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Backward stochastic differential equations and its application to stochastic control [1]

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Abstract. In this article, we introduce the concept of Backward Stochastic Differential Equations (BSDE), provide fundamental theorems of existence and uniqueness of the solution for some essential cases and we show by example its important connections to financial mathematics. Finally, we focus on vast applications of BSDE to stochastic control via Pontryagin's maximum principle.

1 Introduction

The domain of BSDE, in its full generality, was first studied in 1990 by Pardoux and Peng who formulated the general problem of BSDE and proved some fundamental theorems including the central one - the existence and uniqueness of the solution, see [3]. Since then, BSDE have found a variety of applications in finance, in physics but also in even more theoretical fields such as stochastic control, theory of random processes probability distributions, probabilistic representation of elliptic and parabolic-type deterministic PDE's, numerical methods for PDE's and many other.
The first section of the article gives an introduction to BSDE - we start by the theorem of Pardoux and Peng for finite time horizon BSDE and then we proceed to infinite time horizon case considering, in addition, Lévy driven stochastic noise. We refer to [5], [7] and [9] for an overview on generalizations of this type. Further, to present an example of a practical model using the BDSE theory. We show how the theory can be applied to the European Call Option hedging problem. In the second section, we formulate the task of stochastic control and associated maximum stochastic maximum principle and discuss some other extension of the model.

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2 Backward stochastic differential equations (BSDE)

2.1 Finite time horizon case

The main motivation for introducing the BSDE is the need for solving problems with terminal condition of the following type

$$\begin{aligned} -dY_t &= f(t, Y_t, Z_t)dt - Z_t dW_t, \quad \forall t \in [0, T), \text{ a.s.} \\ Y_T &= \xi, \text{ a.s.}, \end{aligned} \tag{1}$$

where $0 < T < +\infty$ is a finite time horizon, $\big(\Omega, \mathcal{F}, \mathbf{P}\big)$ is a standard probability space equipped by a standard $\mathbb{R}^d$-valued Wiener process $\big(W_t\big)_{t\in[0,T]}$. Let $\big(\mathcal{F}_t^W\big)_{t\in[0,T]}$ be the canonical filtration of W_t, i.e. $\mathcal{F}_t^W = \sigma\big(W_s; s \le t\big)$ and $\big(\mathcal{F}_t\big)_{t\in[0,T]}$ be its completion. The function f (called *drift*) and the random variable ξ (*terminal condition*) are, in fact, the only inputs of the equation.

Definition 1: *The couple (f, ξ) is called standard parameters of the equation (1) if it holds*

- $\xi \in \mathbf{L}^2(\mathcal{F}_T; \mathbb{R}^n)$, i.e. ξ is an $\mathcal{F}_T$-measurable r.v., $\mathbb{R}^n$-valued, satisfying $\mathbf{E}||\xi||^2 < +\infty$
- $f : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$, i.e. $(\omega, t, y, z) \mapsto f(\omega, t, y, z) \in \mathbb{R}$
- f is an application $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n)$ - progressively measurable
- $\forall t \in [0, T] : f(\cdot, t, 0, 0) \in \mathcal{H}^2(\mathbb{R})$, i.e. $f(\cdot, t, 0, 0)$ is $\mathcal{F}_t$ -progressive with $\mathbf{E} \int_0^T f^2(\cdot, t, 0, 0)dt < +\infty$
- f is uniformly Lipschitz in y and z, i.e. $\exists C > 0$ that $|f(\omega, t, y_1, z_1) - f(\omega, t, y_2, z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|)$ $\forall y_1, y_2 \in \mathbb{R},\ \forall z_1, z_2 \in \mathbb{R}^n,\ d\mathbf{P} \otimes dt$ a.s.

