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On the α-migrativity of semicopulas, quasi-copulas, and copulas

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ABSTRACT

In this paper we address the problem of α-migrativity (for a fixed α) for semicopulas, copulas and quasi-copulas. We introduce the concept of an α-sum of semicopulas. This new concept allows us to completely characterize α-migrative semicopulas and copulas. Moreover, α-sums also provide a means to obtain a partial characterization of α-migrative quasi-copulas.

1. Introduction and preliminaries

For some image processing applications and decision-making problems, it is important to ensure that variations in the value of some functions caused by considering just a given fraction of one of the input variables is independent of the actual choice of variable. For instance, it is sometimes of interest to darken a certain part of an image. In decision-making, the ordering of inputs may be relevant, even though the result of modifying one or another evaluation by a given ratio is the same. The concept of *migrativity* captures this idea. In this paper we focus on the *α-migrativity* for some fixed α; in other words, we consider that the reduction factor is determined by a fixed factor $\alpha \in]0,1[$.

Mathematically, the α-migrative property for a mapping $A : [0,1] \times [0,1] \to [0,1]$ means that the identity

$$A(\alpha x, y) = A(x, \alpha y) \tag{1}$$

holds for all $x, y \in [0,1]$. Property (2) below extended to the class of all bivariate functions on [0,1] was introduced by Durante and Sarkoci [8] and further studied by Fodor and Rudas [9], whereas the particular case of aggregation functions was considered by Beliakov and Calvo [2]. We previously investigated and characterized aggregation functions that are α-migrative for all $\alpha \in]0,1[$ [4]; in particular, we showed that the only migrative function with neutral element 1 is the product $\Pi(x,y) = xy$.

Property (2) for some specific aggregation functions has already been considered in the literature. In particular, the following problem was posed for t-norms by Mesiar and Novák [11].

Problem. Is there any t-norm T different from

$$T(x,y) = \begin{cases} cxy & \text{if } \max(x,y) < 1, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

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such that, for a fixed $\alpha \in]0,1[$

$$T(\alpha x, y) = T(x, \alpha y) \quad \text{for any } x, y < 1? \tag{2}$$

This problem was definitively solved by Budincevic and Kurilic [3]. Based on their ideas, Mesiar and colleagues proposed a t-norm T_* ([10, Exp. 2.11]) studied by Smutná [14]:

$$T_*(x,y) = \begin{cases} \sum_{n=1}^{\infty} \frac{1}{2^{x_n+y_n-n}} & \text{if } x, y \in]0,1], \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where, if $x > 0 (y > 0)$, $(x_n)((y_n))$ is a strictly increasing sequence in $\mathbb{N}$ such that

$$x = \sum_{n=1}^{\infty} \frac{1}{2^{x_n}} \left(y = \sum_{n=1}^{\infty} \frac{1}{2^{y_n}} \right).$$

Evidently, for $\alpha = 2^{-k}$ with $k \in \mathbb{N}$

$$2^{-k}x = \sum \frac{1}{2^{x_n+k}},$$

and analogously for y (here and subsequently, whenever the summation bounds are omitted, the index is understood to vary from 1 to ∞). Thus, we have:

$$T_*(\alpha x, y) = T_*(x, \alpha y),$$

for all $x, y < 1$ (even for all $x, y \leqslant 1$) and for all $\alpha = 2^{-k}$ with $k \in \mathbb{N}$.

The case of α-migrative t-norms has also been considered [6,8,9]. In the particular case of continuous t-norms, they were shown to be strict t-norms generated by an additive generator $t : [0,1] \to [0,\infty]$ satisfying

$$t(x) = kt(\alpha) + t(\alpha^{-k}x), \tag{4}$$

for all $x \in]\alpha^{k+1}, \alpha^k]$, $k \in \{0, 1, 2, \ldots\}$ (i.e., on $[\alpha, 1]$ the choice of t is free).

Inspired by the results mentioned above, we investigated α-migrative semicopulas with a special focus on copulas and quasi-copulas. Observe that for associative copulas α-migrativity forces the strictness of the discussed copula and α-migrative strict copulas are generated by an additive generator t given by (4) [2], additionally satisfying the convexity property, i.e., t is convex on $[\alpha, 1]$ and $t'(1^-) \leqslant \alpha t'(\alpha^+)$. Here and in the following, for a given function $f : [0,1] \to [0,1]$ and for a point $x_0 \in [0,1]$, $f(x_0^+)$ and $f(x_0^-)$ denote the limit from the right and from the left, respectively, of f at x_0.

For convenience, we now describe some of the basic notions involved in our study.

Definition 1.1. A (bivariate) aggregation function is a mapping $A : [0,1] \times [0,1] \to [0,1]$ such that

(i) $A(0,0) = 0$ and $A(1,1) = 1$; and
(ii) A is non-decreasing in both variables.

