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An inequality related to Minkowski type for Sugeno integrals

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ABSTRACT

An inequality related to Minkowski type for the Sugeno integral on abstract spaces is studied in a rather general form. Some previous results on Chebyshev type inequality obtained by the authors are generalized. Several examples are given to illustrate the validity of this inequality. The conditions such that this inequality becomes an equality are also discussed. Finally, conclusions and some problems for further investigations are included.

1. Introduction

Since the concept of fuzzy sets [37] was introduced, it has been comprehensively investigated [11,34,36]. Fuzzy measures and fuzzy integrals can be used for modelling problems in non-deterministic environment. Since Sugeno [32] initiated research on fuzzy measures and fuzzy integrals, this area has been widely developed and a wide variety of topics have been investigated (see, e.g., [3,4,25–28,30,34] and references therein). Fuzzy integrals (also known as Sugeno integrals) have very interesting properties from a mathematical point of view which have been studied by many authors, including Pap [25], Ralescu and Adams [26], Román-Flores et al. [6,27–30] and, Wang and Klir [34], among others. Ralescu and Adams [26] studied several equivalent definitions of fuzzy integrals, while Pap [25] and Wang and Klir [34] provided an overview of fuzzy measures theory. On the other hand, fuzzy measures and Sugeno integrals have also been successfully applied to various fields by many researchers [5,7,13,14,10,18,33,35].

The integral inequalities are useful tools in several theoretical and applied fields. For instance, integral inequalities play a role in the development of a time scales calculus [24]. For more information on classical inequalities, we refer the reader to the recent monograph [9]. The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [6,27–30], and then followed by the authors [1,2,16,19,20]. In [29] Román-Flores et al. studied some properties of Sugeno integral for strictly monotone real functions, they also provided some Yong type inequalities. Based on these results, Flores-Franulič and Román-Flores [6] provided some Chebyshev type inequalities for Sugeno integral of continuous and strictly monotone real functions based on Lebesgue measure. Some other classical inequalities have also been generalized to Sugeno integral by them (see, for example [28,30]). Later on, Ouyang and Fang [19] generalized the main results of [29] to the case of monotone real functions. Based on these results, Ouyang et al. further generalized the fuzzy Chebyshev type inequalities to the case of arbitrary fuzzy measure-based Sugeno integrals [16,20]. In fact, they proved the following result:

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Theorem 1.1. *Let $f, g \in \mathscr{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_A f\,d\mu$ and $(S)\int_A g\,d\mu$ are finite. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$(S)\int_A f\star g\,d\mu \geqslant \left((S)\int_A f\,d\mu\right)\star\left((S)\int_A g\,d\mu\right) \tag{1.1}$$

holds.

In view of the fact that

$$(S)\int_A f\star g\,d\mu \leqslant \left((S)\int_A f\,d\mu\right)\star\left((S)\int_A g\,d\mu\right) \tag{1.2}$$

holds for comonotone functions $f, g \in \mathscr{F}_+(X)$ whenever $\star \geqslant \max$ (for a similar result, see [21]), it is of great interest to determine the operator $\star$ such that

$$(S)\int_A f\star g\,d\mu = \left((S)\int_A f\,d\mu\right)\star\left((S)\int_A g\,d\mu\right) \tag{1.3}$$

holds for any comonotone functions f, g, and for any fuzzy measure μ and any measurable set A. Ouyang et al. [23,22] proved that there are only 18 operators such that (1.3) holds, including the four well-known operators: minimum, maximum, PF ($\star$ is called the first projection, PF for short, if $x \star y = x$ for each pair (x, y)) and PL ($\star$ is called the last projection, PL for short, if $x \star y = y$ for each pair (x, y)).

The classical Minkowski inequality was published by Minkowski [17, PP.115-117] in his famous book 'Geometrie der Zahlen'. This inequality is an important tool for modern analysis. A proof of Minkowski's inequality as well as several extensions, related results, and interesting geometrical interpretations can be found in [31]. Applications of Minkowski's inequality have been studied by many authors, for example Özkan et al. [24] applied Minkowski's inequality on time scales and Lu et al. [12] used Minkowski's inequality for fast full search in motion estimation. So it is of interest to develop its counterpart for Sugeno integrals.

