On the $\alpha$-migrativity of semicopulas, quasi-copulas, and copulas

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Abstract
In this paper we address the problem of $\alpha$-migrativity (for a fixed $\alpha$) for semicopulas, copulas and quasi-copulas. We introduce the concept of an $\alpha$-sum of semicopulas. This new concept allows us to completely characterize $\alpha$-migrative semicopulas and copulas. Moreover, $\alpha$-sums also provide a means to obtain a partial characterization of $\alpha$-migrative quasi-copulas.

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1. Introduction and preliminaries

For some image processing applications and decision-making problems, it is important to ensure that variations in the value of some functions caused by considering just a given fraction of one of the input variables is independent of the actual choice of variable. For instance, it is sometimes of interest to darken a certain part of an image. In decision-making, the ordering of inputs may be relevant, even though the result of modifying one or another evaluation by a given ratio is the same. The concept of migrativity captures this idea. In this paper we focus on the $\alpha$-migrativity for some fixed $\alpha$; in other words, we consider that the reduction factor is determined by a fixed factor $0 < \alpha \leq 1$.

Mathematically, the $\alpha$-migrative property for a mapping $A : [0, 1] \times [0, 1] \rightarrow [0, 1]$ means that the identity

$$ A(\alpha x, y) = A(x, \alpha y) $$

holds for all $x, y \in [0, 1]$. Property (2) below extended to the class of all bivariate functions on $[0,1]$ was introduced by Durante and Sarkoci [8] and further studied by Fodor and Rudas [9], whereas the particular case of aggregation functions was considered by Beliakov and Calvo [2]. We previously investigated and characterized aggregation functions that are $\alpha$-migrative for all $\alpha \in [0, 1]$ [4]; in particular, we showed that the only migrative function with neutral element 1 is the product $\Pi(x, y) = xy$.

Property (2) for some specific aggregation functions has already been considered in the literature. In particular, the following problem was posed for t-norms by Mesiar and Novák [11].

**Problem.** Is there any t-norm $T$ different from

$$ T(x, y) = \begin{cases} 
  cxy & \text{if } \max(x, y) < 1, \\
  \min(x, y) & \text{otherwise},
\end{cases} $$

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such that, for a fixed $x \in [0, 1]$
\[ T(2x, y) = T(x, 2y) \quad \text{for any } x, y < 1 \] (2)

This problem was definitively solved by Budincevic and Kurilić [3]. Based on their ideas, Mesiar and colleagues proposed a t-norm $T$, ([10, Exp. 2.11]) studied by Smutná [14]:
\[ T_*(x, y) = \left\{ \begin{array}{ll}
\sum_{n=1}^\infty \frac{1}{2^n} & \text{if } x, y \in [0, 1], \\
0 & \text{otherwise}.
\end{array} \right. \] (3)

where, if $x > 0 (y > 0)$, $(x_n)(y_n))$ is a strictly increasing sequence in $\mathbb{N}$ such that
\[ x = \sum_{n=1}^\infty \frac{1}{2^n} \quad (y = \sum_{n=1}^\infty \frac{1}{2^n}). \]

Evidently, for $x = 2^{-k}$ with $k \in \mathbb{N}$
\[ 2^{-k}x = \sum_{n=1}^\infty \frac{1}{2^{n-k}}, \]

and analogously for $y$ (here and subsequently, whenever the summation bounds are omitted, the index is understood to vary from 1 to $\infty$). Thus, we have:
\[ T_*(x, y) = T_*(x, 2y), \]

for all $x, y < 1$ (even for all $x, y \leq 1$) and for all $x = 2^{-k}$ with $k \in \mathbb{N}$.

The case of $x$-migrative t-norms has also been considered [6,8,9]. In the particular case of continuous t-norms, they were shown to be strict t-norms generated by an additive generator $t : [0, 1] \to [0, \infty]$ satisfying
\[ t(x) = kt(x) + t(x^{-k}), \] (4)

for all $x \in [x^{k-1}, x^k], k \in \{0, 1, 2, \ldots \}$ (i.e., on $[x, 1]$ the choice of $t$ is free).

