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L^∞-measure of non-exchangeability for bivariate extreme value and Archimax copulas

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ABSTRACT

In the class of bivariate extreme value copulas, an upper bound is calculated for the measure of non-exchangeability μ_∞ based on the L^∞-norm distance between a copula C and its transpose $C^t(x,y) = C(y,x)$. Copulas that are maximally non-exchangeable with respect to μ_∞ are also determined. Moreover, similar upper bounds are given, respectively, for the class of all EV copulas having a fixed upper tail dependence coefficient and for the larger class of Archimax copulas.

1. Introduction

Let (X, Y) be a pair of continuous random variables defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The joint cumulative distribution function (= d.f.) H and the marginals F_X and F_Y of (X, Y) are defined, respectively, by

$$H(x,y) = \mathbb{P}(X \leqslant x, Y \leqslant y), \qquad F_X(x) = \mathbb{P}(X \leqslant x), \qquad F_Y(y) = \mathbb{P}(Y \leqslant y)$$

for all $x, y \in \mathbb{R}$. As shown in [34], H can be represented, for all $x, y \in \mathbb{R}$, in the form $H(x,y) = C(F_X(x), F_Y(y))$, where the function C, called *copula*, is the restriction on $[0,1]^2$ of the joint d.f. of the random pair (U, V), where $U = F_X(X)$ and $V = F_Y(Y)$. For a complete overview about copulas and some of its applications, see [14,21,29,30,33].

A copula C is called an *extreme value copula* (EV copula, for short), if there exists a convex function $A : [0,1] \to [\frac{1}{2}, 1]$, $\max(t, 1-t) \leqslant A(t) \leqslant 1$ for all $t \in [0,1]$, such that for all $(x,y) \in]0,1[^2$,

$$C(x,y) = \exp\left(\log(xy) A\left(\frac{\log(x)}{\log(xy)}\right)\right). \tag{1}$$

The function A is often referred to as the *Pickands' dependence function* of the pair (X, Y). Examples of EV copulas are the independence copula $\Pi(x,y) = xy$ and the comonotonicity copula $M(x,y) = \min(x,y)$. The interest in EV copulas stems from their characterization as the large-sample limits of copulas of componentwise maxima of strongly mixing stationary sequences (for more details, see [2,8,21,29,32,33]).

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For a suitable (parametric) choice of the dependence function A, several EV copulas can be constructed and estimated from some real data (see, for instance, [3,4,15,18–20]). In particular, a more flexible statistical model can be constructed when the EV copula C is not necessarily *exchangeable*, i.e. $C(x,y) \neq C(y,x)$ for some $(x,y) \in]0,1[^2$. In fact, as underlined in [35], "in some contexts exchangeability would not be a reasonable assumption, and the degree of non-exchangeability may be of interest in itself". Examples of non-exchangeable EV copulas can be derived, for instance, from the asymmetric mixed and logistic models [35], the bilogistic and the negative bilogistic models [6,22]. A method for constructing non-exchangeable EV copulas has been proposed in [17], based on the asymmetrization procedure in [23] (see also [12,26,29]). The usefulness of considering asymmetric EV models is also discussed, for example, in [35].

Nowadays, several investigations have been focused on the ways in which copulas can fail to be exchangeable (see [9,24, 31]). In particular, a special interest has been devoted to the problem of finding the so-called maximally non-exchangeable copulas (with respect to a given measure) in a known class $\mathscr{E}$ of copulas, usually defined by means of a dependence property [1,7,10,11]. Such investigations are not only of a theoretical interest, but might also have a practical impact in the choice of a suitable family of copulas that one can use for a certain problem. In fact, families of copulas including the maximally non-exchangeable copulas (eventually as limiting elements) span all the possible degrees of non-exchangeability and, as such, should be more appropriate for describing situations where asymmetries should be considered into the model (see [9] for a discussion).