Recently, Agahi and Yaghoobi [1] proved a Minkowski type inequality for monotone real functions and Lebesgue measure-based Sugeno integral, and then Agahi et al. [2] further generalized it to comonotone functions and arbitrary fuzzy measure-based Sugeno integrals,

Theorem 1.2. *Let $f, g \in \mathscr{F}_+(X)$ and let μ be an arbitrary fuzzy measure such that $(S)\int_A f\star g\,d\mu$ is finite. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from below by maximum. If f, g are comonotone, then the inequality*

$$\left((S)\int_A (f\star g)^s\,d\mu\right)^{\frac{1}{s}} \leqslant \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\star\left((S)\int_A g^s\,d\mu\right)^{\frac{1}{s}} \tag{1.4}$$

holds for all $s > 0$.

In the present paper, we intend to prove a reverse inequality related to (1.4). As we will see, the obtained inequality can also be seen as a generalization of Chebyshev inequality for Sugeno integrals [16]. We think that our result together with Ineq. (1.4) will be useful for those areas in which the classical Minkowski inequality plays a role whenever the environment is non-deterministic. After some preliminaries and summarization of some previous known results in Section 2, Section 3 presents our main results, including several examples. Finally, some conclusions and problems for further investigations are given.

2. Preliminaries

In this section we recall some basic definitions and previous results which will be used in the sequel.

As usual we denote by R the set of real numbers. Let X be a non-empty set, $\mathscr{F}$ be a σ-algebra of subsets of X. Let $\boldsymbol{N}$ denote the set of all positive integers and $\overline{R_+}$ denote $[0, +\infty]$. Throughout this paper, we fix the measurable space $(X, \mathscr{F})$, and all considered subsets are supposed to belong to $\mathscr{F}$.

Definition 2.1. [26]A set function $\mu : \mathscr{F} \to \overline{R_+}$ is called a fuzzy measure if the following properties are satisfied:

(FM1) $\mu(\emptyset) = 0$;
(FM2) $A \subset B$ implies $\mu(A) \leqslant \mu(B)$;
(FM3) $A_1 \subset A_2 \subset \cdots$ implies $\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$; and
(FM4) $A_1 \supset A_2 \supset \cdots$ and $\mu(A_1) < +\infty$ imply $\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty}\mu(A_n)$.

When μ is a fuzzy measure, the triple $(X, \mathscr{F}, \mu)$ then is called a fuzzy measure space.

Let $(X, \mathscr{F}, \mu)$ be a fuzzy measure space, by $\mathscr{F}_+(X)$ we denote the set of all measurable functions $f : X \to [0, \infty)$ with respect to $\mathscr{F}$. In what follows, all considered functions belong to $\mathscr{F}_+(X)$. Let f be a nonnegative real-valued function defined on X, we will denote the set $\{x \in X | f(x) \geqslant \alpha\}$ by F_α for $\alpha \geqslant 0$. Clearly, F_α is nonincreasing with respect to α, i.e., $\alpha \leqslant \beta$ implies $F_\alpha \supseteq F_\beta$.

Definition 2.2. [25,32,34]Let $(X, \mathscr{F}, \mu)$ be a fuzzy measure space and $A \in \mathscr{F}$, the Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined as

$$(S)\int_A f\,d\mu = \bigvee_{\alpha \geqslant 0} (\alpha \wedge \mu(A \cap F_\alpha)).$$

When $A = X$, then

$$(S)\int_X f\,d\mu = (S)\int f\,d\mu = \bigvee_{\alpha \geqslant 0} (\alpha \wedge \mu(F_\alpha)).$$

It is well known that Sugeno integral is a type of nonlinear integral [15], i.e., for general case,

$$(S)\int (af + bg)d\mu = a(S)\int f\,d\mu + b(S)\int g d\mu,$$

does not hold. Some basic properties of Sugeno integral are summarized in [25,34], we cite some of them in the next Theorem.

Theorem 2.3. [25,34]*Let $(X, \mathscr{F}, \mu)$ be a fuzzy measure space, then*

(i) $\mu(A \cap F_\alpha) \geqslant \alpha \Rightarrow (S)\int_A f\,d\mu \geqslant \alpha$;
(ii) $\mu(A \cap F_\alpha) \leqslant \alpha \Rightarrow (S)\int_A f\,d\mu \leqslant \alpha$;
(iii) $(S)\int_A f\,d\mu < \alpha \iff$ *there exists* $\gamma < \alpha$ *suchthat* $\mu(A \cap F_\gamma) < \alpha$;
(iv) $(S)\int_A f\,d\mu > \alpha \iff$ *there exists* $\gamma > \alpha$ *suchthat* $\mu(A \cap F_\gamma) > \alpha$;
(v) If $\mu(A) < \infty$, *then* $\mu(A \cap F_\alpha) \geqslant \alpha \iff (S)\int_A f\,d\mu \geqslant \alpha$;
(vi) If $f \leqslant g$, *then* $(S)\int f\,d\mu \leqslant (S)\int g\,d\mu$.

