the players themselves are able to recognize the necessity of cooperation but they are not able to agree spontaneously with one single imputation distributing the common profit among them.

Hence, it is inevitable to include an additional element of the game, an arbiter, who decides which distribution of the profit is righteous.

In the practical negotiation, the arbiter can be a person whose authority is confirmed by all agents (players). But it can be an abstract scheme, too, accepted by all players even before the negotiations process. The game theoretical models usually do with a set of principles respected and accepted by the players. These principles were formulated in [16] and the distribution of profits forms a real-valued vector $t = (t_i)_{i \in I}$ called a vector of *Shapley values* (see, e. g., [15, 7, 9]). The principles mentioned above are as follows.

- The values $t_i,\ i \in I$, do not depend on the ordering of players.
- Vector of Shapley values $(t_i)_{i \in I}$ is to be an imputation, such that

$$\sum_{i=1}^{n} t_i = v(I).$$

- If (I, v_1), (I, v_2) are two TU-games over the set of players I and $(t_i(v_1))_{i \in I}$, $(t_i(v_2))_{i \in I}$ are vector of Shapley values, respectively, if $(I, v_1 + v_2)$ is a TU-game such that for each $K \in \mathscr{P}(I)$,

$$(v_1 + v_2)\,(K) = v_1(K) + v_2(K),$$

  and if $(t_i(v_1 + v_2))_{i \in I}$ is the vector of Shapley values for $(I, v_1 + v_2)$ then

$$t_i(v_1 + v_2) = t_i(v_1) + t_i(v_2) \quad \text{for all } i \in I.$$

Note that the non-emptiness of Core is not demanded. Shapley (see, [16]) has constructed an effective formula for the evaluation of t_i, $i = 1, \ldots, n$, namely, if for every $K \in \mathscr{P}(I)$, k is the number of players in K, then

$$t_i = \sum_{K \in \mathscr{P}(I)} \frac{(n-k)!(k-1)!}{n!} (v(K) - v(K - \{i\})), \quad i \in I. \tag{2.2}$$

Let us note that formula (2.2) defines a vector of Shapley values $(t_i)_{i\in I}$ fulfilling the above conditions even if the core C of the game (I, v) is empty, under the assumption that for any K, $L \subset I$ such that $K \cap L = \emptyset$, the inequality

$$v(K \cup L) \geq v(K) + v(L) \tag{2.3}$$

holds. Of course, in such case $(t_i)_{i\in I} \notin C$. If $C \neq \emptyset$ then $(t_i)_{i\in I} \in C$,

2.2 Fuzzy Quantities

The above deterministic TU-game model is well know, relatively simple, but its correspondence with real cooperative behaviour is rather limited by the latent assumption that the values $v(K)$, $K \subset I$, are deterministic real numbers. Such precise knowledge preceding the proper realization of the game appears too optimistic. This discrepancy can be avoided by using the concepts of fuzzy quantities theory.

In the rest of this paper, if M is a set then we denote by $\mathscr{F}(M)$ the class of all *fuzzy subsets* of M (cf. [17]).

If $A \in \mathscr{F}(M)$ then $\mu_A : M \to [0,1]$ is the *membership* function of A. Any $a \in \mathscr{F}(R)$ with $\mu_a R \to [0,1]$ such that

$$\begin{aligned}
&\mu_a(x_a) = 1 \quad \text{for at least one } x_a \in R, && (2.4)\\
&\text{there exist } x_1,\, x_2 \in R \text{ such that } x_1 \leq x_a \leq x_2 && (2.5)\\
\text{and}\ \ &\mu_a(x) = 0 \quad \text{for all } x \notin [x_1, x_2], &&
\end{aligned}$$

is called a *fuzzy quantity.* Each real number x_a fulfilling (2.4) is called a *modal value* of a. The set of all fuzzy quantities will be denoted by $\mathscr{F}^*$. As shown, e. g., in [4, 8] and many other works, it is possible to define algebraic operations over $\mathscr{F}^*$, using so called *extension principle.* In this paper, we use two of algebraic operations over fuzzy quantities. Let us consider a, $b \in \mathscr{F}^*$ and $r \in R$, then the *sum* $a \oplus b$ and *crisp product* $r \cdot a$ are fuzzy quantities, too. Their membership functions are

$$\begin{aligned}
&\mu_{a\oplus b}(x) = \sup_{y\in R} \left[\min\left(\mu_a(y), \mu_b(x-y)\right)\right], \quad \text{for } x \in R, && (2.6)\\
&\mu_{r\cdot a}(x) = \mu_a(x/r) \quad \text{if } r \neq 0, \text{ and} && (2.7)\\
&\mu_{0\cdot a}(0) = 1, \quad \mu_{0\cdot a}(x) \;=\; 0 \quad \text{for } x \neq 0. &&
\end{aligned}$$