Along these lines of investigations, we determine an upper bound for the measure of non-exchangeability μ_∞ (as defined below) of an EV copula and we give EV copulas that are maximally non-exchangeable with respect to μ_∞, i.e., EV copulas that reach this upper bound (Section 2). Analogous considerations are, hence, made in Section 3 for the larger class of *Archimax* copulas, introduced in [5].

2. Non-exchangeability for EV copulas

For any copula C, the measure of non-exchangeability μ_∞ of C is defined by

$$\mu_\infty(C) = 3\Big(\max_{(x,y)\in[0,1]^2} \big|C(x,y) - C(y,x)\big|\Big).$$

It takes values on $[0,1]$ (see, e.g., [24,31]) and, in particular, $\mu_\infty(C) = 0$ if, and only if, C is exchangeable. This measure may be considered as a starting point for estimating other measures of non-exchangeability based on some L^p-norm distance ($p \geqslant 1$) between the copula C and its transpose $C^t(x,y) = C(y,x)$. Moreover, it could be easily computed from some real data by using, for instance, the empirical versions of C and C^t. For more details, see [9].

The upper bound for the measure μ_∞ in the class of all EV copulas is given by the following result.

Theorem 1. *For every EV copula C,*

$$\mu_\infty(C) \leqslant 3 \cdot \frac{4^4}{5^5}.$$

Moreover, $\mu_\infty(C) = 3 \cdot \frac{4^4}{5^5}$ if, and only if, $C \in \{C_1, C_2\}$ where

$$C_1(x,y) = \begin{cases} y, & x^2 \geqslant y, \\ x\sqrt{y}, & \text{otherwise}, \end{cases} \quad \text{or} \quad C_2(x,y) = \begin{cases} x, & y^2 \geqslant x, \\ y\sqrt{x}, & \text{otherwise}. \end{cases} \tag{2}$$

Proof. Let C be an EV copula with dependence function A,

$$C(x,y) = \exp\left(\log(xy) A\left\{\frac{\log(x)}{\log(xy)}\right\}\right) = (xy)^{A(\frac{\log(x)}{\log(xy)})}.$$

We aim at determining the value

$$\begin{aligned}\delta_C &= \max_{(x,y)\in[0,1]^2} \big|C(x,y) - C(y,x)\big| \\ &= \max_{(x,y)\in]0,1[^2} \big|(xy)^{A(\frac{\log(x)}{\log(xy)})} - (xy)^{A(1-\frac{\log(x)}{\log(xy)})}\big|.\end{aligned}$$

Suppose that δ_C is reached at a point $(x,y) \in]0,1[^2$ and set $t = \frac{\log(x)}{\log(xy)}$. Without loss of generality, we assume that $0 < x < y < 1$, and, hence, $t \in]0, \frac{1}{2}[$ and $A(t) \leqslant A(1-t)$.

Let A_α be the dependence function given by $A_\alpha(s) = \max\{1-s, (1-\alpha)s + \alpha\}$ for $\alpha, s \in [0,1]$, where $g(s) = (1-\alpha)s + \alpha$ is the line passing through $(t, A(t))$ and $(1,1)$. Due to the convexity of A, we have that

$$\big|A(t) - A(1-t)\big| \leqslant \big|A_\alpha(t) - A_\alpha(1-t)\big|.$$

Since, for every $(x, y) \in]0, 1[^2$, $(xy)^s$ is a decreasing function in s, we have

$$\left|(xy)^{A(t)} - (xy)^{A(1-t)}\right| \leqslant \left|(xy)^{A_\alpha(t)} - (xy)^{A_\alpha(1-t)}\right|.$$

Let $(t_\alpha, A_\alpha(t_\alpha))$ be the intersection point of the lines $f(s) = 1-s$ and $g(s) = (1-\alpha)s+\alpha$. It is easy to see that $t_\alpha = \frac{1-\alpha}{2-\alpha} \leqslant \frac{1}{2}$, and

$$A_\alpha(t_\alpha) = \frac{1}{2-\alpha}, \qquad A_\alpha(1-t_\alpha) = \frac{1+\alpha-\alpha^2}{2-\alpha}.$$