Inspired by the results mentioned above, we investigated $x$-migrative semicopulas with a special focus on copulas and quasi-copulas. Observe that for associative copulas $x$-migrativity forces the strictness of the discussed copula and $x$-migrative strict copulas are generated by an additive generator $t$ given by (4) [2], additionally satisfying the convexity property, i.e., $t$ is convex on $[x, 1]$ and $t(1^-) \leq 2t(x^+).$ Here and in the following, for a given function $f : [0, 1] \to [0, 1]$ and for a point $x_0 \in [0, 1], f(x_0^+)$ and $f(x_0^-)$ denote the limit from the right and from the left, respectively, of $f$ at $x_0$.

For convenience, we now describe some of the basic notions involved in our study.

**Definition 1.1.** A (bivariate) aggregation function is a mapping $A : [0, 1] \times [0, 1] \to [0, 1]$ such that

(i) $A(0, 0) = 0$ and $A(1, 1) = 1$; and
(ii) $A$ is non-decreasing in both variables.

**Definition 1.2.** Let $x \in [0, 1].$ An aggregation function $A : [0, 1] \times [0, 1] \to [0, 1]$ is $x$-migrative if the identity
\[ A(ax, y) = A(x, ay) \]

holds for all $x, y \in [0, 1].$

**Definition 1.3.** An aggregation function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called a semicopula if 1 is its neutral element, i.e., $S(x, 1) = S(1, x) = x$ for all $x \in [0, 1].$ A 1-Lipschitz semicopula, i.e., a semicopula $Q$ satisfying
\[ |Q(x, y) - Q(x', y')| \leq |x - x'| + |y - y'|, \]

for all $x, y, x', y' \in [0, 1],$ is called a quasi-copula.

A semicopula $C$ that is 2-increasing, that is:
\[ C(x, y) + C(x', y') - C(x, y') - C(x', y) \geq 0, \]

for all $0 \leq x \leq x' \leq 1$ and $0 \leq y \leq y' \leq 1,$ is called a copula.

More details are available elsewhere [1,12]. Note that each copula is also a quasi-copula and that quasi-copulas that are not copulas are termed proper.

The remainder of the paper is organized as follows. In the next section, $x$-migrative semicopulas are characterized and the $x$-migrative sum of $x$-migrative semicopulas is introduced. Section 3 is devoted to the study of $x$-migrative copulas and quasi-copulas. In particular, we describe expression (3) for $T_*$. In Section 4, some final considerations are discussed.
2. \( \alpha \)-Migrative semicopulas

Throughout the remainder of the paper, \( \alpha \in [0, 1] \) is fixed. Because semicopulas possess a neutral element 1, it is evident that each \( \alpha \)-migrative semicopula \( S \) satisfies:

\[
S(\alpha, x) = S(x, \alpha) = \alpha x,
\]

and by induction

\[
S(\alpha^k, x) = S(x, \alpha^k) = \alpha^k x,
\]

for \( k = 2, 3, \ldots \) Hence, for any \( \alpha \)-migrative semicopula \( S \) the following result holds.

**Lemma 2.1.** Let \( S : [0, 1] \times [0, 1] \) be an \( \alpha \)-migrative semicopula. Then, for any \( x, y \in [0, 1] \) such that \( \{ x, y \} \cap \{ 1, \alpha, \alpha^2, \ldots \} \neq \emptyset \), it holds that

\[
S(x, y) = \Pi(x, y) = xy.
\]

The next result can also be obtained by induction.

**Lemma 2.2.** Let \( S : [0, 1] \times [0, 1] \) be an \( \alpha \)-migrative semicopula. Then, for any \( x, y \in [0, 1] \) and \( k, m, i, j \in \{ 0, 1, 2, \ldots \} \), it holds that

\[
S(\alpha^k x, \alpha^m y) = S(\alpha^k x, \alpha^j y),
\]

whenever \( k + m = i + j \).

These two lemmas have a crucial impact on the following definition. We denote \( \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} \).

**Definition 2.1.** Let \( (S_i)_{i \in \mathbb{N}_0} \) be a system of \( \alpha \)-migrative semicopulas. Then the function \( S : [0, 1] \times [0, 1] \to [0, 1] \) given by

\[
S(x, y) = \begin{cases} 
S_i(x, y) & \text{if } (x, y) \in E_i \text{ for some } i \in \mathbb{N}_0, \\
0 & \text{otherwise},
\end{cases}
\]

where for \( i \in \mathbb{N}_0, E_i = \bigcup_{m,k \in \mathbb{N}_0, m+k=i} \alpha^{k+1} \times \alpha^m \), is called \( \alpha \)-migrative sum of \( (S_i)_{i \in \mathbb{N}_0} \). This is denoted by \( S = \alpha - (S_i)_{i \in \mathbb{N}_0} \).