Generally, we denote as $\mathcal{H}^2(\mathcal{X})$ the set of stochastic processes $(\varphi_t)_{t\in[0,T]}$, $\mathcal{F}_t$ - progressive, with values in Banach space $\mathcal{X}$, satisfying $\mathbf{E} \int_0^T ||\varphi_t||^2_{\mathcal{X}} dt < +\infty$.

The properties of standard parameters are sufficient conditions for the existence and uniqueness of the solution which is an assertion of the following theorem proved by Pardoux and Peng in [3].

Theorem 1: *Let (f, ξ) be standard parameters. Then the BSDE (1) has a unique solution $(Y_t, Z_t)_{t\in[0,T]} \in \mathcal{H}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$.*

Idea of the proof: We define an application $\Phi : \mathcal{H}^2(\mathbb{R}^n)\times\mathcal{H}^2(\mathbb{R}^{n\times d}) \to \mathcal{H}^2(\mathbb{R}^n)\times\mathcal{H}^2(\mathbb{R}^{n\times d})$ so that $\Phi(U, V) = (Y, Z)$ where

$$\begin{aligned} -dY_t &= f(t, U_t, V_t)dt - Z_t dW_t, \quad \forall t \in [0, T) \text{ a.s.} \\ Y_T &= \xi, \text{ a.s.} \end{aligned} \tag{2}$$

To have Φ defined correctly, one must show that there exists a unique solution to (2) belonging to the product space $\mathcal{H}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$. Note that in (2), the driver f does not depend on Y_t and Z_t.

Further, we realize that (Y,Z) solves (1) iff $\Phi(Y,Z) = (Y,Z)$ therefore, (Y,Z) is a fixed point of Φ (on a Banach space $\mathcal{H}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$). It is possible to show that Φ is a contraction on $\mathcal{H}^2(\mathbb{R}^n) \times \mathcal{H}^2(\mathbb{R}^{n\times d})$ for the norm $||\cdot||_\beta$ where $\beta > 0$ is chosen properly and

$$||(Y,Z)||_\beta^2 = \mathbf{E}\int_0^T e^{\beta s}||Y_s||^2 ds + \mathbf{E}\int_0^T e^{\beta s}||Z_s||^2 ds.$$

Then the solution to BSDE (1) exists uniquely by the fixed point theorem.

Remark 1: 1) The process $(Z_t)_{t\in[0,T]}$, introduced by Theorem 1, ensures the adaptability of the process $(Y_t)_{t\in[0,T]}$.

2) The uniqueness of the solution means that if (Y_t, Z_t) and $(\tilde{Y}_t, \tilde{Z}_t)$ are two solutions to (1) then $\mathbf{E}\int_0^T ||Y_t - \tilde{Y}_t||^2 dt = \mathbf{E}\int_0^T ||Z_t - \tilde{Z}_t||^2 dt = 0$.

3) Since the process $(Y_t)_{t\in[0,T]}$ has continuous trajectories a.s., the space $\mathcal{H}^2(\mathbb{R}^n)$ in Definition 1 can be replaced with the space $\mathcal{S}^2(\mathbb{R}^n)$ which is a set of $\mathcal{F}_t$-adapted processes $(Y_t)_{t\in[0,T]}$ with $\mathbf{E}\big[\underset{0\le t\le T}{sup}||Y_t||^2\big] < +\infty$.

Theorem 1, in general, says nothing about the form of the solution even if it exists. Nevertheless, it is possible to express and compute it in some special cases. One such a case is a linear model, i.e. $f(t, Y_t, Z_t) = \beta_t Y_t + \gamma_t' Z_t + \varphi_t$ where $(\beta_t)_{t\in[0,T]}$ and $(\gamma_t)_{t\in[0,T]}$ are two processes $\mathcal{F}_t$ - progressively measurable, bounded, with values in $\mathbb{R}$ and $\mathbb{R}^n$, respectively. $(\varphi_t)_{t\in[0,T]}$ is a $\mathcal{F}_t$ - progressively measurable, $\mathbb{R}$-valued process, square-integrable. We suppose that Y_t and Z_t have corresponding dimensions, i.e. they are $\mathbb{R}$ and $\mathbb{R}^n$ - valued, respectively. Then we have, due to Pardoux and Peng [3],