Moreover, $s \mapsto |A_\alpha(s) - A_\alpha(1-s)|$ reaches its maximum at t_α, and $s \mapsto A_\alpha(s)$ reaches its minimum at t_α. Since $(xy)^s$ is a decreasing and convex function in s, we have

$$\left|(xy)^{A_\alpha(t)} - (xy)^{A_\alpha(1-t)}\right| \leqslant \left|(xy)^{A_\alpha(t_\alpha)} - (xy)^{A_\alpha(1-t_\alpha)}\right|;$$

and, as a consequence,

$$\delta_C = \max_{(x,y)\in]0,1[^2} \left(\max_{\alpha\in[0,1]} \left|(xy)^{\frac{1}{2-\alpha}} - (xy)^{\frac{1+\alpha-\alpha^2}{2-\alpha}}\right| \right).$$

Consider the mapping

$$\psi_\alpha : [0,1] \to [0,1], \qquad \psi_\alpha(k) = k^{\frac{1}{2-\alpha}} - k^{\frac{1+\alpha-\alpha^2}{2-\alpha}},$$

which attains the maximum at the point k_α satisfying $\psi'_\alpha(k_\alpha) = 0$ and given by

$$k_\alpha = \left(1+\alpha-\alpha^2\right)^{\frac{2-\alpha}{\alpha(\alpha-1)}}.$$

After easy calculations, we can show that

$$\delta_C = \max_{\alpha\in[0,1]} \psi_\alpha(k_\alpha) = \frac{4^4}{5^5},$$

and the maximum is attained when $\alpha = \frac{1}{2}$. In particular, the dependence function A producing the value δ_C is

$$A(t) = \max\left(1-t, \frac{t+1}{2}\right). \tag{3}$$

The corresponding copula has the following expression:

$$C(x,y) = \begin{cases} y, & x^2 \geqslant y, \\ x\sqrt{y}, & \text{otherwise}, \end{cases} \tag{4}$$

and δ_C is obtained when $\alpha = \frac{1}{2}$, $\frac{\log(x)}{\log(xy)} = t_{1/2} = \frac{1}{3}$, $xy = k_{1/2} = (\frac{4}{5})^6$, and, hence, at the point $(\frac{4^2}{5^2}, \frac{4^4}{5^4})$.

The other EV copula that attains δ_C is given by the transpose of the copula C given by (4) and it is generated by the dependence function $A_1(t) = A(1-t)$. It can be also obtained from the above proof by assuming that $t \in [\frac{1}{2}, 1]$. □

Remark 1. In [16], it has been proved that EV copulas satisfy a strong notion of positive dependence, namely they are *stochastically increasing* (*shortly, SI*) *in each variable*. Recently, it was proved in [10] that the upper bound for the measure of non-exchangeability μ_∞ in the family of SI copulas is given by $3 \cdot \frac{5\sqrt{5}-11}{2} \approx 0.271$ and, thus, it is greater than the analogous upper bound in the family of EV copulas (approximately equal to 0.246).

Remark 2. Notice that the exchangeability of an EV copula C reflects on the symmetry of its dependence function A with respect to the line $\{x = \frac{1}{2}\}$. Based on this fact, some other possible "ad hoc" measures of non-exchangeability for EV copulas can be considered. However, in order to compare the asymmetry of EV copulas with the asymmetry of other classes of copulas already considered in the literature, we prefer to restrict ourselves to the general approach based on the measure μ_∞.

