Dynamic model of Loan Portfolio with Lévy Asset Prices

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Abstract. We generalize the well known Merton-Vasicek (KMV) model of a loan portfolio value in two ways: we assume a Lévy process of the debtors' assets' value (instead of the Gaussian one) and we model a dynamics of the portfolio value so that the debts may last several periods (instead of a single one). Our model is computable by simulation.

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1 The Merton-Vasicek Model

Vasicek, in his famous paper [O.A.Vasicek(2002)], assumes a large portfolio of loans. The assets A^i of the i-th debtor follow the geometric Brownian motion

$$\frac{dA^i_\tau}{A^i_\tau} = \mu d\tau + \sigma dW^i_\tau \tag{1}$$

where μ and σ are constants and W^i_t is the standard Wiener process. It is further assumed that the debtor is obliged to pay a (non-random) instalment B^t_i at each of the times $t \in \mathbb{N}$.

By solving (1) and subtracting the annual payment, we get

$$a^i_t = a^i_{t-1} + \eta^i_t + X^i_t, \qquad a^i_t = log A^i_t, \quad \eta^i_t = \mu - \frac{1}{2}\sigma^2 - \log B^i_t \tag{2}$$

where $X^i_t = \sigma(W^i_t - W^i_{t-1})$ is centred normal random variable with variance σ^2. Clearly, the (conditional) probability p^i_t of the default of the i-th debtor is

$$p^i_t \quad = \quad \mathbb{P}(a^i_t < 0 | a^i_{t-1}) = \xi^i_t(-a^i_{t-1} - \eta^i_{t-1})$$

where ξ^i_t is the cumulative distribution function (c.d.f.) of X^i_t.

Further, it is assumed by Vasicek, that

$$corr(X^i_t, X^j_t) = \rho \tag{3}$$

for any $i \neq j$ and some ρ. It is easy to see that the latter assumption is fulfilled, for instance, if there exist mutually independent (normal) variables $Y_t, Z^1_t, Z^2_t, \dots,$ (independent of the evolution of the debtors' assets up to $t-1$) such that $Z^1_t, Z^2_t, \dots$ are equally distributed and

$$X^i_t = Y_t + Z^i_t, \qquad i \in \mathbb{N}, \tag{4}$$

such that $\text{var} Y_t = \rho \text{var} X^1_t$ and $\text{var} Z^1_t = (1-\rho)\text{var} X^1_t$ - to see it, note that

$$corr(X^i_t, X^j_t) = \frac{\text{var} Y_t}{\text{var} Z^i_t + \text{var} Y_t}. \tag{5}$$

Further, if the number of the debtors is very large, and if, for all i, $a^i_t = a_t$ and $\eta^i_t = \eta_t$ for some a_t and η_t (or, more generally, $p^i_t = p_t$ for some p_t) then, by the Law of Large Numbers, the percentage loss L_t of the bank is conditionally constant given Y_t, a_{t-1}:

$$L_t \doteq \psi_t \left(-a_{t-1} - \eta_t - Y_t\right) \tag{6}$$

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where ψ_t is the c.d.f. of Z_t^1 and, consequently, by the Complete Probability Theorem,

$$\begin{aligned}\mathbb{P}(L_t &< \theta | p_t) \doteq 1 - \phi_t\left(-a_{t-1} - \eta_t - \psi_t^{-1}(\theta)\right) \\ &= 1 - \phi_t\left(\xi_t^{-1}(p_t) - \psi_t^{-1}(\theta)\right)\end{aligned} \tag{7}$$

(see [O.A.Vasicek(2002)] for details).

2 A Generalization

We generalize the Merton-Vasicek model two ways: we release the assumption of the normality of the factors and we take the dynamics of the system into account, making p_t's endogenous and no longer requiring the identical initial states of the borrowers' assets. Our goal is to describe the distribution of the losses by specifying the conditional distribution function of L_t given

$$\bar{L}_t = (L_1, L_2, \ldots L_t)$$

for each t.

For a greater mathematical rigour, we assume the infinite number of loans in the portfolio rather than their "large amount".

To model the dynamics realistically, we assume that there is a certain amount of newly acquired deals at the beginning each period: we denote π_t the $\bar{L}_t$-measurable variable determining the ratio of the loans new at t to their overall amount (the overall amount including the new ones plus the existing ones excluding those which have defaulted at the time t). We assume the assets of a newly coming debtor to be distributed according to a common continuous and strictly increasing conditional c.d.f. γ_t given $\bar{L}_t$ and we suppose the assets of all the newcomers be mutually conditionally independent given and $\bar{L}_t$.