In [19], Ouyang and Fang proved the following result which generalized the corresponding one in [29].

Lemma 2.4. *Let m be the Lebesgue measure on R and let $f : [0, \infty) \to [0, \infty)$ be a nonincreasing function. If $(S)\int_0^a f\,dm = p$, then*

$$f(p-) \geqslant (S)\int_0^a f\,dm = p$$

for all $a \geqslant 0$, where $f(p-) = \lim_{x \to p^-} f(x)$.

Moreover, if $p < a$ and f is continuous at p, then $f(p-) = f(p) = p$.

Notice that if m is the Lebesgue measure and f is nonincreasing, then $f(p-) \geqslant p$ implies $(S)\int_0^a f\,dm \geqslant p$ for any $a \geqslant p$. In fact, the monotonicity of f and the fact $f(p-) \geqslant p$ imply that $[0, p) \subset F_p$. Thus, $m([0, a] \cap F_p) \geqslant m([0, a] \cap [0, p)) = m([0, p)) = p$. Now, by Theorem 2.3(i), we have $(S)\int_0^a f\,dm \geqslant p$.

Based on Lemma 2.4, Ouyang et al. proved some Chebyshev type inequalities [20] and their united form [16] (i.e., Theorem 1.1 in this paper). Notice that when proving these Theorems, the following lemma, which is derived from the transformation theorem for Sugeno integrals (see [34]), plays a fundamental role.

Lemma 2.5. *Let $(S)\int_A f\,d\mu = p$. Then $\forall\ r \geqslant p, (S)\int_A f\,d\mu = (S)\int_0^r \mu(A \cap F_\alpha)dm$, where m is the Lebesgue measure.*

In this contribution, we will prove an inequality related to Chebyshev type and Minkowski type inequalities (Theorems 1.1 and 1.2 in this paper) for the Sugeno integral of comonotone functions. Recall that two functions $f, g : X \to R$ are said to be comonotone if for all $(x, y) \in X^2, (f(x) - f(y))(g(x) - g(y)) \geqslant 0$. Clearly, if f and g are comonotone, then for all non-negative real numbers p, q either $F_p \subset G_q$ or $G_q \subset F_p$. Indeed, if this assertion does not hold, then there are $x \in F_p \setminus G_q$ and $y \in G_q \setminus F_p$. That is,

$$f(x) \geqslant p, g(x) < q \quad \text{and} \quad f(y) < p, g(y) \geqslant q,$$

and hence $(f(x) - f(y))(g(x) - g(y)) < 0$, contradicts with the comonotonicity of f and g. Notice that comonotone functions can be defined on any abstract space.

3. Main results

The following theorem, which is related to the Minkowski type inequality for Sugeno integral [1,2] (see also Theorem 1.2), is our main result.

Theorem 3.1. *Let $f,g \in \mathscr{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_A f\,d\mu$ and $(S)\int_A g\,d\mu$ are finite. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f, g are comonotone, then the inequality*

$$((S)\int_A (f\star g)^s\,d\mu)^{\frac{1}{s}} \geqslant \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}} \star \left((S)\int_A g^s\,d\mu\right)^{\frac{1}{s}} \tag{3.1}$$

holds for all $0 < s < \infty$.

Proof. First of all, notice that the finiteness of $(S)\int_A f\,d\mu$ $((S)\int_A g\,d\mu)$ implies that of $(S)\int_A f^s\,d\mu$ $((S)\int_A g^s\,d\mu)$. Put $(S)\int f^s\,d\mu = p$ and $(S)\int g^s\,d\mu = q$ and $r > \max\{p,q\}$. Clearly, if $p^{\frac{1}{s}}\star q^{\frac{1}{s}} = 0$ then In Eq. (3.1) holds readily. So we need only to prove the conclusion for $p^{\frac{1}{s}}\star q^{\frac{1}{s}} > 0$. Since $\star$ is bounded from above by minimum, we have that $0 < p,q < r$. Denote $A(\alpha) = \mu(A\cap\{x|f^s(x) \geqslant \alpha\}), B(\alpha) = \mu(A\cap\{x|g^s(x) \geqslant \alpha\})$, then, by applying Lemma 2.5, it holds

$$(S)\int_0^r A(\alpha)dm = p \quad \text{and,} \quad (S)\int_0^r B(\alpha)dm = q,$$