Definition 1.2. Let $\alpha \in [0,1]$. An aggregation function $A : [0,1] \times [0,1] \to [0,1]$ is α-migrative if the identity

$$A(\alpha x, y) = A(x, \alpha y)$$

holds for all $x, y \in [0,1]$.

Definition 1.3. An aggregation function $S : [0,1] \times [0,1] \to [0,1]$ is called a semicopula if 1 is its neutral element, i.e., $S(x,1) = S(1,x) = x$ for all $x \in [0,1]$. A 1-Lipschitz semicopula, i.e., a semicopula Q satisfying

$$|Q(x,y) - Q(x',y')| \leqslant |x - x'| + |y - y'|,$$

for all $x, y, x', y' \in [0,1]$, is called a quasi-copula.

A semicopula C that is 2-increasing, that is:

$$C(x,y) + C(x',y') - C(x,y') - C(x',y) \geqslant 0,$$

for all $0 \leqslant x \leqslant x' \leqslant 1$ and $0 \leqslant y \leqslant y' \leqslant 1$, is called a copula.

More details are available elsewhere [1,12]. Note that each copula is also a quasi-copula and that quasi-copulas that are not copulas are termed proper.

The remainder of the paper is organized as follows. In the next section, α-migrative semicopulas are characterized and the *α-migrative sum of α-migrative semicopulas* is introduced. Section 3 is devoted to the study of α-migrative copulas and quasi-copulas. In particular, we describe expression (3) for T_*. In Section 4, some final considerations are discussed.

2. α-Migrative semicopulas

Throughout the remainder of the paper, $\alpha \in]0, 1[$ is fixed. Because semicopulas possess a neutral element 1, it is evident that each α-migrative semicopula S satisfies:

$$S(\alpha, x) = S(x, \alpha) = \alpha x,$$

and by induction

$$S(\alpha^k, x) = S(x, \alpha^k) = \alpha^k x,$$

for $k = 2, 3, \ldots$. Hence, for any α-migrative semicopula S the following result holds.

Lemma 2.1. *Let $S : [0, 1] \times [0, 1]$ be an α-migrative semicopula. Then, for any $x, y \in [0, 1]$ such that $\{x, y\} \cap \{1, \alpha, \alpha^2, \ldots\} \neq \emptyset$, it holds that*

$$S(x, y) = \Pi(x, y) = xy.$$

The next result can also be obtained by induction.

Lemma 2.2. *Let $S : [0, 1] \times [0, 1]$ be an α-migrative semicopula. Then, for any $x, y \in [0, 1]$ and $k, m, i, j \in \{0, 1, 2, \ldots\}$, it holds that*

$$S(\alpha^k x, \alpha^m y) = S(\alpha^i x, \alpha^j y),$$

whenever $k + m = i + j$.

These two lemmas have a crucial impact on the following definition. We denote $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 2.1. Let $(S_i)_{i \in \mathbb{N}_0}$ be a system of α-migrative semicopulas. Then the function $S : [0, 1] \times [0, 1] \to [0, 1]$ given by

$$S(x, y) = \begin{cases} S_i(x, y) & \text{if } (x, y) \in E_i \quad \text{for some } i \in \mathbb{N}_0, \\ 0 & \text{otherwise}, \end{cases}$$

where for $i \in \mathbb{N}_0$, $E_i = \cup_{m, k \in \mathbb{N}_0; m+k=i}]\alpha^{k+1}, \alpha^k] \times]\alpha^{m+1}, \alpha^m]$, is called α-migrative sum of $(S_i)_{i \in \mathbb{N}_0}$. This is denoted by $S = \alpha - (S_i)_{i \in \mathbb{N}_0}$.

Fig. 1 provides a representation of the sets E_i.

Proposition 2.3. *The α-migrative sum of semicopulas is a semicopula.*

Proof. First, we consider the neutrality of 1. Clearly $S(1, 0) = S(0, 1) = 0$, and if $x \in]\alpha^{k+1}, \alpha^k]$ for some $k \in \mathbb{N}_0$, then

$$S(1, x) = S_k(1, x) = x = S_k(x, 1) = S(x, 1),$$

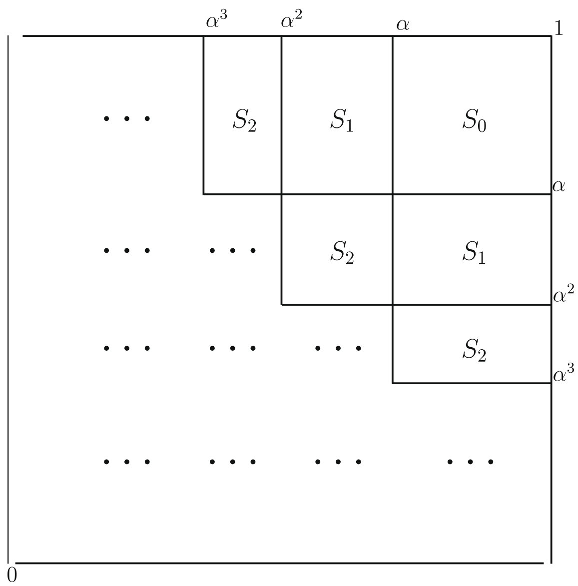

Fig. 1. Structure of an α-migrative sum of semicopulas, with $\alpha = 0.7$.