**Proposition 2.3.** The \( \alpha \)-migrative sum of semicopulas is a semicopula.

**Proof.** First, we consider the neutrality of 1. Clearly \( S(1, 0) = S(0, 1) = 0 \), and if \( x \in \alpha^{k+1}, \alpha^k \) for some \( k \in \mathbb{N}_0 \), then

\[
S(1, x) = S_k(1, x) = x = S_k(x, 1) = S(x, 1),
\]

The structure of the sets \( E_i \) and the \( \alpha \)-migrative sum of semicopulas is illustrated in Fig. 1.

**Fig. 1.** Structure of an \( \alpha \)-migrative sum of semicopulas, with \( \alpha = 0.7 \).
Proof. The fact that 

\[ S(x, y) \leq S(x', y) \]

when \((x', y) \in E_i, (x, y) \in E_j \) and \( j > i \) (and similarly for the other variable). Suppose \( y \in [x_{m-1}, x_m] \). Then

\[ x \in [x_i^{i-m}, x_{i-m}] \quad \text{and} \quad x' \in [x_{i-m}^{i-1}, x_{i-1-m}], \]

and hence

\[ S(x, y) = S_j(x, y) \leq S_j(x', y) = x'^{-m}y = S_i(y', x) = S(x', y), \]

as required. \( \square \)

**Proposition 2.4.** A semicopula \( S \) is \( \alpha \)-migrative if and only if there exists a system \((S_i)_{i \in \mathbb{N}_0}\) of \( \alpha \)-migrative semicopulas such that \( S \) is the \( \alpha \)-migrative sum of \( S_i \), i.e., \( S = \alpha - (S_i)_{i \in \mathbb{N}_0} \).

**Proof.** Necessity is obvious, as it is enough to consider the constant system \((S_i)_{i \in \mathbb{N}_0}\). To prove the sufficiency, by the previous proposition we already have that the \( \alpha \)-migrative sum \( S = \alpha - (S_i)_{i \in \mathbb{N}_0} \) is a semicopula. Now observe that if \((x, y) \in E_i\), then \((x, y) \) and \((x, xy)\) belong to \(E_{i+1}\). Thus, owing to the \( \alpha \)-migrativity of \( S_{i+1} \), it holds that:

\[ S(x, y) = S(x, xy). \]

Moreover, \( xy = 0 \) if and only if \( axy = 0 \), and in this case:

\[ S(x, y) = S(x, 2y) = 0. \]

Consequently, each \( \alpha \)-migrative sum is \( \alpha \)-migrative. \( \square \)

The next result follows directly from **Definition 1.2 and Lemma 2.1.**

**Proposition 2.5.** Let \( S \) be an \( \alpha \)-migrative semicopula. Then the mapping \( D : [0, 1] \times [0, 1] \to \mathbb{R} \) given by

\[
D(x, y) = \frac{S(x + (1 - \alpha)x, x + (1 - \alpha)y) - \alpha(1 - \alpha)(x + y) - \alpha^2}{(1 - \alpha)^2}
\]

satisfies

(i) \( D(x, 0) = D(0, x) = 0 \) for any \( x \in [0, 1] \) (i.e., \( 0 \) is an annihilator of \( D \));

(ii) \( D(x, 1) = D(1, x) = x \) for any \( x \in [0, 1] \), (i.e., \( 1 \) is a neutral element for \( D \)); and

(iii) \( (1 - \alpha)D(x, y) + \alpha(x + y) \leq (1 - \alpha)D(x', y') + \alpha(x' + y') \) for any \( x, y, x', y' \in [0, 1] \) such that \( x \leq x' \) and \( y \leq y' \).

**Proof.** First, note that for \((u, v) \in [x, 1] \times [x, 1]\):

\[ S(u, v) = x(u + v) - \alpha^2 + (1 - \alpha)^2 D \left( \frac{u - x}{1 - \alpha}, \frac{v - x}{1 - \alpha} \right). \]

The result follows from the non-decreasing property of \( S \) and the identities \( S(x, v) = x v, S(u, x) = x u, S(1, v) = v \) and \( S(u, 1) = 1 \). \( \square \)

**Definition 2.2.** A function \( D : [0, 1] \times [0, 1] \to \mathbb{R} \) satisfying the properties given in **Proposition 2.5** is called an \( \alpha \)-generating function.