where m is the Lebesgue measure. Thus we need only to show that $(S)\int_A (f\star g)^s\,d\mu \geqslant \left(p^{\frac{1}{s}}\star q^{\frac{1}{s}}\right)^s$. For any sufficient small $\varepsilon, \mu(A\cap F_{(p-\varepsilon)^{\frac{1}{s}}}) = A(p-\varepsilon) \geqslant p$ and $\mu\left(A\cap G_{(q-\varepsilon)^{\frac{1}{s}}}\right) = B(q-\varepsilon) \geqslant q$. By the monotonicity of $\star$ and the comonotonicity of f,g, we have

$$\begin{aligned}\mu\left(A\cap H_{(p-\varepsilon)^{\frac{1}{s}}\star(q-\varepsilon)^{\frac{1}{s}}}\right) &\geqslant \mu\left(A\cap F_{(p-\varepsilon)^{\frac{1}{s}}}\cap G_{(q-\varepsilon)^{\frac{1}{s}}}\right) = \min\left(\mu\left(A\cap F_{(p-\varepsilon)^{\frac{1}{s}}}\right), \quad \mu\left(A\cap G_{(q-\varepsilon)^{\frac{1}{s}}}\right)\right) = \min(A(p-\varepsilon), B(q-\varepsilon)) \\ &\geqslant \min(p,q) \geqslant \left(p^{\frac{1}{s}}\star q^{\frac{1}{s}}\right)^s,\end{aligned}$$

where $H_\alpha = \{x|f(x)\star g(x) \geqslant \alpha\}$. Notice that if we denote $C(\alpha) = \mu(A\cap\{x|(f(x)\star g(x))^s \geqslant \alpha\})$, then

$$C\left(\left((p-\varepsilon)^{\frac{1}{s}}\star(q-\varepsilon)^{\frac{1}{s}}\right)^s\right) = \mu\left(A\cap H_{(p-\varepsilon)^{\frac{1}{s}}\star(q-\varepsilon)^{\frac{1}{s}}}\right) \geqslant \left(p^{\frac{1}{s}}\star q^{\frac{1}{s}}\right)^s$$

holds for any ε. Letting $\varepsilon\to 0$, by the continuity of $\star$ we obtain that $C((p^{\frac{1}{s}}\star q^{\frac{1}{s}})^s-) \geqslant (p^{\frac{1}{s}}\star q^{\frac{1}{s}})^s$. Thus we have $(S)\int_A (f\star g)^s\,d\mu \geqslant (p^{\frac{1}{s}}\star q^{\frac{1}{s}})^s$ and which implies that

$$\left((S)\int_A (f\star g)^s\,d\mu\right)^{\frac{1}{s}} \geqslant p\star q = \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\star((S)\int_A g^s\,d\mu)^{\frac{1}{s}}.$$

This completes the proof. □

Remark 3.2. Let $(S)\int_A f^s\,d\mu = p$ and $(S)\int_A g^s\,d\mu = q$, and let $c \geqslant \max\left(p,q,p^{\frac{1}{s}},q^{\frac{1}{s}}\right)$. Then the requirement of $\star|_{[0,c]^2} \leqslant \min$ is enough to ensure the validity of Theorem 3.1. If $\star = \min$, then In Eq. (3.1) remains true when $(S)\int_A f^s\,d\mu$ and/or $(S)\int_A g^s\,d\mu$ are/is finite. Therefore In Eq. (3.1) together with the monotonicity of Sugeno integral (Theorem 2.3(vi)) imply that

$$\left((S)\int_A (f\wedge g)^s\,d\mu\right)^{\frac{1}{s}} = \left((S)\int_A f^s d\mu\right)^{\frac{1}{s}} \wedge \left((S)\int_A g^s\,d\mu\right)^{\frac{1}{s}}.$$

This property is equivalent to the comonotone minitivity of Sugeno integral.

Example 3.3. Let $X = [0,10]$ and the fuzzy measure μ be defined as $\mu(A) = m^2(A)$, where m is the Lebesgue measure. Let $f(x) = 3$ and

$$g(x) = \begin{cases} x & x\in[0,5], \\ 10-x & x\in[5,10], \end{cases}$$

then f,g are comonotone and $h = f\wedge g$ is defined as

$$h(x) = \begin{cases} x & x\in[0,3], \\ 3 & x\in(3,7), \\ 10-x & x\in[7,10]. \end{cases}$$

For any $s > 0$, a simple calculation reveals that

$$(S)\int f^s\,d\mu = 3^s\wedge 100, \quad (S)\int g^s d\mu = \bigvee_{\alpha\in[0,5^s]} \alpha\wedge\left(10-2\alpha^{\frac{1}{s}}\right)^2$$

and

$$(S)\int h^s d\mu = \bigvee_{\alpha\in[0,3^s]} \alpha \wedge \left(10 - 2\alpha^{\frac{1}{s}}\right)^2.$$