so the result follows. The non-decreasing nature of S on E_i follows from the non-decreasing property of S_i. It remains to show that $S(x,y) \leqslant S(x',y)$ when $(x',y) \in E_i$, $(x,y) \in E_j$ and $j > i$ (and similarly for the other variable). Suppose $y \in]\alpha^{m+1}, \alpha^m]$. Then

$$x \in]\alpha^{j-m+1}, \alpha^{j-m}] \quad \text{and } x' \in]\alpha^{i-m+1}, \alpha^{i-m}],$$

and hence

$$S(x,y) = S_j(x,y) \leqslant S_j(\alpha^{j-m}, y) = \alpha^{j-m} y = S_i(\alpha^{j-m}, y) \leqslant S_i(x', y) = S(x', y),$$

as required. □

Proposition 2.4. *A semicopula S is α-migrative if and only if there exists a system $(S_i)_{i\in\mathbb{N}_0}$ of α-migrative semicopulas such that S is the α-migrative sum of S_i, i.e., $S = \alpha - (S_i)_{i\in\mathbb{N}_0}$.*

Proof. Necessity is obvious, as it is enough to consider the constant system $(S)_{i\in\mathbb{N}_0}$. To prove the sufficiency, by the previous proposition we already have that the α-migrative sum $S = \alpha - (S_i)_{i\in\mathbb{N}_0}$ is a semicopula. Now observe that if $(x,y) \in E_i$, then $(\alpha x, y)$ and $(x, \alpha y)$ belong to E_{i+1}. Thus, owing to the α-migrativity of S_{i+1}, it holds that:

$$S(\alpha x, y) = S(x, \alpha y).$$

Moreover, $xy = 0$ if and only if $\alpha xy = 0$, and in this case:

$$S(\alpha x, y) = S(x, \alpha y) = 0.$$

Consequently, each α-migrative sum is α-migrative. □

The next result follows directly from Definition 1.2 and Lemma 2.1.

Proposition 2.5. *Let S be an α-migrative semicopula. Then the mapping $D : [0,1] \times [0,1] \to \mathbb{R}$ given by*

$$D(x,y) = \frac{S(\alpha + (1-\alpha)x, \alpha + (1-\alpha)y) - \alpha(1-\alpha)(x+y) - \alpha^2}{(1-\alpha)^2} \tag{5}$$

satisfies

(i) $D(x,0) = D(0,x) = 0$ for any $x \in [0,1]$ (i.e., 0 is an annihilator of D);
(ii) $D(x,1) = D(1,x) = x$ for any $x \in [0,1]$, (i.e., 1 is a neutral element for D); and
(iii) $(1-\alpha)D(x,y) + \alpha(x+y) \leqslant (1-\alpha)D(x',y') + \alpha(x'+y')$ for any $x, y, x', y' \in [0,1]$ such that $x \leqslant x'$ and $y \leqslant y'$.

Proof. First, note that for $(u, v) \in [\alpha, 1] \times [\alpha, 1]$

$$S(u, v) = \alpha(u+v) - \alpha^2 + (1-\alpha)^2 D\left(\frac{u-\alpha}{1-\alpha}, \frac{v-\alpha}{1-\alpha}\right).$$

The result follows from the non-decreasing property of S and the identities $S(\alpha, v) = \alpha v$, $S(u, \alpha) = \alpha u$, $S(1, v) = v$ and $S(u, 1) = 1$. □

Definition 2.2. A function $D : [0,1] \times [0,1] \to \mathbb{R}$ satisfying the properties given in Proposition 2.5 is called an α-generating function.

Note that, regardless of the value of $\alpha \in]0,1[$, any semicopula S is an α-generating function. Moreover, we have the following two results.

Proposition 2.6. *The strongest semicopula $M(x,y) = \min(x,y)$ is also the strongest α-generating function. That is, M is an α-generating function and, for any other α-generating function D, the inequality*

$$D(x,y) \leqslant M(x,y)$$

holds for all $x, y \in [0,1]$.