There exist numerous approaches to the ordering relation between fuzzy quantities (see, e. g., [6]). Here we use the one of them which is defined as a fuzzy relation $\geq$ with membership function $\nu_{\geq} : \mathscr{F}^* \times \mathscr{F}^* \to [0,1]$, where for a, $b \in \mathscr{F}^*$

$$\nu_{\geq}(a, b) = \sup\left[\min\left(\mu_a(x), \mu_b(y)\right) : x,\, y \in R,\, x \geq y\right] \tag{2.8}$$

is the possibility that $a \geq b$.

2.3 Fuzzy Extension of a TU-game

As we have mentioned above, we consider here the fuzzification of the characteristic function v. If for every coalition $K \in \mathscr{P}(I)$ there exists a fuzzy quantity $w(K) \in \mathscr{F}^*$ such that $v(K)$ is a modal value of $w(K)$, then we say that the pair (I, w) is a *fuzzy extension* of the TU-game (I, v), and we call w the *fuzzy characteristic function* of (I, w).

It is not difficult to define the set of *fuzzy imputations* in (I, w) as a fuzzy subset $\mathscr{V}$ of R^n with membership function $\mu_{\mathscr{V}} : R^n \to [0, 1]$, where

$$\mu_{\mathscr{V}}((x_1, \ldots, x_n)) = \nu_{\geq}\left(w(I), \sum_{i=1}^{n} x_i\right), \quad (x_1, \ldots, x_n) \in R^n, \tag{2.9}$$

where (2.8) was used.

Similarly, the *fuzzy core* of (I, w) is a fuzzy subset of R^n (see [9]) denoted by C_w and with membership function $\mu_C : R^m \to [0, 1]$, where for any $\boldsymbol{x} = (x_1, \ldots, x_n) \in R^n$

$$\mu_C(\boldsymbol{x}) = \min\left[\mu_{\mathscr{V}}(\boldsymbol{x}), \min\left(\mu_{\geq}\left(\sum_{i \in K} x_i, w(K)\right) : K \in \mathscr{P}(I)\right).\right] \tag{2.10}$$

Remark 1. Previous definition, together with (2.8) immediately mean that if $\mathscr{V}$ is a fuzzy imputation and C_w is a core of a fuzzy extension (I, w) of some *TU*-game (I, v), and if $\alpha \geq \beta$, α, $\beta \in [0, 1]$ then

$$\{\boldsymbol{x} \in R^n : \mu_{\mathscr{V}}(\boldsymbol{x}) = \alpha\} \subset \{\boldsymbol{x} \in R^n : \mu_{\mathscr{V}}(\boldsymbol{x}) = \beta\},$$

and

$$\{\boldsymbol{x} \in R^n : \mu_C(\boldsymbol{x}) = \alpha\} \subset \{\boldsymbol{x} \in R^n : \mu_C(\boldsymbol{x}) = \beta\}.$$

Comment 3. Analogously to the deterministic concept of imputation as an accessible distribution of profit among all players, the fuzzy imputation represents the accessibility of profit distribution structured by uncertainty connected with particular incomes expected by the coalition of all players I.

Comment 4. Similarly, the fuzzy core represents the distributions of global profit among all players, which cannot be effectively protested by any coalition, and which is structured by uncertainty connected with particular incomes expected by the coalition of all players.

There exists one concept, more, whose deterministic form we know and which turns into its fuzzy counterpart if a fuzzy extension of a TU-game is considered. Namely, the Shapley value (2.2). Its fuzzification can be constructed in two ways.