So,

$$\left((S)\int f^s\,d\mu\right)^{\frac{1}{s}} \bigwedge \left((S)\int g^s\,d\mu\right)^{\frac{1}{s}} = \left(3\wedge 100^{\frac{1}{s}}\right) \bigwedge \left(\bigvee_{\alpha\in[0,5^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}}\right)$$

and

$$\left((S)\int h^s\,d\mu\right)^{\frac{1}{s}} = \bigvee_{\alpha\in[0,\ 3^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}}.$$

Noting that for any $\alpha\in[0,5^s]$, $\left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} \leqslant 100^{\frac{1}{s}}$, thus

$$\left((S)\int f^s\,d\mu\right)^{\frac{1}{s}} \bigwedge \left((S)\int g^s\,d\mu\right)^{\frac{1}{s}} = 3 \bigwedge \left(\bigvee_{\alpha\in[0,5^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}}\right).$$

Now, if $\bigvee_{\alpha\in[0,5^s]}\alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} \geqslant 3$, i.e., $\left(10-2(3^s)^{\frac{1}{s}}\right)^{\frac{2}{s}} \geqslant 3$, then

$$\left((S)\int f^s\,d\mu\right)^{\frac{1}{s}} \bigwedge \left((S)\int g^s\,d\mu\right)^{\frac{1}{s}} = 3 = (3^s)^{\frac{1}{s}} \bigwedge \left(10-2(3^s)^{\frac{1}{s}}\right)^{\frac{2}{s}} = \bigvee_{\alpha\in[0,3^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} = \left((S)\int h^s d\mu\right)^{\frac{1}{s}}$$
$$= \left((S)\int (f\wedge g)^s d\mu\right)^{\frac{1}{s}},$$

if $\bigvee_{\alpha\in[0,5^s]}\alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} < 3$, i.e., $\left(10-2(3^s)^{\frac{1}{s}}\right)^{\frac{2}{s}} < 3$, then

$$\left((S)\int f^s\,d\mu\right)^{\frac{1}{s}} \bigwedge \left((S)\int g^s\,d\mu\right)^{\frac{1}{s}} = \bigvee_{\alpha\in[0,5^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} = \left(\bigvee_{\alpha\in[0,3^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}}\right) \bigvee \left(\bigvee_{\alpha\in(3^s,5^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}}\right)$$
$$= \bigvee_{\alpha\in[0,3^s]} \alpha^{\frac{1}{s}} \wedge \left(10-2\alpha^{\frac{1}{s}}\right)^{\frac{2}{s}} = \left((S)\int h^s d\mu\right)^{\frac{1}{s}} = \left((S)\int (f\wedge g)^s d\mu\right)^{\frac{1}{s}}.$$

Remark 3.4. Let $\star$ be continuous and nondecreasing. If $\star|_{[0,1]^2}$ is a triangular subnorm [8], then Ineq. (3.1) works for any comonotone functions f, g with $(S)\int_A f\,d\mu \leqslant 1$ and $(S)\int_A g\,d\mu \leqslant 1$.

Example 3.5. Let $\star$ be the usual product and the two comonotone functions $f,g:[0,3]\to R^+$ be defined as $f(x)=\frac{1}{3}x, g(x)=\frac{1}{4}(x+1)$. If the fuzzy measure μ be defined as $\mu(A)=m(A)$, where m denotes the Lebesgue measure on R, then

$$(S)\int f^{\frac{1}{2}}d\mu = \bigvee_{\alpha\in[0,1]} \alpha\wedge(3-3\alpha^2) = \frac{\sqrt{37}-1}{6}, (S)\int g^{\frac{1}{2}}d\mu = \bigvee_{\alpha\in[0,1]} \alpha\wedge(4-4\alpha^2) = \frac{\sqrt{65}-1}{8}$$

and

$$(S)\int (fg)^{\frac{1}{2}}d\mu = \bigvee_{\alpha\in[0,1]} \alpha\wedge\frac{7-\sqrt{1+48\alpha^2}}{2} = \frac{\sqrt{577}-7}{22}.$$

Thus,

$$\left((S)\int (fg)^{\frac{1}{2}}d\mu\right)^2 \approx 0.599 > 0.559 \approx \left((S)\int f^{\frac{1}{2}}d\mu\right)^2\left((S)\int g^{\frac{1}{2}}d\mu\right)^2$$

The following example shows that the comonotonicity of f, g in Theorem 3.1 is inevitable.