Remark 1. We express this (more rigorously) as follows: Let $t \in \mathbb{N}$ and let, for each $i \in \mathbb{N}$, there exist a Bernoulli variable I_t^i with a parameter π_t; if $I_t^i = 1$ then a newly coming loan will be indexed by i during the period from t to $t+1$; if, on the other hand, $I_t^i = 0$, then the first existing loan out of those which did not default at t and which are still not indexed will be indexed by i. Naturally, $I_t^1, I_2^1, \ldots$ are required to be mutually conditionally independent given $\bar{L}_t$ and independent of all $Y_\bullet$ and $Z_\bullet^\bullet$.

Let $t, i \in \mathbb{N}$. Coping with the reindexing, we reformulate (2) as

$$\tilde{a}_t^i = a_{t-1}^i + \eta_t^i + X_t^i, \tag{8}$$

where η_t^i is an arbitrary value (including the installment if there is any), a_{t-1}^i and $\tilde{a}_t^i$ are the values of the log assets of the i-th debtor at the time $t-1$, t respectively[1] and X_t^i is given by (4). The variables Y_t and Z_t^1 do not have to be normal now but it is only required that their conditional distribution functions ϕ_t and ψ_t are strictly monotonic and continuous.[2] Later (Section 3) we show how (8) arises naturally from a generazitation of (1).

Naturally, we define the percentage loss as

$$L_t = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} 1[\tilde{a}_t^i < 0].$$

$t \in \mathbb{N}$, in our new infinite setting.

Getting to the distribution of the series $L_1, L_2, \ldots,$, let us assume $a_0^1, a_0^2, \ldots$ are i.i.d (with a strictly increasing continous c.d.f. α_0).

To describe the distribution, we shall proceed by induction:

Let $t \in \mathbb{N}$ and assume that

[1] Remember that a debtor may be indexed by different index in different periods; here, i is the index valid from $t-1$ to t.

[2] Depending on the normallity no way, both (7) and (5) keep holding (with $=$ instead of $\doteq$) even under the generalized conditions.

(1) we know the the (joint) distribution of $\bar{L}_{t-1}$

(2) the variables $a^1_{t-1}, a^2_{t-1}, \dots$ are identically conditinally distributed and mutually conditionally independent given $\bar{L}_{t-1}$

(3) the conditional distribution function α_{t-1} of $a^1_{t-1}|\bar{L}_{t-1}$ is known to us.

Based on the assumptions, let us determine the distributions of $L_t|\bar{L}_{t-1}$ and $a^\bullet_t|\bar{L}_t$.

Note first that, since $Y_t, Z^1_t, Z^2_t, \dots$ are independent of $(\bar{L}_{t-1}, a^1_{t-1}, a^2_{t-1}, \dots)$, all the variables

$$Z^1_t, a^1_{t-1}, Z^2_t, a^2_{t-1} \dots$$

are mutually conditionally independent given $(\bar{L}_{t-1}, Y_t)$. In particular, for any $i \in \mathbb{N}$, the conditional distribution of $\dot{Z}^i_t = a^i_{t-1} + Z^i_t$ given $\bar{L}_{t-1}$ is given by the convolution of the distributions of the summands, hence, for any $s \in \mathbb{R}$,

$$\mathbb{P}[\tilde{a}^i_t < s|\bar{L}_{t-1}, Y_t] = \mathbb{P}[\dot{Z}^i_t < s - \eta_t - Y_t|\bar{L}_{t-1}, Y_t] = \Psi_t(s)$$

$$\Psi_t(s) = \tilde{\psi}_t(s - \eta_t - Y_t), \qquad \tilde{\psi}_t = \alpha_{t-1} \circ \psi_t,$$

and, similarly,

$$\dot{p}_t = \mathbb{P}[\tilde{a}^i_t < 0|\bar{L}_{t-1}] = \Xi_t(-\eta_t)$$

where $\Xi_t = \alpha_{t-1} \circ \psi_t \circ \phi_t$.

