Dynamic model of Loan Portfolio with Lévy Asset Prices

Martin Šmíd

Abstract. We generalize the well known Merton-Vasicek (KMV) model of a loan portfolio value in two ways: we assume a Lévy process of the debtors’ assets’ value (instead of the Gaussian one) and we model a dynamics of the portfolio value so that the debts may last several periods (instead of a single one). Our model is computable by simulation.

Keywords: risk management, loan portfolio, dynamic model

JEL classification: G32

AMS classification: 90C15

1 The Merton-Vasicek Model

Vasicek, in his famous paper [O.A.Vasicek(2002)], assumes a large portfolio of loans. The assets $A_i$ of the $i$-th debtor follow the geometric Brownian motion

$$\frac{dA_i^t}{A_i^t} = \mu dt + \sigma dW_i^t$$

(1)

where $\mu$ and $\sigma$ are constants and $W_i^t$ is the standard Wiener process. It is further assumed that the debtor is obliged to pay a (non-random) instalment $B_i^t$ at each of the times $t \in \mathbb{N}$.

By solving (1) and subtracting the annual payment, we get

$$a_i^t = a_i^{t-1} + \eta_i^t + X_i^t, \quad a_i^t = \log A_i^t, \quad \eta_i^t = \mu - \frac{1}{2}\sigma^2 - \log B_i^t$$

(2)

where $X_i^t = \sigma(W_i^t - W_i^{t-1})$ is centred normal random variable with variance $\sigma^2$. Clearly, the (conditional) probability $p_i^t$ of the default of the $i$-th debtor is

$$p_i^t = \mathbb{P}(a_i^t < 0 | a_i^{t-1}) = \xi_i^t(-a_i^{t-1} - \eta_i^{t-1})$$

where $\xi_i^t$ is the cumulative distribution function (c.d.f.) of $X_i^t$.

Further, it is assumed by Vasicek, that

$$\text{corr}(X_i^t, X_j^t) = \rho$$

(3)

for any $i \neq j$ and some $\rho$. It is easy to see that the latter assumption is fulfilled, for instance, if there exist mutually independent (normal) variables $Y_t, Z_1^t, Z_2^t, \ldots$, (independent of the evolution of the debtors’ assets up to $t - 1$) such that $Z_i^t, Z_2^t, \ldots$ are equally distributed and

$$X_i^t = Y_t + Z_i^t, \quad i \in \mathbb{N},$$

(4)

such that $\text{var} Y_t = \rho \text{var} X_1^t$ and $\text{var} Z_i^t = (1 - \rho)\text{var} X_1^t$ - to see it, note that

$$\text{corr}(X_i^t, X_j^t) = \frac{\text{var} Y_t}{\text{var} Z_i^t + \text{var} Y_t}.$$

(5)

Further, if the number of the debtors is very large, and if, for all $i$, $a_i^t = a_t$ and $\eta_i^t = \eta$ for some $a_t$ and $\eta$ (or, more generally, $p_i^t = p_t$ for some $p_t$) then, by the Law of Large Numbers, the percentage loss $L_t$ of the bank is conditionally constant given $Y_t, a_{t-1}$:

$$L_t = \psi_t(-a_{t-1} - \eta - Y_t)$$

(6)

1Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, e-mail: smid@utia.cas.cz
where \( \psi_t \) is the c.d.f. of \( Z_1^t \) and, consequently, by the Complete Probability Theorem,

\[
P(L_t < \theta | p_t) = 1 - \phi_t \left( -a_{t-1} - \eta_t - \psi_t^{-1}(\theta) \right)
= 1 - \phi_t \left( \xi_t^{-1}(p_t) - \psi_t^{-1}(\theta) \right)
\]

(7)

(see [O.A.Vasicek(2002)] for details).

2 A Generalization

We generalize the Merton-Vasicek model two ways: we release the assumption of the normality of the factors and we take the dynamics of the system into account, making \( p_t \)'s endogenous and no longer requiring the identical initial states of the borrowers' assets. Our goal is to describe the distribution of the losses by specifying the conditional distribution function of \( L_t \) given

\[
\bar{L}_t = (L_1, L_2, \ldots, L_t)
\]

for each \( t \).

For a greater mathematical rigour, we assume the infinite number of loans in the portfolio rather than their "large amount".

To model the dynamics realistically, we assume that there is a certain amount of newly acquired deals at the beginning each period: we denote \( \pi_t \) the \( \bar{L}_t \)-measurable variable determining the ratio of the loans new at \( t \) to their overall amount (the overall amount including the new ones plus the existing ones excluding those which have defaulted at the time \( t \)). We assume the assets of a newly coming debtor to be distributed according to a common continuous and strictly increasing conditional c.d.f. \( \gamma_t \) given \( \bar{L}_t \) and we suppose the assets of all the newcomers be mutually conditionally independent given and \( \bar{L}_t \).