Proof. The fact that $M(x,y) = \min(x,y)$ is an α-generating function follows from easy calculations. To prove that for any other α-generating function D the inequality $D(x,y) \leqslant M(x,y)$ holds, just observe that $D(x,y) \leqslant D(1,y) = y$ and $D(x,y) \leqslant D(x,1) = x$ for all $x, y \in [0,1]$. □

Proposition 2.7. *The weakest α-generating function $D_*^{(\alpha)}$ is given by*

$$D_*^{(\alpha)}(x,y) = \begin{cases} -\frac{\alpha}{1-\alpha}\min(x,y) & \text{if } \max(x,y) < 1, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Proof. Let D be an α-generating function. From (ii) and (iii) in Proposition 2.5 it follows that, for any $x, y \in [0, 1]$

$$\alpha x = (1-\alpha)D(x, 0) + \alpha x \leqslant (1-\alpha)D(x, y) + \alpha(x+y),$$

and

$$\alpha y = (1-\alpha)D(0, y) + \alpha y \leqslant (1-\alpha)D(x, y) + \alpha(x+y),$$

so we have that

$$D(x, y) \geqslant \max\left(-\frac{\alpha}{1-\alpha}x, -\frac{\alpha}{1-\alpha}y\right) = -\frac{\alpha}{1-\alpha}\min(x, y),$$

as required. □

Now we are ready to give a complete characterization of α-migrative semicopulas.

Theorem 2.8. *Let S be a bivariate function. Then S is an α-migrative semicopula if and only if there exists a system $(D_i)_{i\in\mathbb{N}_0}$ of α-generating functions such that*

$$S(x, y) = \begin{cases} \alpha^{m+1}x + \alpha^{k+1}y - \alpha^{k+m+2} + (1-\alpha)^2\alpha^{k+m}D_{k+m}\left(\frac{x-\alpha^{k+1}}{\alpha^k-\alpha^{k+1}}, \frac{y-\alpha^{m+1}}{\alpha^m-\alpha^{m+1}}\right), \\ \text{if } (x, y) \in]\alpha^{k+1}, \alpha^k]\times]\alpha^{m+1}, \alpha^m] \quad \text{for some } k, m \in \mathbb{N}_0, \\ 0 \quad \text{otherwise.} \end{cases} \tag{6}$$

Proof. To see the necessity, observe that, as S is non-decreasing, from Proposition 2.4, it follows that S can be written as an α-migrative sum $(S_i)_{i\mathbb{N}_0}$. However, from Eq. (5) in Proposition 2.5, each of the terms S_i can be written in terms of an α-generating function $D_{k,m}$ for $(x, y) \in [\alpha^{k+1}, \alpha^k] \times [\alpha^{m+1}, \alpha^m]$, with $k, m \in \mathbb{N}_0$ as follows:

$$S_i(x, y) = \alpha^{m+1}x + \alpha^{k+1}y - \alpha^{k+m+2} + (1-\alpha)^2\alpha^{k+m}D_{k,m}\left(\frac{x-\alpha^{k+1}}{\alpha^k-\alpha^{k+1}}, \frac{y-\alpha^{m+1}}{\alpha^m-\alpha^{m+1}}\right).$$

Moreover, from Proposition 2.4 it also holds that $D_{k,m} = D_{k',m'}$ whenever $k+m = k'+m'$. Thus, the condition is necessary.

To prove the sufficiency, observe that, if $(x, y) \in]\alpha^{k+1}, \alpha^k]\times]\alpha^{m+1}, \alpha^m]$, then

$$S(\alpha x, y) = \alpha^{m+1}\alpha x + \alpha^{k+1}y - \alpha^{k+m+3} + (1-\alpha)^2\alpha^{k+m+1}D_{k+m+1}\left(\frac{\alpha x-\alpha^{k+2}}{\alpha^{k+1}-\alpha^{k+2}}, \frac{y-\alpha^{m+1}}{\alpha^m-\alpha^{m+1}}\right),$$

whereas

$$S(x, \alpha y) = \alpha^{m+2}x + \alpha^k\alpha y - \alpha^{k+m+3} + (1-\alpha)^2\alpha^{k+m+1}D_{k+m+1}\left(\frac{x-\alpha^{k+1}}{\alpha^k-\alpha^{k+1}}, \frac{\alpha y-\alpha^{m+2}}{\alpha^{m+1}-\alpha^{m+2}}\right).$$

Evidently, $S(\alpha x, y) = S(x, \alpha y)$, ensuring the α-migrativity of S. To prove the monotonicity, we can use arguments similar to those for the proof of Proposition 2.4 . Finally, the fact that $S(0, 0) = 0$ is obvious from the definition of S, whereas $S(x, 1) = S(1, x) = 1$ for all $x \in [0, 1]$ follows from property (ii) in Proposition 2.5. □

Definition 2.3. Let D be an α-generating function. For the constant system $(D)_{i\in\mathbb{N}_0}$, the α-migrative copula given by (6) is denoted as $S_{(D,\alpha)}$.

The next result is an easy corollary of Proposition 2.4.