Example 3.6. Let $X = [0, 1]$, $f(x) = x$, $g(x) = 1 - x$ and the fuzzy measure μ be defined as $\mu(A)=m^2(A)$, where m denotes the Lebesgue measure on R. Then

$$(S)\int f^2\,d\mu = (S)\int g^2\,d\mu = \bigvee_{\alpha\in[0,1]} \alpha\wedge\left(1-\sqrt{\alpha}\right)^2 = \frac{1}{4}$$

and

$$(S)\int (f\wedge g)^2\,d\mu = \bigvee_{\alpha\in[0,\frac{1}{4}]} \alpha\wedge\left(1-2\sqrt{\alpha}\right)^2 = \frac{1}{9}.$$

Hence

$$\left((S)\int (f\wedge g)^2\,d\mu\right)^{\frac{1}{2}} = \frac{1}{3} < \frac{1}{2} = \left((S)\int f^2\,d\mu\right)^{\frac{1}{2}} \bigwedge \left((S)\int g^2\,d\mu\right)^{\frac{1}{2}},$$

which violates Theorem 3.1.

The following example shows that the condition of $\star \leqslant \min$ in Theorem 3.1 cannot be omitted.

Example 3.7. Let $X = [0,5]$, and $f(x) = g(x) \equiv 2$. Then

$$(S)\int f^2\,dm = (S)\int g^2\,dm = 4,\ (S)\int (fg)^2\,dm = 5,$$

where m denotes the Lebesgue measure on R. Thus,

$$\left((S)\int (fg)^2\,dm\right)^{\frac{1}{2}} = \sqrt{5} < 4 = \left((S)\int f^2\,dm\right)^{\frac{1}{2}}\left((S)\int g^2\,dm\right)^{\frac{1}{2}},$$

which violates Theorem 3.1.

We close this section with the following two results.

Corollary 3.8. *Let $f_1, f_2, \ldots, f_n$ be such that any two of them are comonotone and $\star$ be as in Theorem 3.1 . Then*

$$\left((S)\int_A ((\cdots((f_1\star f_2)\star f_3)\star\cdots)\star f_n)^s\,d\mu\right)^{\frac{1}{s}} \geqslant \left(\left(\cdots\left(\left(\left((S)\int_A f_1^s d\mu\right)^{\frac{1}{s}}\star\left((S)\int_A f_2^s\,d\mu\right)\right)^{\frac{1}{s}}\right)\star\cdots\right)\star\left((S)\int_A f_n^s\,d\mu\right)^{\frac{1}{s}}.$$

Proof. Since f_1, f_2 are comonotone, then by Theorem 3.1 we have

$$\left((S)\int (f_1\star f_2)^s\,d\mu\right)^{\frac{1}{s}} \geqslant \left((S)\int f_1^s\,d\mu\right)^{\frac{1}{s}}\star\left((S)\int f_2^s\,d\mu\right)^{\frac{1}{s}}.$$

Moreover, the comonotonicity of $f_1\star f_2$ and f_3 (see the proof of Corollary 3.8 in [16]) implies that

$$\begin{aligned}\left((S)\int ((f_1\star f_2)\star f_3)^s\right)d\mu)^{\frac{1}{s}} &\geqslant \left((S)\int (f_1\star f_2)^s\,d\mu\right)^{\frac{1}{s}}\star((S)\int f_3^s\,d\mu)^{\frac{1}{s}}\\ &\geqslant \left(\left((S)\int f_1^s\,d\mu\right)^{\frac{1}{s}}\star\left((S)\int f_2^s\,d\mu\right)^{\frac{1}{s}}\right)\star\left((S)\int f_3^s\,d\mu\right)^{\frac{1}{s}}.\end{aligned}$$

Thus we can prove the conclusion by induction. □

Let $s = 1$, we then get the Chebyshev type inequality:

Corollary 3.9 [16]. *Let $f, g \in \mathscr{F}_+(X)$ and μ be an arbitrary fuzzy measure such that both $(S)\int_A f\,d\mu$ and $(S)\int_A g\,d\mu$ are finite. And let $\star : [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments and bounded from above by minimum. If f,g are comonotone, then the inequality*

$$(S)\int_A f\star g\,d\mu \geqslant \left((S)\int_A f\,d\mu\right)\star\left((S)\int_A g\,d\mu\right) \tag{1.2}$$

holds.