The first way consists in the passive application of formula (2.2) where fuzzy quantities $w(K)$, $K \in \mathscr{P}(I)$ are used instead of the crisp values $v(K)$. This method was analyzed in [9] and it is evident that it results into fuzzy quantities for every Shapley value. More precisely, let $i \in I$ and let us numerate the coalitions from $\mathscr{P}(I)$ as

$$\{K_0, K_1, K_2, \ldots, K_N\}, \quad \text{where } K_0 = \emptyset \text{ and } N = 2^n - 1. \tag{2.11}$$

Then the fuzzy quantity $s_i \in \mathscr{F}^*$ with $\mu_{s(i)}$, defined by

$$\begin{aligned} s_i = {} & \frac{(n-k_1)!(k_1-1)!}{n!} \left(w(K_1) \oplus (-1 \cdot w(K_1 - \{i\}))\right) \oplus \cdots \\ & \cdots \oplus \frac{(n-k_N)!(k_N-1)!}{n!} \left(w(K_N) + (-1 \cdot w(K_N - \{i\}))\right) \end{aligned} \tag{2.12}$$

can be considered for the i-th component of the vector of *fuzzy Shapley values.* Here, k_j is the number of members of the coalition K_j and all k_j, $j = 1, \ldots, N$, are crisp numbers. It means that operations used in (2.12) are fully characterized by (2.6) and (2.7) and their properties are analyzed, e. g., in [8] and [4] and recollected also in [9]. This method, however lucid it is, displays one significant discrepancy. Namely, if $i \notin K$ for some $K \in \mathscr{P}(I)$ then in the deterministic case $v(K) - v(K - \{i\}) = 0$, and formula (2.2) deals with coalitions including i, only. As shown in [8], this conclusion is not correct in the case of fuzzy extension (I, w) of (I, v). If $i \notin K$ then $w(K) \oplus (-1 \cdot w(K - \{i\})) = w(K) \oplus (-1 \cdot w(K)) = a(i, K)$, where $a(i, K)$ is a fuzzy quantity from $\mathscr{F}^*$ with at least one modal value equal to 0,

$$\mu_{a(i,K)}(0) = 1,$$

and with symmetric membership function where

$$\mu_{a(i,K)}(x) = \mu_{a(i,K)}(-x) \quad \text{for all } x \in R.$$

Usually, except very special cases with degenerated fuzziness, $a(i, K)$ used in (2.12) extends the uncertainty of the resulting fuzzy value $s_i \in \mathscr{F}^*$ and in this sense it influences the stability of eventually achieved results of negotiation. Namely, it symmetrically increases the extent of uncertainty connected with s_i.

This, in certain sense formal, discrepancy can be avoided if we limit the summation in (2.12) on the coalitions from $\mathscr{P}(I)$ for which $i \in K$. More formally, we may define a fuzzy quantity $q_i \in \mathscr{F}^*$ with membership function $\mu_{q(i)} : R \to [0, 1]$ by means of modified (2.12)

$$q_i = \frac{(n-k_1)!(k_1-1)!}{n!} \cdot \sigma(i,k_1) \cdot (w(K_1) \oplus (-1 \cdot w(K_1 - \{i\}))) \oplus \cdots \quad (2.13)$$
$$\cdots \oplus \frac{(n-k_1N)!(k_N-1)!}{n!} \cdot \sigma(i,k_N) \cdot (w(K_N) \oplus (-1 \cdot w(K_N - \{i\}))),$$

where

$$\sigma(i,K_j) = 1 \quad \text{iff } i \in K_j, \qquad \sigma(i,K_j) = 0 \quad \text{iff } i \notin K_j,\ j = 1,\ldots,N$$

and where the notations used in (2.12) are preserved.

The second way of constructing *fuzzy Shapley value* is based on the general extension principle, as well. For every $i \in I$ and for every K_j, $j = 0,1,\ldots,N$, we denote $\mu_j : R \to [0,1]$ the membership function of fuzzy quantity $w(K_j)$. Then we define fuzzy quantity $u_i \in \mathscr{F}^*$ with membership function $\mu_{u(i)} : R \to [0,1]$ by means of

$$\mu_{u(i)}(x) = \sup \Big[\min(\mu_1(y_1), \mu_2(y_2), \ldots, \mu_n(y_N)) : \quad (2.14)$$
$$y_1,\ldots,y_N \in R,\ x = \sum_{j=1,\ldots,N} \frac{(n-k_j)!(k_j-1)!}{n!} \left(y_j - y_{\ell_j}\right) \Big],$$

where n and k_j are interpreted in agreement with (2.12) and for every $j = 1,\ldots,N$, $K_{\ell_j} - \{i\}$.