In [25] it has been considered a dependence coefficient θ for an EV copula C with dependence function A, given by $\theta = 1 - A(\frac{1}{2})$. In the following, we refer to such a $\theta = \theta_{KM}$ as the *Klüppelberg–Mai dependence coefficient* (shortly, KM-dependence coefficient). For maximally non-exchangeable EV copulas of type (2), we have that $\theta_{KM} = \frac{1}{4}$, i.e., the KM-dependence coefficient takes the mean value among the two extremal case, $\theta_{KM} = 0$ and $\theta_{KM} = \frac{1}{2}$ that correspond to the exchangeable EV copulas Π and M, respectively. Notice that it is easy to show that θ_{KM} is also related to the upper tail dependence coefficient λ_U of the copula C by means of the formula $2\theta_{KM} = \lambda_U$ (see [21]). By using the same ideas of Theorem 1, one can find the maximally non-exchangeable EV copulas with a given KM-dependence coefficient, or, equivalently, a given upper tail dependence coefficient, as shown by the following result.

Proposition 2. *Let C be an EV copula with dependence function A. Suppose that the KM-dependence coefficient of C is given by $\theta \in]0, \frac{1}{2}[$. Then*

$$\mu_\infty(C) \leqslant 3\left(\left(1+2\theta-4\theta^2\right)^{\frac{1}{2\theta(-1+2\theta)}} - \left(1+2\theta-4\theta^2\right)^{\frac{1+2\theta-4\theta^2}{2\theta(-1+2\theta)}}\right). \tag{5}$$

Moreover, $\mu_\infty(C)$ attains the upper bound in (5) *if, and only if, $C \in \{C_1, C_2\}$ where C_1 and C_2 are the EV copulas with dependence functions given, respectively, by*:

$$A_\theta^1(t) = \max(1-t, 2\theta t + 1 - 2\theta) \quad or \quad A_\theta^2(t) = \max(t, 1-2\theta t). \tag{6}$$

Proof. Let C be an EV copula with dependence function A such that $A(\frac{1}{2}) = 1-\theta$ for a given $\theta \in]0, \frac{1}{2}[$. We aim at determining the value

$$\delta_C = \max_{(x,y)\in[0,1]^2} \left|C(x,y) - C(y,x)\right|.$$

We consider that δ_C is reached at a point $(x,y) \in]0,1[^2$ and set $t = \frac{\log(x)}{\log(xy)}$. Without loss of generality, we assume that $t \in]0, \frac{1}{2}[$ and $A(t) \leqslant A(1-t)$. By repeating the same procedure as in the proof of Theorem 1 and considering the constraint on the value $A(\frac{1}{2})$, we obtain that the dependence function that describes the maximally non-exchangeable case is given by

$$A_\theta(t) = \max(1-t, 2\theta t + 1 - 2\theta).$$

In particular, δ_C is obtained when $t_\theta = \frac{2\theta}{2\theta+1}$ and, hence,

$$A_\theta(t_\theta) = \frac{1}{2\theta+1}, \qquad A_\theta(1-t_\theta) = \frac{2\theta+1-4\theta^2}{2\theta+1}.$$

Now, δ_C is the maximum of

$$\psi_\theta : [0,1] \to [0,1], \qquad \psi_\theta(k) = k^{\frac{1}{2\theta+1}} - k^{\frac{2\theta+1-4\theta^2}{2\theta+1}},$$

which is attained at the point k_θ satisfying $\psi_\theta'(k_\theta) = 0$ and given by

$$k_\theta = \left(1+2\theta-4\theta^2\right)^{\frac{1+2\theta}{2\theta(-1+2\theta)}}.$$