Theorem 2: *The linear BSDE*

$$\begin{aligned} -dY_t &= (\beta_t Y_t + \gamma_t' Z_t + \varphi_t)dt - Z_t' dW_t, \quad \forall t \in [0,T) \text{ a.s.} \\ Y_T &= \xi \quad \text{a.s.} \end{aligned} \tag{3}$$

has a unique solution $Y_t = \mathbf{E}\big[H_T\xi + \int_t^T H_s\varphi_s ds|\mathcal{F}_t\big]$, $\forall t \in [0,T]$ a.s., *where the process* $(H_t)_{t\in[0,T]}$ *is a solution to the following SDE*

$$dH_t = H_t(\beta_t dt + \gamma_t' dW_t); H_0 = 1.$$

Remark 2: 1) The second solution process $(Z_t)_{t\in[0,T]}$ is obtained by applying the integral representation theorem for square-integrable continuous martingales (see e.g. [1]) to the martingale $M_t = Y_t + \int_0^t H_s\varphi_s ds$.

2.2 Example

To see one possible application of BSDE, we give a classical example. It concerns the hedging task for a European Call Option in a complete market.

We consider a financial market model with $n+1$ assets $(S^0, S^1, ..., S^n)$ whose price dynamics is given by the following SDE's

- $dS_t^0 = S_t^0 r_t dt$ (one non-risky asset)

- $dS_t^i = S_t^i\big(b_t^i dt + \sigma_t^i dW_t\big),\ i = 1, ..., n$ (n risky assets) where

$(r_t)_{t\in[0,T]},\ (b_t)_{t\in[0,T]}$ and $(\sigma_t)_{t\in[0,T]}$ are $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n,n}$ - valued bounded processes, $\mathcal{F}_t$ - progressive. Moreover, we assume that there is a bounded process $(\theta_t)_{t\in[0,T]}$ with values in $\mathbb{R}^n$. $(\theta_t)_{t\in[0,T]}$ is called *market price of risk* and it ensures the absence of arbitrage in the market.

The portfolio process π is an $\mathbb{R}^n$ - valued process, $\mathcal{F}_t$ - progressive whose i^{th} component π_t^i represents the amount invested into the i^{th} asset in time t. Moreover, we assume that $\mathbf{E}\int_0^T ||\sigma_t'\pi_t||^2 dt < +\infty$.

The wealth process $Y^{y_0,\pi}$, associated to the initial amount y_0 and the portfolio process π, is given as a solution to the following (forward) SDE

$$\begin{aligned} dY_t^{y_0,\pi} &= r_t Y_t^{y_0,\pi} dt + \pi_t'[b_t - r_t\mathbf{1}]dt + \pi_t'\sigma_t dW_t,\ t\in(0,T] \\ Y_0^{y_0,\pi} &= y_0,\ \text{a.s.} \end{aligned} \tag{4}$$

This approach is very intuitive for the wealth process simply expresses our wealth gained by applying our investment strategy π starting with an initial deposit y_0. What is, nevertheless, more interesting is a task of hedging a financial instrument, concretely a European Call option (EC), i.e. we look for an investment strategy π so that the terminal value Y_T^π of the corresponding wealth process would be equal to the EC pay-off which means $Y_T^\pi = (S_T - K)^+$ where K is an exercise price of the EC and S_T is the price of an underlying asset at time T. Less formally said, we can imagine EC pay-off as a random amount (contingent claim) which we will have to pay (cover) in the future (at time T). Our goal is to invest now (at $t_0 < T$) so that our wealth at time T is equal to that random amount. Formally, it means that we need to find a solution $(Y, Z) = (Y, \sigma'\pi)$ to the following BSDE

$$\begin{aligned} dY_t^{\pi} &= r_t Y_t^{\pi} dt + \pi_t'[b_t - r_t\mathbf{1}]dt + \pi_t'\sigma_t dW_t,\ t\in[0,T) \\ Y_T^{\pi} &= (S_T - K)^+,\ \text{a.s.} \end{aligned} \tag{5}$$

Then, if we assume,in addition, that the matrix σ is invertible, we can express our investment strategy as $\pi = \sigma^{-1}Z$.