The fuzzy number u_i is the i-th component of *fuzzy Shapley value* of (I,w).

Remark 2. It is easy to see that for $K_j \in \mathscr{P}(I)$ and $i \in I$ such that $i \notin K$ then $y_j - y_{\ell_j}$ and, consequently, the relevant element of the sum in (2.13) vanishes.

Lemma 1. Let (I,w) be fuzzy extension of a TU-game (I,v), let for any $i \in I$, $s_i \in \mathscr{F}^*$ be defined by (2.12), $q_i \in \mathscr{F}^*$ be defined by (2.13), and let $u_i \in \mathscr{F}^*$ be defined by (2.14). Then

$$\mu_{u(i)}(x) = \mu_{q(i)}(x) \quad \text{for all } i \in I,\ x \in R$$

and there exist b, $d \in \mathscr{F}^*$, such that

$$\mu_b(0) = \mu_d(0) = 1, \qquad \mu_b(x) = \mu_b(-x), \qquad \mu_d(x) = \mu_d(-x), \qquad x \in R,$$

and

$$u_i \oplus b = s_i \oplus d.$$

Proof. The first statement,

$$\mu_{u(i)}(x) = \mu_{q(i)}(x)$$

for all $i \in I$ and $x \in R$ follows from (2.13) and (2.14), immediately, as for any $K_j \in \mathscr{P}(I)$ such that $i \notin K_j$, $\sigma(K_j, i) = 0$ and $\mu_j(0) = 1$. In this sense, the values of μ_j do not influence the value of $\mu_{u_i}(x)$.

The second statement, namely the additive equivalence of u_i and s_i in the sense of [8], follows from (2.13) and (2.12). Namely, $s_i = q_i \oplus b$, where $b \in \mathscr{F}^*$, and b is the sum of fuzzy quantities

$$\frac{(n-k_j)!(k_j-1)!}{n!}\,(w(K_j) \oplus (-1 \cdot w(K_j)))$$

for those K_j for which $i \notin K_j$. Then each of such fuzzy quantities is symmetric, i. e., they fulfil the properties formulated in the proved statement. It means that their sum b is symmetric, as well (see [8]), and the second statement is proven. $\square$

If (I, v) is a TU-game and (I, w) its fuzzy extension, then the fundamental fuzzy solution concepts of (I, w) are fuzzy extensions of their crisp counterparts in (I, v). It is not difficult to formulate this conclusion by means of the following statements.

Theorem 1. Let (I, v) be a TU-game and (I, w) its fuzzy extension. If C and C_w are the core of (I, v) and fuzzy core of (I, w), respectively, then C_w is fuzzy extension of C. It means that for any $\boldsymbol{x} \in R^n$

$$\mu_C(\boldsymbol{x}) = 1 \quad \text{iff } \boldsymbol{x} \in C.$$

Proof. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in C$. Then

$$\begin{aligned} &\nu_{\geq}\left(\sum_{i\in K} x_i, w(K)\right) = 1 \quad \text{for all } K \in \mathscr{P}(I), \\ &\text{and} \quad \nu_{\geq}\left(w(I), \sum_{i=1}^{n} x_i\right) = 1 \end{aligned} \tag{2.15}$$

as follows from (2.8) and from the fact that each fuzzy imputation is a fuzzy extension from some crisp imputation. Consequently, $\mu_C(\boldsymbol{x}) = 1$.