4. Further discussions

Combining (1.4) and (3.1), one will find it is of interest to examine the operations $\star$ such that

$$\left((S)\int_A (f\star g)^s\,d\mu\right)^{\frac{1}{s}} = \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\star\left((S)\int_A g^s\,d\mu\right)^{\frac{1}{s}} \tag{4.1}$$

for any fuzzy measure space $(X, \mathscr{F}, \mu)$ and any measurable set A, and for any comonotone functions $f, g : X \to [0, \infty)$. We have the following result:

Theorem 4.1. *Eq. (4.1) holds for any fuzzy measure space $(X, \mathscr{F}, \mu)$ and any measurable set A, and for any comonotone functions $f, g : X \to [0, \infty)$ if and only if $\star$ belongs to one of the 18 operators in [22].*

Proof. In [22], the equality (4.1) for fixed power $s = 1$ was solved. Thus there are at most 18 operations $\star$ described in [22] which can satisfy (4.1). We sketch how to prove that all of them are solutions of (4.1). For $\star \in \{\min, \max, PF, PL\}$, it suffices to note that $(f \star g)^t = f^t \star g^t$ holds for any $t > 0$, then (4.1) can be proven by using (1.3). For other 14 operators, the proofs are similar to those in [22]. Here we give in detail the proof for one of these special operations $\star$, the case examplified in Fig. 1.

Let $(X, \mathscr{F}, \mu)$ be an arbitrary fuzzy measure space and A an arbitrary measurable set. Let $f, g \in \mathscr{F}_+(X)$ be two arbitrary comonotone functions. Suppose $(S)\int_A f^s\, d\mu = a^s$ and $(S)\int_A g^s\, d\mu = b^s$. Thus, for any n, $\mu(A \cap F_{a-\frac{1}{n}}) \geqslant a^s$ and $\mu(A \cap G_{b-\frac{1}{n}}) \geqslant b^s$. Moreover, for any $c > a$, $\mu(A \cap F_c) \leqslant a^s$ and for any $c > b$, $\mu(A \cap G_c) \leqslant b^s$. We distinguish the following two possible cases:

Case 1: $a \wedge b \geqslant r$. In this case we have $a \star b = r$. To show (1.3), we need only to show $(S)\int_A (f \star g)^s\, d\mu = r^s$. For each n we have $\mu\left(A \cap F_{r-\frac{1}{n}}\right) \geqslant \mu\left(A \cap F_{a-\frac{1}{n}}\right) \geqslant a^s \geqslant r^s$ and $\mu\left(A \cap G_{r-\frac{1}{n}}\right) \geqslant \mu\left(A \cap G_{b-\frac{1}{n}}\right) \geqslant b^s \geqslant r^s$. By the fact of $H_{r-\frac{1}{n}} \supset F_{r-\frac{1}{n}} \cap G_{r-\frac{1}{n}}$ as well as the comonotonicity of f, g, we have that $\mu\left(A \cap H_{r-\frac{1}{n}}\right) \geqslant \mu\left(A \cap F_{r-\frac{1}{n}}\right) \wedge \mu\left(A \cap G_{r-\frac{1}{n}}\right) \geqslant r^s$, where $H_\alpha = \{x \in X | f(x) \star g(x) \geqslant \alpha\}$. Thus $(S)\int_A (f \star g)^s\, d\mu \geqslant r^s$. On the other hand, for any $x \in X, f(x) \star g(x) \leqslant r$, so $(S)\int_A (f \star g)^s\, d\mu \leqslant r^s$. Hence $(S)\int_A (f \star g)^s\, d\mu = r^s$.

Case 2: $a \wedge b < r$. We suppose $a \wedge b = b < r$, the case of $a \wedge b = a < r$ can be shown analogously. This case can be further divided into three subcases:

2(i): $r > b \geqslant r_1'$. For each $n, F_{a-\frac{1}{n}} \cap G_{b-\frac{1}{n}} \subset H_{b-\frac{1}{n}}$, and thus the comonotonicity of f, g implies $\mu\left(A \cap H_{b-\frac{1}{n}}\right) \geqslant \mu\left(A \cap F_{a-\frac{1}{n}}\right) \wedge \mu\left(A \cap G_{b-\frac{1}{n}}\right) \geqslant a^s \wedge b^s = b^s$. Whence $(S)\int_A (f \star g)^s\, d\mu \geqslant b^s$. Furthermore, for any $c > b, H_c \subset G_c$. Hence $\mu(A \cap H_c) \leqslant \mu(A \cap G_c) \leqslant b^s$, which implies that $(S)\int_A (f \star g)^s\, d\mu \leqslant b^s$. Therefore