Moreover, $\dot{Z}^1_t, \dot{Z}^2_t, \dots$ are mutually conditionally independent given $(\bar{L}_{t-1}, Y_t)$, which implies, by the Law of Large Numbers, applied to the conditional distributions of $\dot{Z}^\bullet_t$, that L_t is conditionally constant given $(\bar{L}_{t-1}, Y_t)$ with

$$L_t = \mathbb{P}[\tilde{a}^i_t < 0|\bar{L}_{t-1}, Y_t] = \Psi_t(0)$$

Applying the Complete Probability Theorem and the independence of $\bar{L}_{t-1}$ and Y_t, we finally get

$$\begin{aligned}\mathbb{P}[L_t < \theta|\bar{L}_{t-1}] &= \int \mathbb{P}[L_t < \theta|\bar{L}_{t-1}, y] d\mathbb{P}_{Y_t}(y) \\ &= \int \mathbb{P}[\tilde{\psi}_t(-\eta_t - y) < \theta|\bar{L}_{t-1}, y] d\mathbb{P}_{Y_t}(y) \\ &= \int \mathbb{P}[-\tilde{\psi}^{-1}_t(\theta) - \eta_t < y|\bar{L}_{t-1}, y] d\mathbb{P}_{Y_t}(y) \\ &= \int_{-\eta_t - \tilde{\psi}^{-1}_t(\theta)} d\mathbb{P}_{Y_t}(y) = 1 - \phi_t(-\eta_t - \tilde{\psi}^{-1}_t(\theta)) \\ &= 1 - \phi_t(\Xi^{-1}_t(\dot{p}_t) - \tilde{\psi}^{-1}_t(\theta)).\end{aligned} \tag{9}$$

To completely describe the distribution of the losses, it remains to specify α_t (i.e. the c.c.d.f of a^1_t) which we do in the following Lemma:

Lemma 1. *It follows that $a^1_t, a^2_t, \dots$ are conditionally independent given $\bar{L}_t$ and*

$$\alpha_t(s) = \pi_t \gamma_t(s) + (1 - \pi_t)\zeta_t(s), \qquad \zeta_t(s) = \frac{\Psi_t(s - \Psi^{-1}_t(L_t)) - L_t}{1 - L_t}. \tag{10}$$

Proof. Denote $\hat{a}^i_t$ the i-th existing loan which has not defaulted at t. We show that $\hat{a}^1_t, \hat{a}^2_t, \dots$ are mutually conditionally independent given $\bar{L}_t$ with common c.c.d.f. ζ_t ; the Lemma will then follow by the textbook probability calculus. Before doing so, however, let us mention an easily provable fact:

Auxiliary assertion. Let u and v be independent unit uniform and let $c \in (0, 1)$ be an independent random variable. Then the variable

$$\upsilon_{u,v,c} = \begin{cases} u & \textit{if } u < c \\ c + v(1 - c) & \textit{otherwise} \end{cases}$$

is unit uniform independent of c.

Returning to the main proof, let $u_1, v_1, u_2, v_2, \ldots$ be mutually independent unit uniform variables, independent of $\bar{L}_t$ and of the assets of all newcomers. By [Pollard(2002)] p. 238, [Kallenberg(2002)] Theorem 6.10 and our Auxiliary assertion, the distribution of $\bar{L}_t, \tilde{a}_t^1, \tilde{a}_t^2, \ldots$ will not change if $\tilde{a}_t^i = \Psi_t^{-1}(\upsilon_{u_i,v_i,L_t})$. Hence let us assume the last equality and note that

$$\tilde{a}_t^i < 0 \Leftrightarrow \upsilon_{u_i,v_i,L_t} < \Psi_t(0) \Leftrightarrow u_i < \Psi_t(0) \Leftrightarrow u_i < L_t$$

If all $\bar{L}_t, u_1, u_2, \ldots$ were deterministic then, for any i, $\hat{a}_t^i = \Psi_t^1(L_t + v_j(1 - L_t)) = \zeta_t^{-1}(v_j)$ for some j i.e. ζ_t would be a c.d.f. of $\hat{a}_t^i$ and it could be easily checked that $\hat{a}_t^1, \hat{a}_t^2, \ldots$ are independent; however, this implies, by [Hoffmann-Jørgenson(1994)]4.5.2, that ζ_t is a conditional c.d.f. of $\hat{a}_t^i$ for any i and that $\hat{a}_t^1, \hat{a}_t^2, \ldots$ are conditionally independent given $\bar{L}_t$. $\square$

By the initial assumptions, (9) and (10), we have completely described the distribution of L.

3 The Geometric Lévy Assets

In the present Section we show that our generalization is suitable in the case that the assets of the individuals follow a geometric Lévy, instead of a geometric Brownian, motion.