Remark 1. We express this (more rigorously) as follows: Let \( t \in \mathbb{N} \) and let, for each \( i \in \mathbb{N} \), there exist a Bernoulli variable \( I_{i,t} \) with a parameter \( \pi_t \); if \( I_{i,t} = 1 \) then a newly coming loan will be indexed by \( i \) during the period from \( t \) to \( t+1 \); if, on the other hand, \( I_{i,t} = 0 \), then the first existing loan out of those which did not default at \( t \) and which are still not indexed will be indexed by \( i \). Naturally, \( I_{1,t}, I_{2,t}, \ldots \) are required to be mutually conditionally independent given \( \bar{L}_t \) and independent of all \( Y \cdot \) and \( Z \cdot \).

Let \( t, i \in \mathbb{N} \). Coping with the reindexing, we reformulate (2) as

\[
\tilde{a}_{i,t} = a_{i-1} + \eta_t + X_{i,t},
\]

(8)

where \( \eta_{t} \) is an arbitrary value (including the installment if there is any), \( a_{i-1} \) and \( \tilde{a}_{i,t} \) are the values of the log assets of the \( i \)-th debtor at the time \( t-1 \), \( t \) respectively\(^1\) and \( X_{i,t} \) is given by (4). The variables \( Y_t \) and \( Z_{i,t} \) do not have to be normal now but it is only required that their conditional distribution functions \( \phi_t \) and \( \psi_t \) are strictly monotonic and continuous.\(^2\) Later (Section 3) we show how (8) arises naturally from a generalization of (1).

Naturally, we define the percentage loss as

\[
L_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} 1[\tilde{a}_{i,t} < 0].
\]

\( t \in \mathbb{N} \), in our new infinite setting.

Getting to the distribution of the series \( L_1, L_2, \ldots \), let us assume \( a_{0,1}, a_{0,2}, \ldots \) are i.i.d (with a strictly increasing continuous c.d.f. \( \alpha_0 \)).

To describe the distribution, we shall proceed by induction:

Let \( t \in \mathbb{N} \) and assume that

\(^1\)Remember that a debtor may be indexed by different index in different periods; here, \( i \) is the index valid from \( t-1 \) to \( t \).

\(^2\)Depending on the normality no way, both (7) and (5) keep holding (with = instead of \( \geq \) ) even under the generalized conditions.
(1) we know the the (joint) distribution of $\bar{L}_{t-1}$

(2) the variables $a_{t-1}^1, a_{t-1}^2, \ldots$ are identically conditionally distributed and mutually conditionally independent given $\bar{L}_{t-1}$

(3) the conditional distribution function $\alpha_{t-1}$ of $a_{t-1}^i | \bar{L}_{t-1}$ is known to us.

Based on the assumptions, let us determine the distributions of $L_t | \bar{L}_{t-1}$ and $a_t^i | \bar{L}_{t}$.

Note first that, since $Y_t, Z_t^1, Z_t^2, \ldots$ are independent of ($\bar{L}_{t-1}, a_{t-1}^1, a_{t-1}^2, \ldots$), all the variables

\[ Z_t^1, a_{t-1}^1, Z_t^2, a_{t-1}^2, \ldots \]

are mutually conditionally independent given $\bar{L}_{t-1}$, $Y_t$. In particular, for any $i \in \mathbb{N}$, the conditional distribution of $\bar{Z}_t^i = a_{t-1}^i + Z_t^i$ given $\bar{L}_{t-1}$ is given by the convolution of the distributions of the summands, hence, for any $s \in \mathbb{R}$,

\[ P[\bar{a}_t^i < s | \bar{L}_{t-1}, Y_t] = P[\bar{Z}_t^i < s - \eta_t - Y_t | \bar{L}_{t-1}, Y_t] = \Psi_t(s) \]

\[ \Psi_t(s) = \tilde{\psi}_t(s - \eta_t - Y_t), \quad \tilde{\psi}_t = \alpha_{t-1} \circ \psi_t, \]

and, similarly,

\[ \hat{\psi}_t = P[\bar{a}_t^i < 0 | \bar{L}_{t-1}] = \Xi_t(-\eta_t) \]

where $\Xi_t = \alpha_{t-1} \circ \psi_t \circ \phi_t$.