Let, on the other hand, $\mu_C(\boldsymbol{x}) = 1$. Then, due to (2.10), all membership values in (2.15) are necessarily equal to 1, which immediately implies that $\boldsymbol{x} \in C$. $\square$

Theorem 2. Let (I, v) be a TU-game and (I, w) its fuzzy extension. Let $\boldsymbol{t} = (t_1, t_2, \ldots, t_n) \in R^n$ be (crisp) Shapley value of (I, v) defined by (2.2). Then the vector of fuzzy quantities $\boldsymbol{s} = (s_i)_{i\in I}$ defined by (2.12), vector of fuzzy numbers $\boldsymbol{u} = (u_i)_{i\in I}$ defined by (2.14), and vector of fuzzy quantities $\boldsymbol{q} =$

$(q_i)_{i \in I}$ defined by (2.15) are vectors of fuzzy extensions of t_i, $i = 1, 2, \ldots, n$, respectively.

Proof. Formulas (2.2) and (2.14) immediately imply that $u_i(t_i) = 1$ for all $i \in I$ and, consequently, fuzzy quantity u_i is a fuzzy extension of t_i for all $i = 1, 2, \ldots, n$. The first statements of Lemma 1 means that the above result is true for fuzzy quantities q_i and crisp Shapley values t_i, $i \in I$, as well. Moreover, formula

$$u_i \oplus b = s_i \oplus d$$

where $\mu_b(0) = \mu_d(0) = 1$, used in the second statement of Lemma 1, implies that if some $x \in R$ is a modal value of u_i then it is a modal value of s_i, as well (see (2.6)). Hence, t_i is a modal value of s_i for all $i \in I$, and the statement of the theorem is true. □

Remark 3. The relevant definitions immediately imply that if for all $j = 0, 1, \ldots, N$

$$\mu_j(v(k_j)) = 1, \qquad \mu_j(x) = 0 \quad \text{for all } x \neq v(K_j),$$

then C_w is identical with the crisp core C of (I, v), and all fuzzy Shapley values $(u_i)_{i \in I}$, $(q_j)_{i \in I}$ and $(s_i)_{i \in I}$ are equal and identical with crisp Shapley values $(t_i)_{i \in I}$ of (I, v).

3 Vague Willingness to Cooperation

After introducing or remembering the main concepts of interest, i. e., the fuzzy extensions of cooperative game with transferable utility, its core and Shapley value, we aim to transfer the ideas of crisp cooperation model into their fuzzy counterparts. Let us summarize the fundamental knowledge achieved in the deterministic theory:

- The non-emptiness of the core suffices to the recognition that rational cooperation is the optimal behaviour of players.
- But it does not suffice to identify which cooperation (partition of the common profit of I) is the rational one. Identification of this rationality demands the acceptance of an additional rule, the value (Shapley value) of the game.

The general properties of the value are summarized in subsection 2.1.

The main purpose of this section is to discuss the validity of the previous, rather methodological, consequences for the case of the fuzzy extension of a TU-game.

The vagueness of the expected pay-offs, i. e., the substitution of the crisp numbers $v(K)$ by fuzzy quantities $w(K)$ for all $K \in \mathscr{P}(I)$, may appear like a degradation of the conditions under which the players form the coalitions – our traditional thinking percepts the uncertainty as a discrepancy, in general. In fact, the existence of vagueness in the expectation of coalitional profits, enriches the analysis of the game and its structure. The fuzzy extension (I, w) of (I, v), with many levels of possibilities regarding the players' expectations, is not only much more realistic but also much more effective in the process of forming the most rational cooperative behaviour.

The fuzzy core is a lucid demonstration of the above rule. If (I, v) and (I, w) are a deterministic TU-game and its fuzzy extension, respectively, then the core C and fuzzy core C_w respect analogous relation. The fuzzy core C_w does not grind the crisp willingness for cooperation based on the deterministic expectations of profit but, on the contrary, it enlarges the potential possibilities of agreement by the (usually quite wide) class of not completely sure but possible variants. If we consider the fact that the "fully deterministic" expectations of profit made before the realization of the game cannot be as doubtless as they appear to be the fuzzy extension of the cooperation model is more realistic (and more precise) than the crisp one.

More formally, let us consider the fuzzy core C_w of a fuzzy extension (I, w) of TU-game (I, v). Then we may define the number

$$m_C = \sup\left(\mu_C(\boldsymbol{x}) : \boldsymbol{x} \in R^n\right), \tag{3.1}$$

which we call the *cooperative potential* of (I, w).