Note that, regardless of the value of \( \alpha \in [0, 1] \), any semicopula \( S \) is an \( \alpha \)-generating function. Moreover, we have the following two results.

**Proposition 2.6.** The strongest semicopula \( M(x, y) = \min(x, y) \) is also the strongest \( \alpha \)-generating function. That is, \( M \) is an \( \alpha \)-generating function and, for any other \( \alpha \)-generating function \( D \), the inequality

\[ D(x, y) \leq M(x, y) \]

holds for all \( x, y \in [0, 1] \).

**Proof.** The fact that \( M(x, y) = \min(x, y) \) is an \( \alpha \)-generating function follows from easy calculations. To prove that for any other \( \alpha \)-generating function \( D \) the inequality \( D(x, y) \leq M(x, y) \) holds, just observe that \( D(x, y) \leq D(1, y) = y \) and \( D(x, y) \leq D(x, 1) = x \) for all \( x, y \in [0, 1] \). \( \square \)

**Proposition 2.7.** The weakest \( \alpha \)-generating function \( D^{(\alpha)} \) is given by

\[
D^{(\alpha)}(x, y) = \begin{cases} \frac{1}{1 - \alpha} \min(x, y) & \text{if } \max(x, y) < 1, \\ \min(x, y) & \text{otherwise.} \end{cases}
\]
**Proof.** Let $D$ be an $\alpha$-generating function. From (ii) and (iii) in Proposition 2.5 it follows that, for any $x, y \in [0, 1]$

$$ax = (1 - \alpha)D(x, 0) + ax \leq (1 - \alpha)D(x, y) + \alpha(x + y),$$

and

$$ay = (1 - \alpha)D(0, y) + ay \leq (1 - \alpha)D(x, y) + \alpha(x + y),$$

so we have that

$$D(x, y) \geq \max \left( -\frac{\alpha}{1 - \alpha} x, -\frac{\alpha}{1 - \alpha} y \right) = -\frac{\alpha}{1 - \alpha} \min(x, y),$$

as required. □

Now we are ready to give a complete characterization of $\alpha$-migrative semicopulas.

**Theorem 2.8.** Let $S$ be a bivariate function. Then $S$ is an $\alpha$-migrative semicopula if and only if there exists a system $(D_i)_{i \in \mathbb{N}_0}$ of $\alpha$-generating functions such that

$$S(x, y) = \left\{ \begin{array}{ll}
\alpha^{m+1}x + \alpha^k y - \alpha^{k+m+2} + (1 - \alpha)^2 \alpha^{k+m} D_{k,m} \left( \frac{x - \alpha^k}{\alpha^k - \alpha^{k+1}}, \frac{y - \alpha^{m+1}}{\alpha^m - \alpha^{m+1}} \right), & \text{if } (x, y) \in [\alpha^{k+1}, \alpha^k] \times [\alpha^{m+1}, \alpha^m], \\
0, & \text{otherwise.}
\end{array} \right. \tag{6}$$

**Proof.** To see the necessity, observe that, as $S$ is non-decreasing, from Proposition 2.4, it follows that $S$ can be written as an $\alpha$-migrative sum $(S_i)_{i \in \mathbb{N}_0}$. However, from Eq. (5) in Proposition 2.5, each of the terms $S_i$ can be written in terms of an $\alpha$-generating function $D_{k,m}$ for $(x, y) \in [\alpha^{k+1}, \alpha^k] \times [\alpha^{m+1}, \alpha^m]$, with $k, m \in \mathbb{N}_0$ as follows:

$$S_i(x, y) = \alpha^{m+1}x + \alpha^k y - \alpha^{k+m+2} + (1 - \alpha)^2 \alpha^{k+m} D_{k,m} \left( \frac{x - \alpha^k}{\alpha^k - \alpha^{k+1}}, \frac{y - \alpha^{m+1}}{\alpha^m - \alpha^{m+1}} \right).$$

Moreover, from Proposition 2.4 it also holds that $D_{k,m} = D_{k',m'}$ whenever $k + m = k' + m'$. Thus, the condition is necessary.