Corollary 2.9. *A semicopula S is α-migrative if and only if S is the α-migrative sum $S = \alpha - (S_{(D_i,\alpha)})_{i\in\mathbb{N}_0}$, where $(D_i)_{i\in\mathbb{N}_0}$ is a system of α-generating functions.*

Remark 1.

(i) In Theorem 2.8 and Corollary 2.9, the choice of the system $(D_i)_{i\in\mathbb{N}_0}$ of α-generating functions has no restriction and it is evident that different systems generate different α-migrative semicopulas.
(ii) The symmetry of an α-migrative semicopula S is equivalent to the symmetry of each α-generating function D_i in representation (6).
(iii) Owing to Corollary 2.9, a prominent role in the study of α-migrative semicopulas is played by those generated by a single generating function.

Example 1.

(i) The strongest α-migrative semicopula is:

$$S_{(M,\alpha)} : [0, 1] \times [0, 1] \to [0, 1]$$

given, for $(x, y) \in]\alpha^{k+1}, \alpha^k] \times]\alpha^{m+1}, \alpha^m]$, by

$$\begin{aligned} S_{(M,\alpha)}(x, y) &= \alpha^{m+1}x + \alpha^{k+1}y - \alpha^{k+m+2}x + (1-\alpha)\min(\alpha^m x - \alpha^{m+k+1}, \alpha^k y - \alpha^{m+k+1}) \\ &= \alpha^{m+1}x + \alpha^{k+1}y - \alpha^{k+m+1} + (1-\alpha)\min(\alpha^m x, \alpha^k y). \end{aligned}$$

Its support is depicted in Fig. 2 for $\alpha = 0.7$. Remember that the support of a semicopula S, by analogy with that of a copula, is its support when considered as a probability distribution function on $[0, 1]^2$; that is, the complement of the union of all open sets in $[0, 1]^2$ with S-measure zero [12]. Observe also that $S_{(M,\alpha)}$ is a singular copula (i.e., a copula with support having zero Lebesgue measure).

(ii) The weakest α-migrative semicopula is

$$S_{(D_*^{(\alpha)},\alpha)} : [0, 1] \times [0, 1] \to [0, 1]$$

given, for $(x, y) \in [\alpha^{k+1}, \alpha^k[\times [\alpha^{m+1}, \alpha^m[$, by

$$S_{(D_*^{(\alpha)},\alpha)}(x, y) = \max(\alpha^{m+1}x, \alpha^{k+1}y),$$

whereas, if $\{x, y\} \cap \{0, 1\} \neq \emptyset$

$$S_{(D_*^{(\alpha)},\alpha)}(x, y) = \min(x, y).$$

(iii) For the product Π, it holds that $\Pi = S_{\Pi,\alpha}$.

(iv) The $(1/2)$-migrative t-norm T_*, introduced in Section 1, satisfies $T_* = S_{(D,\frac{1}{2})}$, where the $(1/2)$-generating function D is given by

$$D(x, y) = \begin{cases} 2T_*(x, y) - x - y + 1 & \text{if } \min(x, y) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. α-Migrative copulas and quasi-copulas

As observed in Example 1, an α-migrative semicopula $S_{(D,\alpha)}$ can be generated by an α-generating function D that is not a semicopula, as the non-decreasing property of D may be violated. By contrast, for any semicopula H, $S_{(H,\alpha)}$ is again an (α-migrative) semicopula. In the case of t-norms (associative and symmetric semicopulas), we have all possible situations. As shown in Example 1 (iv), there are t-norms generated by non-associative and non-monotonic α-generating functions. By contrast, there are t-norms of the form $S_{(T,\alpha)}$, where T is a t-norm. For example, for the Lukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$, the corresponding α-migrative semicopula $S_{(T_L,\alpha)}$ is the weakest α-migrative 1-Lipschitz t-norm and its additive generator $t : [0, 1] \to [0, \infty]$ is a piecewise linear function determined by points (α^i, i), $i \in \mathbb{N}_0$ (cf. Ref. [2]). The support of $S_{(T_L,\alpha)}$ (for $\alpha = 0.7$) is depicted in Fig. 3. Example 1 (i) shows that not every t-norm T generates an α-migrative t-norm (associativity of $S_{(M,\alpha)}$ is violated). A different situation occurs for the class of copulas and quasi-copulas.