Remark 4. Evidently, $0 \leq m_C \leq 1$, and if (I, w), (I, w') are two fuzzy extensions of (I, v) such that $\mu_w(\boldsymbol{x}) \geq \mu_{w'}(\boldsymbol{x})$ for all $\boldsymbol{x} \in R^n$ and if m_C, m'_C are their cooperative potentials, respectively, then $m_C \geq m'_C$.

The cooperative potential can be accepted for the measure of ability of the players in I to accept the global all-players' coalition. The previous remark stresses the obvious fact that the more the fuzzy extension of (I, w) differs from its crisp base, the higher is the possibility that the players find a common agreement.

The question to be answered about the fuzzy Shapley value is rather different. Namely, it is important to know if, and in which way, the fuzzy Shapley value respects the general demands on values, formulated in subsection 2.1.

Here, we focus our attention on the fuzzy Shapley values defined by (2.14) and denoted by $u_i \in \mathscr{F}^*$. Due to Lemma 1, we know that its properties are identical with the properties of $q_i \in \mathscr{F}^*$ (defined by (2.13)) and in some sense equivalent with the properties of $s_i \in \mathscr{F}^*$. This is valid for all $i \in I$.

Remark 5. As follows from (2.14), immediately, the membership functions $\mu_{u(i)}$ for $i = 1, 2, \ldots, n$, are independent on the ordering of their computation.

Lemma 2. The modal values $t_1, t_2, \ldots, t_n$, $v(I)$ of the fuzzy quantities $u_1, u_2, \ldots, u_n$, $w(I)$ fulfil the equality

$$\sum_{i\in I} t_i = v(I).$$

Proof. The statement follows from (2.14) and (2.2), immediately. □

Lemma 3. Let (I, v_1) and (I, v_2) be TU-games and (I, w_1), (I, w_2) be their fuzzy extensions, respectively. Let $(I, w_1 \oplus w_2)$ be a fuzzy cooperative game such that for every $K \in \mathscr{P}(I)$

$$(w_1 \oplus w_2)(K) = w_1(K) \oplus w_2(K),$$

and, finally, let $(I, v_1 + v_2)$ be a TU-game such that for every $K \in \mathscr{P}(K)$

$$(v_1 + v_2)(K) = v_1(K) + v_2(K).$$

Then $(I, w_1 + w_2)$ is a fuzzy extension of $(I, w_1 \oplus w_2)$, modal values $(v_1 + v_2)(K)$ of $(w_1 \oplus w_2)(K)$, $K \in \mathscr{P}(I)$.

Proof. The statement follows from the definition of fuzzy extension of TU-game, and from the assumptions of this lemma. □

Lemma 4. Under the assumptions of Lemma 3 let us denote by t_i, $i = 1, 2, \ldots, n$, the Shapley values of (I, v). Then for every $i \in I$, t_i is a modal value of $u_i \in \mathscr{F}^*$.

Proof. The statement follows from (2.13) and (2.2), immediately. For every $K \in \mathscr{P}(I)$, $\mu_w(v(K)) = 1$ and, consequently, $\mu_{u(i)}(t_i) = 1$, $i = 1, 2, \ldots, n$. □

Note that the validity of the general properties of fuzzy Shapley values is in more detailed way investigated in [9], Chapter 9.

Comment 5. The vagueness included in the concept of fuzzy extension of a TU-game influences also the validity of the general principles connected with the concept of the value, especially of the Shapley value. Relative generality of the fuzzy characteristic function implies also rather free formal structure of fuzzy Shapley value, and the guaranteed fulfillment of the basic properties of value for the modal values of its fuzzified form, only.

4 Conclusive Remark

The previous brief analysis of the fuzzified TU-games where the fuzzification regards the characteristic function and concepts derived from it, allows to formulate the following heuristic conclusion.

The vagueness of expected pay-offs, which is natural, rather influences the formal properties of the core and Shapley value, but it does not violate their very important functions – namely, to indicate the motivation of player for cooperation, and to show an acceptable distribution of the common profit among cooperating players. Or course, the vagueness of expectations causes certain vagueness of the concepts of core and Shapley value, but this vagueness does not limit the information hidden in the core, and it rather modifies than limits similar information in the Shapley value.

In other words the fuzzification of pay-offs, in principle, does not significantly influence the ability and willingness of the players to cooperate.

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