To prove the sufficiency, observe that, if $(x, y) \in [\alpha^{k+1}, \alpha^k] \times [\alpha^{m+1}, \alpha^m]$, then

$$S(ax, ay) = \alpha^{m+1}ax + \alpha^k ay - \alpha^{k+m+3} + (1 - \alpha)^2 \alpha^{k+m+1} D_{k,m+1} \left( \frac{ax - \alpha^k}{\alpha^k - \alpha^{k+1}}, \frac{ay - \alpha^{m+1}}{\alpha^m - \alpha^{m+1}} \right),$$

wheras

$$S(x, y) = \alpha^{m+1}x + \alpha^k y - \alpha^{k+m+3} + (1 - \alpha)^2 \alpha^{k+m+1} D_{k,m+1} \left( \frac{x - \alpha^k}{\alpha^k - \alpha^{k+1}}, \frac{y - \alpha^{m+1}}{\alpha^m - \alpha^{m+1}} \right).$$

Evidently, $S(ax, ay) = S(x, y)$, ensuring the $\alpha$-migrativity of $S$. To prove the monotonicity, we can use arguments similar to those for the proof of Proposition 2.4. Finally, the fact that $S(0, 0) = 0$ is obvious from the definition of $S$, whereas

$$S(x, 1) = S(1, x) = 1 \text{ for all } x \in [0, 1]$$

follows from property (ii) in Proposition 2.5. □

**Definition 2.3.** Let $D$ be an $\alpha$-generating function. For the constant system $(D_i)_{i \in \mathbb{N}_0}$, the $\alpha$-migrative copula given by (6) is denoted as $S_{(D, \alpha)}$.

The next result is an easy corollary of Proposition 2.4.

**Corollary 2.9.** A semicopula $S$ is $\alpha$-migrative if and only if $S$ is the $\alpha$-migrative sum $S = \alpha - (S_{(D, \alpha)})_{i \in \mathbb{N}_0}$, where $(D_i)_{i \in \mathbb{N}_0}$ is a system of $\alpha$-generating functions.

**Remark 1.**

(i) In Theorem 2.8 and Corollary 2.9, the choice of the system $(D_i)_{i \in \mathbb{N}_0}$ of $\alpha$-generating functions has no restriction and it is evident that different systems generate different $\alpha$-migrative semicopulas.

(ii) The symmetry of an $\alpha$-migrative semicopula $S$ is equivalent to the symmetry of each $\alpha$-generating function $D_i$ in representation (6).

(iii) Owing to Corollary 2.9, a prominent role in the study of $\alpha$-migrative semicopulas is played by those generated by a single generating function.

**Example 1.**

(i) The strongest $\alpha$-migrative semicopula is:

$$S_{(M, \alpha)} : [0, 1] \times [0, 1] \to [0, 1]$$
given, for \((x, y) \in [x^{k+1}, x^k] \times [x^{m+1}, x^m]\), by
\[
S_{(M, a)}(x, y) = x^{m+1}x + x^{k+1}y - x^{k+m+2}x + (1 - x) \min(x^{m}x - x^{m+1}, x^{k}y - x^{k+m+1})
\]
\[
= x^{m+1}x + x^{k+1}y - x^{k+m+1} + (1 - x) \min(x^{m}x, x^{k}y).
\]

Its support is depicted in Fig. 2 for \(x = 0.7\). Remember that the support of a semicopula \(S\), by analogy with that of a copula, is its support when considered as a probability distribution function on \([0, 1]^2\); that is, the complement of the union of all open sets in \([0, 1]^2\) with \(S\)-measure zero [12]. Observe also that \(S_{(M, a)}\) is a singular copula (i.e., a copula with support having zero Lebesgue measure).

(ii) The weakest \(x\)-migrative semicopula is
\[
S_{(D, \alpha^2)} : [0, 1] \times [0, 1] \rightarrow [0, 1]
\]
given, for \((x, y) \in [x^{k+1}, x^k] \times [x^{m+1}, x^m]\), by
\[
S_{(D, \alpha^2)}(x, y) = \max(x^{m+1}x, x^{k+1}y),
\]
whereas, if \((x, y) \cap \{0, 1\} \neq \emptyset\)
\[
S_{(D, \alpha^2)}(x, y) = \min(x, y).
\]

(iii) For the product \(II\), it holds that \(II = S_{(H, \alpha)}\).