2.3 Infinite time horizon and Lévy driven BSDE

Since the end of 1990's, there has been a huge progress in introducing jumps into BSDE models. First, just by considering an additional Poisson process but gradually, the theory was built up for general Lévy processes. The reason was, beside some specific physical tasks, that it was more and more clear that real financial asset prices do not follow normal (or better log-normal) distribution naturally obtained by using geometrical Brownian motion. Lévy-driven stochastic models were capable to improve (yet not to solve completely) the problem of heavy tails and to incorporate intuitively expected (and observed) jumps, see [4]. In this subsection we work only with $\mathbb{R}$- valued Lévy processes and we adopt the notation from [2].

Definition 2: *An adapted process $X = \big(X_t\big)_{t\geq 0}$ with $X_0 = 0$ a.s. is a Lévy process if*

1. *X has increments independent of the past, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leq s < t < +\infty$; and*

2. *X has stationary increments, i.e. $X_t - X_s$ has the same distribution as X_{t-s} for $0 \leq s < t < +\infty$; and*

3. *X is continuous in probability, that is $\mathbf{P} - \lim\limits_{s \to t} X_s = X_t$.*

Remark 3: Since every Lévy process Y has a càdlàg modification X (i.e. right continuous with left limit) which is again Lévy process (see [2], Theorem 30), we will always work with this càdlàg process X.

When considering Lévy process in the model, one must specify what filtration is he or she using. In our case, we take a natural filtration of X, i.e. $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$ and we proceed to completion and augmentation $\left(\mathcal{F}_t\right)_{t\geq 0}$ of the natural filtration. We lay $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t \stackrel{def}{=} \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

Before pronouncing the existence and uniqueness theorem for infinite time horizon BSDE, we remind a crucial lemma due to Nualart and Schoutens in [8]. First, we denote as l^2 the space of real-valued sequences $(x_i)_{i\geq 1}$ such that $\sum_{i=1}^{+\infty} |x_i|^2 < +\infty$ and as $\mathcal{H}^2(l^2)$ we denote the space of l^2- valued predictable processes $\psi = \left(\psi_t\right)_{t\geq 0}$ such that

$$||\psi||^2_{\mathcal{H}^2(l^2)} = \mathbf{E} \int_0^{+\infty} \sum_{i=1}^{+\infty} |\psi_t^{(i)}|^2 dt, \tag{6}$$

Lemma 1: *Let X be a Lévy process whose associated Lévy measure ν fulfills*

1. *$\int_{\mathbb{R}} (1 \wedge z^2)\nu(dz) < +\infty$,*

2. *$\int_{(-\varepsilon,\varepsilon)^c} e^{\lambda|z|}\nu(dz) < +\infty$ for every $\varepsilon > 0$ and for some $\lambda > 0$.*

Then every square-integrable random variable $F \in L^2(\mathcal{F}_\infty)$ has a representation of the form

$$F = \mathbf{E}[F] + \int_0^{+\infty} \sum_{i=1}^{+\infty} \psi_t^{(i)} dH_t^{(i)}, \tag{7}$$

where $\left\{\left(H_t^{(i)}\right)_{t\geq 0}\right\}_{i=1}^{+\infty}$ are strongly orthogonal martingales such that each $H^{(i)}$ is a linear combination of the Teugels martingales $Y^{(j)},\ j = 1, ..., i$ associated to the Lévy process X.

Remark 4: See [8] and [2] for more details on this orthogonalization.