Moreover, $\bar{Z}_t^1, \bar{Z}_t^2, \ldots$ are mutually conditionally independent given ($\bar{L}_{t-1}, Y_t$), which implies, by the Law of Large Numbers, applied to the conditional distributions of $\bar{Z}_t^i$, that $L_t$ is conditionally constant given ($\bar{L}_{t-1}, Y_t$) with

\[ L_t = P[\bar{a}_t^i < 0 | \bar{L}_{t-1}, Y_t] = \Psi_t(0) \]

Applying the Complete Probability Theorem and the independence of $\bar{L}_{t-1}$ and $Y_t$, we finally get

\[
\mathbb{P}[L_t < \theta | \bar{L}_{t-1}] = \int \mathbb{P}[L_t < \theta | \bar{L}_{t-1, y}] \mathbb{P}[Y_t(y)] dy
\]

\[
= \int \mathbb{P}[\tilde{\psi}_t(-\eta_t - Y_t) < \theta | \bar{L}_{t-1, y}] \mathbb{P}[Y_t(y)] dy
\]

\[
= \int \mathbb{P}[-\tilde{\psi}_t^{-1}(\theta) - \eta_t < Y_t | \bar{L}_{t-1, y}] \mathbb{P}[Y_t(y)] dy
\]

\[
= \int \mathbb{P}[Y_t(y) = 1 - \phi_t(-\eta_t - \tilde{\psi}_t^{-1}(\theta))]
\]

\[
= 1 - \phi_t(\Xi_t^{-1}(\hat{\psi}_t) - \tilde{\psi}_t^{-1}(\theta)).
\]

To completely describe the distribution of the losses, it remains to specify $\alpha_t$ (i.e. the c.c.d.f of $a_t^i$) which we do in the following Lemma:

**Lemma 1.** It follows that $a_t^1, a_t^2, \ldots$ are conditionally independent given $\bar{L}_t$ and

\[ \alpha_t(s) = \pi_t \gamma_t(s) + (1 - \pi_t) \zeta_t(s), \quad \zeta_t(s) = \frac{\Psi_t(s - \tilde{\psi}_t^{-1}(L_t)) - L_t}{1 - L_t}. \]

**Proof.** Denote $\bar{a}_t^i$ the i-th existing loan which has not defaulted at $t$. We show that $\bar{a}_t^1, \bar{a}_t^2, \ldots$ are mutually conditionally independent given $\bar{L}_t$ with common c.c.d.f. $\zeta_t$ ; the Lemma will then follow by the textbook probability calculus. Before doing so, however, let us mention an easily provable fact:

**Auxiliary assertion.** Let $u$ and $v$ be independent unit uniform and let $c \in (0, 1)$ be an independent random variable. Then the variable

\[ u_{w, v, c} = \begin{cases} u & \text{if } u < c \\ c + v(1 - c) & \text{otherwise} \end{cases} \]
is unit uniform independent of $c$.

Returning to the main proof, let $u_1, v_1, u_2, v_2, \ldots$ be mutually independent unit uniform variables, independent of $L_t$ and of the assets of all newcomers. By [Pollard(2002)] p. 238, [Kallenberg(2002)] Theorem 6.10 and our Auxiliary assertion, the distribution of $\tilde{L}_t, \tilde{a}_1, \tilde{a}_2, \ldots$ will not change if $\tilde{a}_i = \Psi^{-1}_t(v_{u_i,v_i,L_t})$.

Hence let us assume the last equality and note that

$$\tilde{a}_i < 0 \Leftrightarrow v_{u_i,v_i,L_t} < \Psi_t(0) \Leftrightarrow u_i < \Psi_t(0) \Leftrightarrow u_i < L_t$$

If all $\tilde{L}_t, u_1, u_2, \ldots$ were deterministic then, for any $i,$ $\tilde{a}^i = \Psi^i_t(L_t + v_j(1-L_t)) = \zeta_t(1)$ for some $j$ i.e. $\zeta_t$ would be a c.d.f. of $\tilde{a}_i$ and it could be easily checked that $\tilde{a}_1, \tilde{a}_2, \ldots$ are independent; however, this implies, by [Hoffmann-Jørgensen(1994)]4.5.2, that $\zeta_t$ is a conditional c.d.f. of $\tilde{a}_i$ for any $i$ and that $\tilde{a}_i, \tilde{a}_2, \ldots$ are conditionally independent given $L_t.$

By the initial assumptions, (9) and (10), we have completely described the distribution of $L.$

3 The Geometric Lévy Assets

In the present Section we show that our generalization is suitable in the case that the assets of the individuals follow a geometric Lévy, instead of a geometric Brownian, motion.