Before doing so, let us stay with the Brownian model for a while and note, even if it is not a necessary condition for the validity of the assumptions of the model, that it is quite natural to assume that $W^i = U^0 + U^i$ where U^0 and U^i are independent Brownian motions (the first playing role of the common factor, the latter being the individual one) - clearly, $U^1, U^2, \ldots$ have to be equally distributed. Reflecting this and preparing for discontinuous paths, we may rewrite the evolution of the assets of the i-th borrower as

$$\frac{dA_t^i}{A_{t-}^i} = \mu dt + dU_t^0 + dU_t^i, \tag{11}$$

Getting back to our generalization, we no longer require $U^0, U^1, \ldots$ to be Brownian but we allow them to be Lévy processes with $\mathbb{E}U_1^i = 0, \mathrm{var}U_1^i < \infty$. Since the variances of both U^1, U^2 at the unit time are finite, so have to be their absolute moments at finite times implying the i-th process to possess a Lévy decomposition

$$U_t^i = \varsigma_i W_t^i + J_t, \qquad J_t = \int_{-\infty}^{\infty} z N^i(t.dz)$$

where W_t^i is a standard Wiener process N^i is a compensated Poisson measure given by a Lévy measure ν_i (see e.g. [Oksendal and Sulem(2004)] or [Kallenberg(2002)] for the notions of Poisson and Lévy measures and the Lévy decomposition). Clearly $\nu_1 = \nu_2 = \ldots$.

Lemma 2. *Under our assumptions, (8) holds with*

$$\eta_t^i = \mu - \frac{1}{2}\left(|\nu_0| + \varsigma_0^2 + |\nu_1| + \varsigma_1^2\right) - \log B_t^i$$

$$Y_t = \varsigma_0 H_t^0 + \int_0^1 \int_{-\infty}^{\infty} [\log(1+s) - s] N_t^0(ds, dz)$$

$$Z_t^i = \varsigma_0 H_t^i + \int_0^1 \int_{-\infty}^{\infty} [\log(1+s) - s] N_t^i(ds, dz)$$

where, for any $i \in \mathbb{N}$, H_t^i is standard normal and N_t^i is a compensated Poisson measure given by ν_i such that $H_t^0, N_t^0, H_t^1, N_t^1, \ldots$ are mutually independent.

Proof. Fix $i \in \mathbb{N}$ and denote $V_\tau = \mu(\tau - t) + U^0_{\tau-t} + U^i_{\tau-t}$, $\tau \geq 0$. Clearly, V is a Lévy process with Lévy decomposition

$$V_\tau = \mu t + \theta W + J_t, \qquad \theta = \sqrt{\varsigma_0^2 + \varsigma_1^2}, \quad W_\tau = \theta^{-1}(\varsigma_0 W^0_{t-\tau} + \varsigma_1 W^i_{\tau-t}),$$
$$J_\tau = J^0_{\tau-t} + J^i_{\tau-t}.$$

Evidently, W is a standard Wiener process and J is a compensated Lévy jump process with Lévy measure $\nu = \nu_0 + \nu_i$.

The evolution of the assets clearly fulfils

$$\frac{dA^i_{t+\tau}}{A^i_{t+\tau-}} = dV_\tau$$

with the solution, according to [Oksendal and Sulem(2004)] 1.15, given by

$$\begin{aligned} A_{t+\tau} &= A_t \exp\{(\mu - \frac{1}{2}\theta^2)\tau + \theta W_\tau \\ &+ \int_0^\tau \int_{-\infty}^\infty (\log(1+s) - s)\nu(dz)ds \\ &+ \int_0^\tau \int_{-\infty}^\infty (\log(1+s) - s)[N^0_t(ds,dz) + N^i_t(ds,dz)]\} \\ &= A_t \exp\{(\mu - \frac{1}{2}\theta^2)\tau + \varsigma_0 W^0_\tau + \varsigma_1 W^i_\tau \\ &+ [\frac{1}{1+\tau} - 1 - \frac{\tau^2}{2}]|\nu_0 + \nu_1| \\ &+ \int_0^\tau \int_{-\infty}^\infty (\log(1+s) - s)[N^0_t(ds,dz) + N^i_t(ds,dz)]\}. \end{aligned}$$

By putting $\tau = 1$ and subtracting $\log B^i_t$ we get the Lemma. □

In fact, the Lemma says that, as in the original model, the value of a borrower's assets depends on a common and an individual factors. However, the distributions of the factors are not the same as those of an increments of corresponding "driving" processes $U^\bullet$ as at the original model.

It remains to note that, generally, there are not closed formulas for distribution functions corresponding to variables Y_t and Z^1_t hence a MC simulation has to be used, possibly requiring special treatment, especially in the case of "jump" parts of the variables - for more on this topic, see [Cont and Tankov(2008)], Part II.

4 Conclusion

Summarized, we have formulated the dynamical version of the Merton-Vasicek model. Even if our results are not closed form, the model is tractable by simulation.

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