(iv) The \((1/2)\)-migrative t-norm \(T_{\alpha}\), introduced in Section 1, satisfies \(T_{\alpha} = S_{(D, 1/2)}\), where the \((1/2)\)-generating function \(D\) is given by
\[
D(x, y) = \begin{cases} 
2T_{\alpha}(x, y) - x - y + 1 & \text{if } \min(x, y) > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

3. \(x\)-Migrative copulas and quasi-copulas

As observed in Example 1, an \(x\)-migrative semicopula \(S_{(D, \alpha)}\) can be generated by an \(x\)-generating function \(D\) that is not a semicopula, as the non-decreasing property of \(D\) may be violated. By contrast, for any semicopula \(H, S_{(H, \alpha)}\) is again an \((x\)-migrative\) semicopula. In the case of t-norms (associative and symmetric semicopulas), we have all possible situations. As shown in Example 1 (iv), there are t-norms generated by non-associative and non-monotonic \(x\)-generating functions. By contrast, there are t-norms of the form \(S_{(T, \alpha)}\), where \(T\) is a t-norm. For example, for the Lukasiewicz t-norm \(T_{\alpha}(x, y) = \max(x + y - 1, 0)\), the corresponding \(x\)-migrative semicopula \(S_{(T, \alpha)}\) is the weakest \(x\)-migrative 1-Lipschitz t-norm and its additive generator \(r : [0, 1] \rightarrow [0, \infty]\) is a piecewise linear function determined by points \((\alpha^i, i), i \in \mathbb{N}_0\) (cf. Ref. [2]). The support of \(S_{(T, \alpha)}\) (for \(x = 0.7\)) is depicted in Fig. 3. Example 1 (i) shows that not every t-norm \(T\) generates an \(x\)-migrative t-norm (associativity of \(S_{(M, \alpha)}\) is violated). A different situation occurs for the class of copulas and quasi-copulas.

![Fig. 2. Support of \(S_{(M, \alpha)}\) (for \(x = 0.7\)).](image-url)
**Corollary 3.2.** Let $D$ be a quasi-copula. It is a proper quasi-copula whenever there is at least one proper quasi-copula in the system 
\[ k \] 
only if $D$ is a copula.

**Proof.** Observe that the 2-increasing property of a function $C : [0, 1] \times [0, 1] \to [0, 1]$, together with 0 being the annihilator of $C$ and 1 being the neutral element of $C$, ensures that $C$ is a copula. Moreover, it is not difficult to check that the 2-increasing property of $C$ over $[0, 1] \times [0, 1]$ is equivalent to the 2-increasing property of $C$ over all rectangles $[x^{k+1}, x^k] \times [y^m, y^{m+1}]$, for $k, m \in \mathbb{N}_0$. These facts, together with Proposition 2.5 and Theorem 2.8, prove the result. \[ \square \]

**Theorem 3.1.** A semicopula $S$ is an $\alpha$-migrative copula if and only if there exists a system $(C_i)_{i \in \mathbb{N}_0}$ of copulas generating $S$ by means of (6).

**Proof.** Observe that the 2-increasing property of a function $C : [0, 1] \times [0, 1] \to [0, 1]$, together with 0 being the annihilator of $C$ and 1 being the neutral element of $C$, ensures that $C$ is a copula. Moreover, it is not difficult to check that the 2-increasing property of $C$ over $[0, 1] \times [0, 1]$ is equivalent to the 2-increasing property of $C$ over all rectangles $[x^{k+1}, x^k] \times [y^m, y^{m+1}]$, for $k, m \in \mathbb{N}_0$. These facts, together with Proposition 2.5 and Theorem 2.8, prove the result. \[ \square \]

**Corollary 3.2.** Let $D : [0, 1] \times [0, 1] \to \mathbb{R}$ be an $\alpha$-generating function. Then the $\alpha$-migrative semicopula $S_{(D, \alpha)}$ is a copula if and only if $D$ is a copula.

It is evident that the strongest $\alpha$-migrative copula is $S_{(M, \alpha)}$. As already mentioned, it is a singular copula for which the support is depicted in Fig. 3. More details on singular copulas are available elsewhere [12].