Using this representation result, it is sufficient to consider infinite time horizon BSDE of the following type

$$Y_t = \xi + \int_t^{+\infty} g(s, Y_{s_-}, Z_s)ds - \int_t^{+\infty} \sum_{i=1}^{+\infty} Z_t^{(i)} dH_t^{(i)},\ \forall t \in [0, +\infty], \tag{8}$$

where the $\xi \in L^2(\mathcal{F}_\infty)$ and the function $g : \Omega \times [0, +\infty] \times \mathbb{R} \times l^2 \to \mathbb{R}$ fulfills

(A1): There exist two positive non-random functions $u(t) \in L^1([0,+\infty])$ and $v(t) \in L^2([0,+\infty])$ such that

$$|g(t,y_1,z_1)-g(t,y_2,z_2)| \leq v(t)|y_1-y_2|+u(t)|z_1-z_2|, a.s.\ \forall t \in [0,+\infty], (y_i,z_i) \in \mathbb{R}\times l^2,\ i=1,2$$

(A2): $\big(g(t,y,z)\big)_{t\geq 0}$ is $\mathcal{F}_t$-progressively measurable $\forall (y,z) \in \mathbb{R}\times l^2$ with

$$\mathbf{E}\Big(\int_0^{+\infty} |g(t,0,0)|dt\Big)^2 < +\infty.$$

Definition 3: *A solution to BSDE (8) is a pair of processes $(Y,Z) \in \mathcal{S}^2(\mathbb{R}) \times \mathcal{H}^2(l^2)$ and satisfying (8).*

For definition of $\mathcal{S}^2(\mathbb{R})$ see Remark 1. Now we have all the tools to pronounce the existence and uniqueness theorem which is due to Zheng [7].

Theorem 3: *Let $\xi \in L^2(\mathcal{F}_\infty)$ and let g satisfy the assumptions (A1) and (A2). Then BSDE (8) has a unique solution.*

In the next section we show how BSDE naturally arise in the domain of optimal control having the meaning of conjugate variables ("generalized Lagrange multiplicators").

3 Stochastic control

3.1 Finite horizon control problem

Let $X_t^{t,x}$ be a controlled diffusion process in $\mathbb{R}^n$, i.e. $X_t^{t,x}$ is a solution to the (forward) SDE

$$\begin{aligned} dX_s^{t,x} &= b(X_s^{t,x},\alpha_s)ds + \sigma(X_s^{t,x},\alpha_s)dW_s, \quad \forall s \in (t,T] \text{ a.s.} \\ X_t^{t,x} &= x, \end{aligned} \tag{9}$$

where $0 < T < +\infty$, $t \in [0,T)$, $x \in \mathbb{R}^n$, $\alpha = (\alpha_s)_{t\leq s\leq T}$ is an $\mathcal{F}_s$-progressively measurable A-valued control process, $A \subset \mathbb{R}^m$, $\big(W_s\big)_{s\in[t,T]}$ is an $\mathbb{R}^d$-valued standard Wiener process, $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ are two measurable functions satisfying a uniform Lipschitz condition in A, that means that there is a positive finite constant K so that

$$||b(x,a)-b(y,a)|| + ||\sigma(x,a)-\sigma(y,a)|| \leq K||x-y||, \quad \forall x,y \in \mathbb{R}^n, \forall a \in A \tag{10}$$

Let us denote as $\mathcal{A}(t,x)$ the set of all admissible controls α such that

$$\mathbf{E}\Big[\int_t^T ||b(0,\alpha_s)|| + ||\sigma(0,\alpha_s)||^2 ds\Big] < +\infty \tag{11}$$

which ensures strong existence of the diffusion process X from (9).