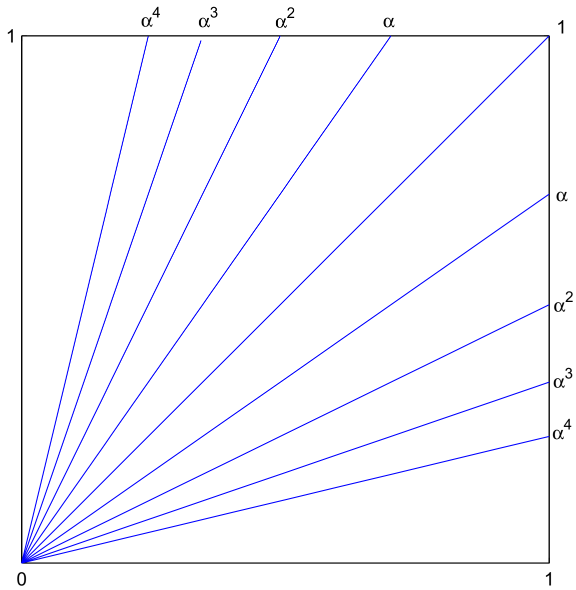

Fig. 2. Support of $S_{(M,\alpha)}$ (for $\alpha = 0.7$).

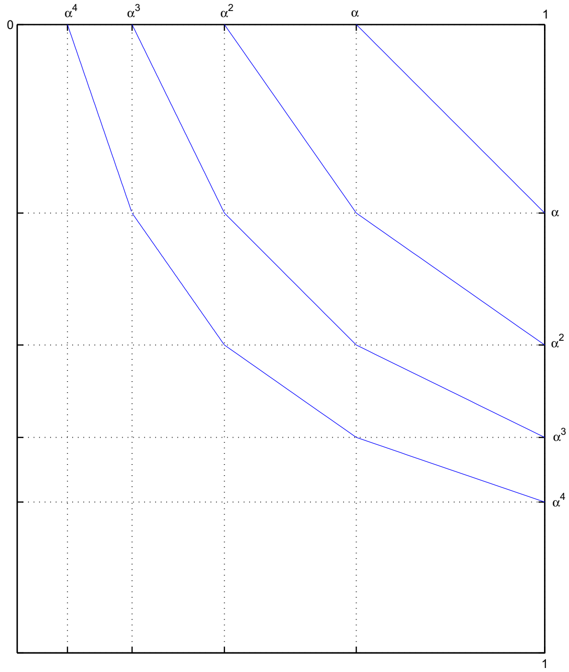

Fig. 3. Support of $S_{(T_L,\alpha)}$ (for $\alpha = 0.7$).

Theorem 3.1. *A semicopula S is an α-migrative copula if and only if there exists a system $(C_i)_{i\in\mathbb{N}_0}$ of copulas generating S by means of (6).*

Proof. Observe that the 2-increasing property of a function $C : [0,1]\times[0,1] \to [0,1]$, together with 0 being the annihilator of C and 1 being the neutral element of C, ensures that C is a copula. Moreover, it is not difficult to check that the 2-increasing property of C over $[0,1]\times[0,1]$ is equivalent to the 2-increasing property of C over all rectangles $[\alpha^{k+1},\alpha^k]\times[\alpha^{m+1},\alpha^m]$, for $k,m\in\mathbb{N}_0$. These facts, together with Proposition 2.5 and Theorem 2.8, prove the result. □

Corollary 3.2. *Let $D : [0,1]\times[0,1]\to\mathbb{R}$ be an α-generating function. Then the α-migrative semicopula $S_{(D,\alpha)}$ is a copula if and only if D is a copula.*

It is evident that the strongest α-migrative copula is $S_{(M,\alpha)}$. As already mentioned, it is a singular copula for which the support is depicted in Fig. 3. More details on singular copulas are available elsewhere [12].

Similarly, the weakest α-migrative copula is $S_{(T_L,\alpha)}$, which is indeed a t-norm, and its additive generator (unique up to a positive multiplicative constant) $t_{(T_L,alpha)} : [0,1]\to[0,\infty]$ is given by

$$t_{(T_L,alpha)}(x) = k(1-\alpha) + \left(1 - \frac{x}{\alpha^k}\right) \quad \text{if } x \in]\alpha^{k+1},\alpha^k].$$

In this case we also have a singular copula with support as shown in Fig. 3.

For α-migrative quasi-copulas we have only a sufficient condition.

Proposition 3.3. *Let $(Q_i)_{i\in\mathbb{N}_0}$ be a system of quasi-copulas. Then the α-migrative semicopula S generated by this system as in (6) is a quasi-copula. It is a proper quasi-copula whenever there is at least one proper quasi-copula in the system $(Q_i)_{i\in\mathbb{N}_0}$.*

Proof. The monotonicity and 1-Lipschitz property of Q_i ensure the same properties for S on the closure of E_i. It is evident that S is a continuous α-migrative semicopula and thus 1-Lipschitzianity of S on the closures of all E_i ensures the 1-Lipschitz property of S on the whole domain $[0,1]\times[0,1] = \cup\overline{E_i}$. To prove this, observe that if $x\in[\alpha^{k+1},\alpha^k]$ and $y\in[\alpha^{k+2},\alpha^{k+1}]$ for some $k\in\mathbb{N}_0$, then we have that $Q(x,z) \geqslant Q(y,z)$ for all $z\in[0,1]$. Thus, in particular

$$|Q(x,z) - Q(y,z)| = Q_i(x,z) - Q_i(\alpha^{k+1},z) + Q_i(\alpha^{k+1},z) - Q_{i+1}(y,z).$$