Elementary calculations show

$$\begin{aligned}
\delta_C &= \psi_\theta(k_\theta) \\
&= \left(\left(1+2\theta-4\theta^2\right)^{\frac{1+2\theta}{2\theta(-1+2\theta)}}\right)^{\frac{1}{1+2\theta}} - \left(\left(1+2\theta-4\theta^2\right)^{\frac{1+2\theta}{2\theta(-1+2\theta)}}\right)^{\frac{1+2\theta-4\theta^2}{1+2\theta}} \\
&= \left(1+2\theta-4\theta^2\right)^{\frac{1}{2\theta(-1+2\theta)}} - \left(1+2\theta-4\theta^2\right)^{\frac{1+2\theta-4\theta^2}{2\theta(-1+2\theta)}},
\end{aligned}$$

which is the desired assertion. □

EV copulas generated by the dependence functions of Eq. (6) belong to the Marshall–Olkin family of copulas. For instance, A_θ corresponds to the survival copula of Marshall–Olkin bivariate exponential distribution [28], which is associated with a random pair (X,Y) with stochastic representation $(X,Y) = (Z, \min(Z, Z_1))$, where Z and Z_1 are independent exponential r.v.'s with rates 1 and $\frac{1+2\theta}{2\theta}$. Although these copulas have a singular component (and, hence, seem to be not very appealing for applications), they have been recently considered as a basis for constructing multivariate copulas with some interesting properties and applications (compare with [12,27]).

Moreover, note that A_θ can be considered as a limiting case of the asymmetric logistic model considered in [35, Eq. (5.3)], which can hence be considered adequate for modeling a wide range of asymmetry.

3. Non-exchangeability for Archimax copulas

Now, let us consider the class of *Archimax* copulas, which has been introduced in [5].

Let $A : [0,1] \to [1/2, 1]$ be a convex function such that $\max(t, 1-t) \leqslant A(t) \leqslant 1$ for every $t \in [0,1]$, and let φ be an additive generator of an Archimedean copula, i.e. $\varphi : [0,1] \to [0,+\infty]$ is continuous, strictly decreasing and convex with $\varphi(1) = 0$ and *pseudo-inverse* $\varphi^{[-1]}$ defined by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leqslant t \leqslant \varphi(0), \\ 0, & \text{otherwise.} \end{cases}$$

The following function

$$C_{\varphi,A}(x,y) := \varphi^{[-1]}\left[\left(\varphi(x)+\varphi(y)\right)A\left(\frac{\varphi(x)}{\varphi(x)+\varphi(y)}\right)\right]$$

is a copula, known as *Archimax* copula (generated by φ and A). The family of Archimax copulas includes both Archimedean copulas and EV copulas. As a matter of fact, it can be also obtained from the family of EV copulas by using the transformation of copulas described in [13].

As we will see in the next result, the previous methods for finding bounds for the non-exchangeability of EV copulas can be further applied to investigate the maximal non-exchangeability of this class of copulas.

Proposition 3. *For every Archimax copula $C_{\varphi,A}$*

$$\mu_\infty(C_{\varphi,A}) \leqslant 3\left(\sup_{u\in]0,+\infty[}\left|\varphi^{[-1]}(u)-\varphi^{[-1]}\left(\frac{5u}{4}\right)\right|\right). \tag{7}$$

Moreover, $\mu_\infty(C_{\varphi,A})$ attains the upper bound in (7) if, and only if, $C \in \{C_1, C_2\}$ where C_1 and C_2 are the Archimax copulas generated by φ and by the dependence functions given, respectively, by:

$$A^1(t) = \max\left(1-t, \frac{t+1}{2}\right), \quad \textit{or} \quad A^2(t) = \max\left(t, \frac{2-t}{2}\right). \tag{8}$$

Proof. Let $C = C_{\varphi,A}$ be an Archimax copula. We aim at determining the value

$$\delta_C = \max_{(x,y)\in[0,1]^2}\left|C(x,y)-C(y,x)\right|.$$

Suppose that δ_C is reached at a point $(x,y) \in]0,1[^2$ and set $t = \frac{\varphi(x)}{\varphi(x)+\varphi(y)}$. Without loss of generality, we assume that $t \in]0, \frac{1}{2}[$ and $A(t) \leqslant A(1-t)$. Then

$$\delta_C = \sup_{(x,t)\in]0,1[\times]0,\frac{1}{2}[}\left|\varphi^{[-1]}\left(\frac{\varphi(x)}{t}\cdot A(t)\right)-\varphi^{[-1]}\left(\frac{\varphi(x)}{t}\cdot A(1-t)\right)\right|.$$