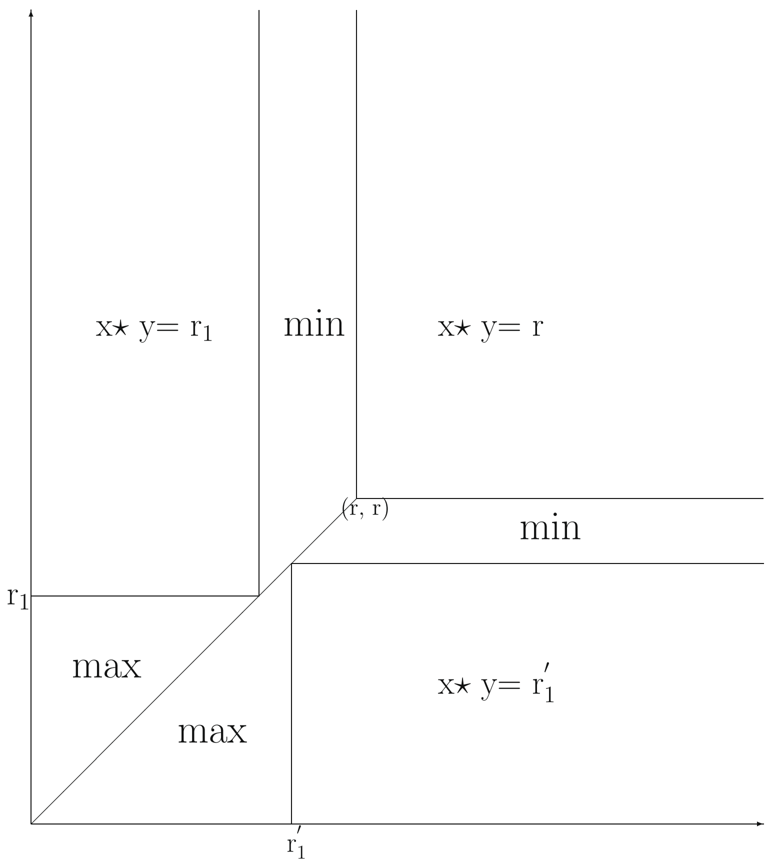

Fig. 1. One of the solutions of (4.1).

$$\left((S)\int_A (f\bigstar g)^s\,d\mu\right)^{\frac{1}{s}} = b = a\bigstar b = \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\bigstar((S)\int_A g^s\,d\mu)^{\frac{1}{s}},$$

i.e., (4.1) holds.

2(ii): $b < r'_1 < a$. Since $H_{r'_1} \supset F_{r'_1}$, we have $\mu\left(A\cap H_{r'_1}\right) \geqslant \mu\left(A\cap F_{r'_1}\right) \geqslant a^s > (r'_1)^s$. Thus $(S)\int_A (f\bigstar g)^s\,d\mu \geqslant (r'_1)^s$. On the other hand, for any $c > r'_1 > b, H_c \subset G_c$. Hence $\mu(A\cap H_c) \leqslant \mu(A\cap G_c) \leqslant b^s < (r'_1)^s$, and which implies that $(S)\int_A (f\bigstar g)^s\,d\mu \leqslant (r'_1)^s$. Thus

$$\left((S)\int_A (f\bigstar g)^s\,d\mu\right)^{\frac{1}{s}} = r'_1 = a\bigstar b = \left((S)\int_A f^s\,d\mu\right)^{\frac{1}{s}}\bigstar\left((S)\int_A g^s\,d\mu\right)^{\frac{1}{s}},$$

i.e., (4.1) holds.

2(iii): $a \leqslant r'_1$. From the facts that $H_{a-\frac{1}{n}} \supset F_{a-\frac{1}{n}}, \forall\, n$ and $H_c \subset F_c \cup G_c, \forall\, c > a$ as well as the comonotonicity of f, g, we conclude that $((S)\int_A (f\bigstar g)^s\,d\mu)^{\frac{1}{s}} = a = a\bigstar b = ((S)\int_A f^s\,d\mu)^{\frac{1}{s}}\bigstar((S)\int g^s\,d\mu)^{\frac{1}{s}}$, again (4.1) holds. □

5. Conclusions and problems for further investigation

We have proved an inequality for the Sugeno integral on an abstract fuzzy measure space $(X, \mathscr{F}, \mu)$ based on a product-like operation ★. As we have seen, this inequality is related to Minkowski type one and Chebyshev type one. Moreover, we have shown all cases for ★ yielding the commuting (for comonotone functions) with "power-root" Sugeno integral.

On the other hand, there are numerous applications of Choquet integral, and thus the study of Minkowski, Chebyshev and similar inequalities for Choquet integral is an important and interesting topic for the further research. Moreover, both Sugeno and Choquet integrals when restricted to [0,1] can be seen as copula-based fuzzy integrals, and then Minkowski-like inequalities is a challenging topic also in this rather general situation.

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