However, owing to the continuity of Q in the closure of E_i and E_{i+1}, $Q_i(\alpha^{k+1},z) = Q_{i+1}(\alpha^{k+1},z)$. As $Q_i(Q_{i+1})$ is 1-Lipschitz in $E_i(E_{i+1})$, we arrive at the inequality

$$|Q(x,z) - Q(y,z)| \leqslant (x-\alpha^{k+1}) + (\alpha^{k+1} - y) = x - y = |x-y|.$$

Since any two points in $\cup E_i$ can be connected by a finite number of steps, like this one, (perhaps also considering the other variable), the 1-Lipschitz property follows. The last claim in the statement of the Proposition is evident. $\square$

Example 2. There are α-migrative quasi-copulas $S_{(D,\alpha)}$ generated by α-generating functions that are not quasi-copulas. As an example, take $D : [0,1] \times [0,1] \to \mathbb{R}$ given by

$$D(x,y) = \begin{cases} -\min(x,y) & \text{if } x, y \in [0, 1/3] \\ x + y - 1 & \text{if } x, y \in [1/3, 1] \\ 3xy - 2\min(x,y) & \text{otherwise.} \end{cases}$$

Then D is a $1/2$-generating function. Moreover, it is easy to see that D is also a continuous and 1-Lipschitz function. If we consider the $1/2$-migrative semicopula $S_{(D,1/2)}$, we have that

$$S_{(D,1/2)}(x,y) = \begin{cases} \frac{\max(2^k x, 2^m y)}{2^{k+m+1}} & \text{if } (x,y) \in J_{k,m}, \\ \frac{2^k x + 2^m y}{2^{k+m}} - \frac{3}{2^{k+m+2}} & \text{if } (x,y) \in L_{k,m}, \\ \frac{1 - 2^k x - 2^m y - \min(2^k x, 2^m y)}{2^{k+m}} + 3xy & \text{if } (x,y) \in I_{k,m} / \{J_{k,m} \cup L_{k,m}\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_{k,m} =]\frac{1}{2^{k+1}}, \frac{1}{2^k}] \times]\frac{1}{2^{m+1}}, \frac{1}{2^m}]$, $J_{k,m} =]\frac{1}{2^{k+1}}, \frac{1}{3}\frac{1}{2^{k-1}}] \times]\frac{1}{2^{m+1}}, \frac{1}{3}\frac{1}{2^{m-1}}]$ and $L_{k,m} = [\frac{1}{3}\frac{1}{2^{k-1}}, \frac{1}{2^k}] \times [\frac{1}{3}\frac{1}{2^{m-1}}, \frac{1}{2^m}]$ for $k, m \in \mathbb{N}_0$. This function is 1-Lipschitz, since D is. Thus, $S_{(D,1/2)}$ is a $(1/2)$-migrative quasi-copula. However, it is clear that D is not a quasi-copula and not even an aggregation function since it is not greater than or equal to zero in its whole domain.

4. Concluding remarks

For a fixed $\alpha \in]0,1[$, we have completely characterized α-migrative semicopulas and α-migrative copulas. As a by-product, a new construction method for copulas was obtained. This assigns to a given copula C an α-migrative copula $S_{(C,\alpha)}$. This new construction method raises a problem: is there a copula C different from the product Π such that $C = S_{(C,\alpha)}$? Other related problems for further investigations arise. For example, if a copula C is β-migrative, what can we say about the α-migrative copula $S_{(C,\alpha)}$?

Recall that Siburg and Stomeinov recently introduced a gluing construction method for copulas [13]. For $p \in]0,1[$ and two copulas C_1 and C_2, the function $C = \mathrm{vg} - (\langle 0, p, C_1 \rangle, \langle 1, p, C_2 \rangle) : [0,1] \times [0,1] \to [0,1]$ (where vg denotes vertical gluing), given by

$$C(x,y) = \begin{cases} pC_1(\frac{x}{p}, y) & \text{if } x \in [0,p] \\ p + (1-p)C_2\left(\frac{x-p}{1-p}, y\right) & \text{otherwise,} \end{cases}$$

is a copula. Similarly, horizontal gluing $C = \mathrm{hg} - (\langle 0, p, C_1 \rangle, \langle p, 1, C_2 \rangle)$ is given by

$$C(x,y) = \begin{cases} pC_1\left(x, \frac{y}{p}\right) & \text{if } y \in [0,p] \\ p + (1-p)C_2\left(x, \frac{y-p}{1-p}\right) & \text{otherwise.} \end{cases}$$

The next result is also of interest for further research.