Setting $u = \frac{\varphi(x)A(t)}{t}$, we have

$$\delta_C = \sup_{u\in]0,+\infty[}\left(\sup_{t\in]0,1/2[}\left|\varphi^{[-1]}(u)-\varphi^{[-1]}\left(u\frac{A(1-t)}{A(t)}\right)\right|\right).$$

Now, for a fixed u, the above difference takes its maximum value for some dependence function A that realizes the maximum value of $\frac{A(1-t)}{A(t)}$. But, as in the proof of Theorem 1, from the convexity of A it follows that:

$$\frac{A(1-t)}{A(t)} \leqslant \frac{A_\alpha(1-t)}{A_\alpha(t)},$$

where, for $\alpha, s \in [0,1]$, $A_\alpha(s) = \max\{1-s, (1-\alpha)s+\alpha\}$ and $g(t) = (1-\alpha)s+\alpha$ is the line passing through $(t, A(t))$ and $(1,1)$. Moreover,

$$\frac{A_\alpha(1-t)}{A_\alpha(t)} \leqslant \frac{A_\alpha(1-t_\alpha)}{A_\alpha(t_\alpha)},$$

where $t_\alpha = \frac{1-\alpha}{2-\alpha}$, $A_\alpha(t_\alpha) = \frac{1}{2-\alpha}$ and $A_\alpha(1-t_\alpha) = \frac{1+\alpha-\alpha^2}{2-\alpha}$, and $\frac{A_\alpha(1-t_\alpha)}{A_\alpha(t_\alpha)}$ takes its maximum value when $\alpha = \frac{1}{2}$. Therefore

$$\begin{aligned}\delta_C &= \sup_{(u,\alpha)\in]0,+\infty[\times]0,1[}\left|\varphi^{[-1]}(u)-\varphi^{[-1]}\left(u\frac{A_\alpha(1-t_\alpha)}{A_\alpha(t_\alpha)}\right)\right| \\ &= \sup_{u\in]0,+\infty[}\left|\varphi^{[-1]}(u)-\varphi^{[-1]}\left(\frac{5u}{4}\right)\right|,\end{aligned}$$

which is the desired assertion. □

Remark 3. Notice that any maximally non-exchangeable Archimax copula (in particular, EV copula) has a singular component. In fact, it can be easily seen that a necessary condition for the absolute continuity of an Archimax copula $C_{\varphi,A}$ is the existence (and continuity) of both first derivatives on $]0,1[^2$, which requires (again as a necessary condition) the differentiability of the dependence function A on $]0,1[$. However, non-exchangeable Archimax copulas are linked with dependence functions of type (8), which either in the point 1/3 or in 2/3 do not possess the derivative.

Table 1
Approximated sharp upper bounds for the measure of non-exchangeability of families of Archimax copulas with a specified additive generator.

Generator $\varphi(t)$	$\max_A(\mu_\infty(C_{\varphi,A}))$
$t^{-1}-1$	0.167
$\log(\frac{3-t}{2t})$	0.274
$(-\log(t))^3$	0.082
$-\log(\frac{e^{-t}-1}{e^{-1}-1})$	0.214
$1-t$	0.600

The maximal non-exchangeability of several families of Archimax copulas additively generated by some φ is computed in Table 1. As can be noted, Archimax copulas have a wider range of asymmetry than EV copulas. Moreover, due to the fact that they are easily tractable and can be conveniently simulated [5], this family seems to have several nice properties for describing stochastic models, which cannot be assumed to be exchangeable and/or to have an EV interpretation.

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