Before doing so, let us stay with the Brownian model for a while and note, even if it is not a necessary condition for the validity of the assumptions of the model, that it is quite natural to assume that the assets of the $i$-th borrower as

$$\frac{dA^i_t}{A^i_{t-}} = \mu dt + dU^0_t + dU^i_t,$$ (11)

Getting back to our generalization, we no longer require $U^0, U^1, \ldots$ to be Brownian but we allow them to be Lévy processes with $E U^i_1 = 0,$ $\text{var} U^i_1 < \infty.$ Since the variances of both $U^1, U^2$ at the unit time are finite, so have to be their absolute moments at finite times implying the $i$-th process to possess a Lévy decomposition

$$U^i_t = \zeta_t W^i_t + J_t, \quad J_t = \int_{-\infty}^{\infty} z N^i(t, dz)$$

where $W^i_t$ is a standard Wiener process $N^i$ is a compensated Poisson measure given by a Lévy measure $\nu$ (see e.g. [Oksendal and Sulem(2004))] or [Kallenberg(2002)] for the notions of Poisson and Lévy measures and the Lévy decomposition). Clearly $\nu_1 = \nu_2 = \ldots$.

**Lemma 2.** Under our assumptions, (8) holds with

$$\eta^i_t = \frac{1}{2} \left( |\nu_0| + \zeta^2_0 + |\nu_1| + \zeta^2_1 \right) - \log B^i_t$$

$$Y_t = \zeta_0 H^0_t + \int_0^1 \int_{-\infty}^{\infty} \log(1+s) - s] N^0_t(ds, dz)$$

$$Z^i_t = \zeta_t H^i_t + \int_0^1 \int_{-\infty}^{\infty} \log(1+s) - s] N^i_t(ds, dz)$$

where, for any $i \in \mathbb{N},$ $H^i_t$ is standard normal and $N^i_t$ is a compensated Poisson measure given by $\nu$ such that $H^0_t, N^0_t, H^1_t, N^1_t, \ldots$ are mutually independent.
Proof. Fix \( i \in \mathbb{N} \) and denote \( V_{\tau} = \mu(\tau - t) + U_{\tau - t}^0 + U_{\tau - t}^i, \ \tau \geq 0 \). Clearly, \( V \) is a Lévy process with Lévy decomposition

\[
V_{\tau} = \mu t + \theta W_{\tau} + J_{\tau} \quad \theta = \sqrt{\varsigma_0^2 + \varsigma_1^2}, \quad W_{\tau} = \theta^{-1}(\varsigma_0 W_{\tau - t}^0 + \varsigma_1 W_{\tau - t}^i),
\]

\[
J_{\tau} = J_{\tau - t}^0 + J_{\tau - t}^i.
\]

Evidently, \( W \) is a standard Wiener process and \( J \) is a compensated Lévy jump process with Lévy measure \( \nu = \nu_0 + \nu_i \).

The evolution of the assets clearly fulfils

\[
\frac{dA_{i+\tau}}{A_{i+\tau}} = dV_{\tau}
\]

with the solution, according to [Oksendal and Sulem(2004)] 1.15, given by

\[
A_{i+\tau} = A_t \exp\{\left(\mu - \frac{1}{2}\theta^2\right)\tau + \theta W_{\tau}
+ \int_0^\tau \int_{-\infty}^{\infty} (\log(1 + s) - s) \nu(ds, dz)ds
+ \int_0^\tau \int_{-\infty}^{\infty} (\log(1 + s) - s) [N_0^i(ds, dz) + N_i^i(ds, dz)]\}
\]

\[
= A_t \exp\{\left(\mu - \frac{1}{2}\theta^2\right)\tau + \varsigma_0 W_{\tau}^0 + \varsigma_1 W_{\tau}^i
+ \left[\frac{1}{1 + \tau} - 1 - \frac{\tau^2}{2}\right] ||\nu_0 + \nu_i||
+ \int_0^\tau \int_{-\infty}^{\infty} (\log(1 + s) - s) [N_0^i(ds, dz) + N_i^i(ds, dz)]\}.
\]

By putting \( \tau = 1 \) and subtracting \( \log B_i \) we get the Lemma. \( \square \)

In fact, the Lemma says that, as in the original model, the value of a borrower’s assets depends on a common and an individual factors. However, the distributions of the factors are not the same as those of an increments of corresponding “driving” processes \( U \) as at the original model.

It remains to note that, generally, there are not closed formulas for distribution functions corresponding to variables \( Y_t \) and \( Z_t \) hence a MC simulation has to be used, possibly requiring special treatment, especially in the case of “jump” parts of the variables - for more on this topic, see [Cont and Tankov(2008)], Part II.

4 Conclusion

Summarized, we have formulated the dynamical version of the Merton-Vasicek model. Even if our results are not closed form, the model is tractable by simulation.

Acknowledgement: This work is supported by the grants no. 402/09/0965 and P402/10/1610 and P402/10/0956 of the Czech Science Foundation.

References