Proposition 4.1. *Let $D : [0,1] \times [0,1] \to [0,1]$ be an α-generating function. Then the following are equivalent.*

(i) $S_{(D,\alpha)}$ is an $\sqrt{\alpha}$-migrative copula.
(ii) There is a copula C such that $D = \mathrm{vg} - (\langle 0, p, \mathrm{hg} - (\langle 0, p, C \rangle, \langle p, 1, C \rangle) \rangle, \langle 1, p, \mathrm{hg} - (\langle 0, p, C \rangle, \langle p, 1, C \rangle)\rangle)$, where $p = \frac{\sqrt{\alpha}}{1+\sqrt{\alpha}}$.

Note that the roles of horizontal and vertical gluing in the above proposition can be exchanged. Moreover, owing to the above proposition, any copula of type $S_{(C,\alpha)}$ can be considered as a fractal structure. Indeed, the following hold:

(i) $S_{(C,\alpha)}(x,y) = \alpha^{-k} S_{(C,\alpha)}(\alpha^k x, \alpha^k y)$ for all $k \in \mathbb{N}_0$ and $x, y \in [0,1]$;
(ii) $S_{(C,\alpha)} = \mathrm{vg} - (\langle \alpha^{k+1}, \alpha^k, C_1 \rangle)$, with $k \in \mathbb{N}_0$, and where $C_1 = \mathrm{hg} - (\langle \alpha^{k+1}, \alpha^k, C \rangle)$; and
(iii) $S_{(C,\alpha)} = \mathrm{hg} - (\langle \alpha^{k+1}, \alpha^k, C_2 \rangle)$, with $k \in \mathbb{N}_0$, and where $C_2 = \mathrm{vg} - (\langle \alpha^{k+1}, \alpha^k, C \rangle)$.

As an example, recall $S_{(M,\alpha)}$ [see Example 1 (i) and Fig. 3] for which the support of the corresponding copulas C_1 and C_2 is depicted in Figs. 3 and 4, respectively, for $\alpha = 0.7$.(See Fig. 5)

Note finally that α-migrative copulas can be viewed as a special type of rectangular patchwork based on the product copula; compare Refs. [7,5].

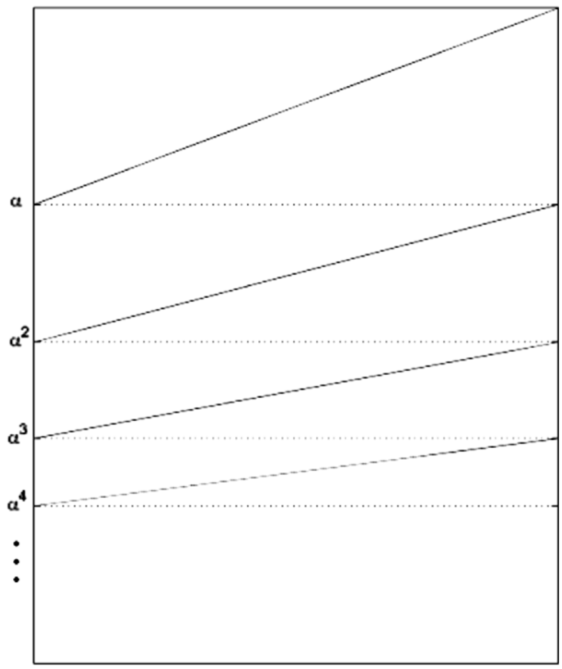

Fig. 4. Support of the vertical gluing copula C_1 for $S_{(M,\alpha)}$ and $\alpha = 0.7$.

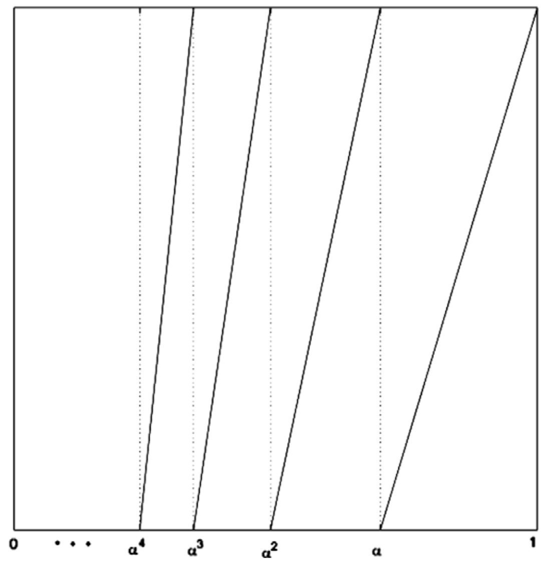

Fig. 5. Support of the horizontal gluing copula C_2 for $S_{(M,\alpha)}$ and $\alpha = 0.7$.

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