Similarly, the weakest $\alpha$-migrative copula is $S_{(I, \alpha)}$, which is indeed a $t$-norm, and its additive generator (unique up to a positive multiplicative constant) $\{T_{(I, \alpha)}\} : [0, 1] \to [0, \infty]$ is given by

\[ T_{(I, \alpha)}(x) = k(1 - x) + \left(1 - \frac{x}{\alpha^k}\right) \quad \text{if } x \in [x^{k+1}, x^k]. \]

In this case we also have a singular copula with support as shown in Fig. 3.

For $\alpha$-migrative quasi-copulas we have only a sufficient condition.

**Proposition 3.3.** Let $(Q_i)_{i \in \mathbb{N}_0}$ be a system of quasi-copulas. Then the $\alpha$-migrative semicopula $S$ generated by this system as in (6) is a quasi-copula. It is a proper quasi-copula whenever there is at least one proper quasi-copula in the system $(Q_i)_{i \in \mathbb{N}_0}$.

**Proof.** The monotonicity and 1-Lipschitz property of $Q_i$ ensure the same properties for $S$ on the closure of $E_i$, It is evident that $S$ is a continuous $\alpha$-migrative semicopula and thus 1-Lipschitzianity of $S$ on the closures of all $E_i$ ensures the 1-Lipschitz property of $S$ on the whole domain $[0, 1] \times [0, 1] = \cup E_i$. To prove this, observe that if $x \in [x^{k+1}, x^k]$ and $y \in [x^{k-2}, x^{k+1}]$ for some $k \in \mathbb{N}_0$, then we have that $Q(x, z) \geq Q(y, z)$ for all $z \in [0, 1]$. Thus, in particular

\[ |Q(x, z) - Q(y, z)| = Q_i(x, z) - Q_i(z^{k+1}, z) + Q_i(x^{k+1}, z) - Q_i(y, z). \]

However, owing to the continuity of $Q$ in the closure of $E_i$ and $E_{i+1}$, $Q_i(x^{k+1}, z) = Q_i(z^{k+1}, z)$. As $Q_i(Q_{i+1})$ is 1-Lipschitz in $E_i(E_{i+1})$, we arrive at the inequality

\[ |Q(x, z) - Q(y, z)| \leq (x - x^{k+1}) + (x^{k+1} - y) = x - y = |x - y|. \]
Since any two points in \( \cup E_i \) can be connected by a finite number of steps, like this one, (perhaps also considering the other variable), the 1-Lipschitz property follows. The last claim in the statement of the Proposition is evident. \( \square \)

**Example 2.** There are \( \alpha \)-migrative quasi-copulas \( S_{\{0, \alpha\}} \) generated by \( \alpha \)-generating functions that are not quasi-copulas. As an example, take \( D : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) given by

\[
D(x, y) = \begin{cases} 
- \min(x, y) & \text{if } x, y \in [0, 1/3] \\
x + y - 1 & \text{if } x, y \in [1/3, 1] \\
3xy - 2 \min(x, y) & \text{otherwise.}
\end{cases}
\]

Then \( D \) is a 1/2-generating function. Moreover, it is easy to see that \( D \) is also a continuous and 1-Lipschitz function. If we consider the 1/2-migrative semicopula \( S_{\{0, 1/2\}} \), we have that

\[
S_{\{0, 1/2\}}(x, y) = \begin{cases} 
\frac{\max(2^x, 2^y)}{2^x + 2^y} & \text{if } (x, y) \in J_{k, m}, \\
\frac{2^x - 2^y}{2^x + 2^y} & \text{if } (x, y) \in L_{k, m}, \\
\frac{1 - 2^x - 2^y - \min(2^x, 2^y)}{2^x + 2^y} + 3xy & \text{if } (x, y) \in I_{k, m}/(J_{k, m} \cup L_{k, m}), \\
0 & \text{otherwise.}
\end{cases}
\]

where \( J_{k, m} = [\frac{1}{2^k}, \frac{1}{2^k} \times [\frac{1}{2^k}, \frac{1}{2^k}] \times [\frac{1}{2^k}, \frac{1}{2^k}], \frac{1}{2^k}] \) and \( L_{k, m} = [\frac{1}{2^k}, \frac{1}{2^k} \times [\frac{1}{2^k}, \frac{1}{2^k}] \times [\frac{1}{2^k}, \frac{1}{2^k}], \frac{1}{2^k}] \) for \( k, m \in \mathbb{N}_0 \). This function is 1-Lipschitz, since \( D \) is. Thus, \( S_{\{0, 1/2\}} \) is a \( (1/2) \)-migrative quasi-copula. However, it is clear that \( D \) is not a quasi-copula and not even an aggregation function since it is not greater than or equal to zero in its whole domain.

### 4. Concluding remarks

For a fixed \( \alpha \in [0, 1] \), we have completely characterized \( \alpha \)-migrative semicopulas and \( \alpha \)-migrative copulas. As a by-product, a new construction method for copulas was obtained. This assigns to a given copula \( C \) an \( \alpha \)-migrative copula \( S_{\{\alpha\}, \alpha} \). This new construction method raises a problem: is there a copula \( C \) different from the product \( P \) such that \( C = S_{\{\alpha\}, \alpha} \)? Other related problems for further investigations arise. For example, if a copula \( C \) is \( \beta \)-migrative, what can we say about the \( \alpha \)-migrative copula \( S_{\{\alpha\}, \alpha} \)?

Recall that Siburg and Stömenov recently introduced a gluing construction method for copulas [13]. For \( p \in [0, 1] \) and two copulas \( C_1 \) and \( C_2 \), the function \( C = \text{vg} - (\langle 0, p, C_1 \rangle, \langle 1, p, C_2 \rangle) : [0, 1] \times [0, 1] \rightarrow [0, 1] \) (where \( \text{vg} \) denotes vertical gluing), given by

\[
C(x, y) = \begin{cases} 
pC_1 \left( \frac{y}{p} \right) & \text{if } x \in [0, p] \\
p + (1 - p) C_2 \left( \frac{y - p}{1 - p} \right) & \text{otherwise.}
\end{cases}
\]

is a copula. Similarly, horizontal gluing \( C = \text{hg} - (\langle 0, p, C_1 \rangle, \langle p, 1, C_2 \rangle) \) is given by

\[
C(x, y) = \begin{cases} 
pC_1 \left( \frac{x}{p} \right) & \text{if } y \in [0, p] \\
p + (1 - p) C_2 \left( \frac{x - p}{1 - p} \right) & \text{otherwise.}
\end{cases}
\]

The next result is also of interest for further research.

**Proposition 4.1.** Let \( D : [0, 1] \times [0, 1] \rightarrow [0, 1] \) be an \( \alpha \)-generating function. Then the following are equivalent.

(i) \( S_{\{\alpha, \alpha\}} \) is an \( \sqrt{\alpha} \)-migrative copula.

(ii) There is a copula \( C \) such that \( D = \text{vg} - (\langle 0, p, C \rangle, \langle 0, p, C \rangle, \langle 0, p, C \rangle, \langle 0, p, C \rangle, \langle 0, p, C \rangle, \langle 0, p, C \rangle) \), where \( p = \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} \).

Note that the roles of horizontal and vertical gluing in the above proposition can be exchanged. Moreover, owing to the above proposition, any copula of type \( S_{\{\alpha\}, \alpha} \) can be considered as a fractal structure. Indeed, the following hold:

(i) \( S_{\{\alpha\}, \alpha}(x, y) = \alpha^{-k} S_{\{\alpha\}, \alpha}(\alpha^k x, \alpha^k y) \) for all \( k \in \mathbb{N}_0 \) and \( x, y \in [0, 1] \);

(ii) \( S_{\{\alpha\}, \alpha} = \text{vg} - (\langle \alpha x^k, \alpha^k C_1 \rangle) \), with \( k \in \mathbb{N}_0 \), and where \( C_1 = \text{hg} - (\langle \alpha x^k, \alpha^k C \rangle, \langle \alpha x^k, \alpha^k C \rangle) \) and\( \langle \alpha x^k, \alpha^k C \rangle \) are the \( k \)-th \( \alpha \)-migrative copulas generated by \( \alpha \)-generating functions.

As an example, recall \( S_{M, \alpha} \) [see Example 1 (i) and Fig. 3] for which the support of the corresponding copulas \( C_1 \) and \( C_2 \) is depicted in Figs. 3 and 4, respectively, for \( \alpha = 0.7 \). (See Fig. 5)

Note finally that \( \alpha \)-migrative copulas can be viewed as a special type of rectangular patchwork based on the product copula; compare Refs. [7, 5].
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References