Furthermore, let $f \in \mathcal{C}([0,T]\times\mathbb{R}^n\times A)$ and $g \in \mathcal{C}^1(\mathbb{R}^n)$ be two functions so that the following functional is meaningful (i.e. it converges)

$$J(t,x,\alpha) = \mathbf{E}\Big[\int_t^T f(s,X_s^{t,x},\alpha_s)ds + g(X_T^{t,x})\Big], \tag{12}$$

and we define cost function $v(t,x)$ by

$$v(t,x) = \underset{\alpha\in\mathcal{A}(t,x)}{sup} J(t,x,\alpha). \tag{13}$$

Our goal is to find such a strategy $\alpha^* \in \mathcal{A}(t,x)$ so that

$$v(t,x) = J(t,x,\alpha^*).$$

Let us define generalized Hamiltonian of the problem $\mathcal{H} : [0,T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}$ by

$$\mathcal{H}(t,x,a,y,z) = b(x,a)'y + trace(\sigma(x,a)'z) + f(t,x,a).$$

We suppose that $\mathcal{H}$ is differentiable in x (with the gradient denoted as $\nabla_x\mathcal{H}$) and we consider the following BSDE

$$\begin{aligned} -dY_s &= \nabla_x\mathcal{H}(s, X_s^{t,x}, \alpha_s, Y_s, Z_s)ds - Z_s dW_s, \quad \forall s\in[t,T) \text{ a.s.} \\ Y_T &= \nabla_x g(X_T^{t,x}) \quad \text{a.s.} \end{aligned} \tag{14}$$

Then we can formulate stochastic Pontryagin's maximum principle providing conditions on the optimal strategy α^*. The proof can be found in [6].

Theorem 4(Stochastic Pontryagin's maximum principle): *Let $\hat\alpha \in \mathcal{A}(t,x)$ and $\hat X$ be the associated controlled diffusion process. Further, let us suppose that there exists a solution $(\hat Y, \hat Z)$ to associated BSDE (14) such that*

1. $\mathcal{H}(t,\hat X_t,\hat\alpha,\hat Y,\hat Z) = \underset{a\in A}{max}\,\mathcal{H}(t,\hat X_t,a,\hat Y,\hat Z), \quad \forall t\in[0,T]$ a.s.

2. $(x,a) \to \mathcal{H}(t,x,a,\hat Y,\hat Z)$ *is a concave function for all* t.

Then $\hat\alpha = \alpha^*$, i.e. $\hat\alpha$ is optimal control strategy to the stochastic control problem (13) which means $v(t,x) = J(t,x,\hat\alpha)$.

3.2 Lévy-driven stochastic control problem

The question now is if we are able to generalize the previous result to Lévy-driven stochastic control problems - both for finite and infinite time horizon. A positive answer to the first part of the question gives us the paper [5]. We note that in case of Lévy diffusion the model is

$$\begin{aligned} dX_s^{t,x} &= b(s,X_s^{t,x},\alpha_s)ds + \sigma(s,X_s^{t,x},\alpha_s)dW_s + \int_{\mathbb{R}^n}\eta(s,X_{s_-}^{t,x},\alpha_{s_-},z)\bar N(ds,dz), \ \forall s\in(t,T] \text{ a.s.} \\ X_t^{t,x} &= x. \end{aligned} \tag{15}$$

The new term is an integral with respect to Poisson random measure

$$\bar N(ds,dz) = \big(\bar N_1(ds,dz),...,\bar N_l(ds,dz)\big)' = \big(N_1(ds,dz) - \chi_1(z)d\nu_1(z),...,N_l(ds,dz) - \chi_l(z)d\nu_l(z)\big)', \tag{16}$$

where $N_i(ds, dz),\ i = 1, ..., l$ are independent Poisson random measures with Lévy measures ν_i respectively, on a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\big)$ satisfying the usual conditions. The indicator functions $\chi_i,\ i = 1, ..., l$ truncate the domain of "small and big jumps". Moreover, we assume that the control process α is predictable, left continuous with right limits. Hand in hand with these corrections, one must change the form of the generalized Hamiltonian to $\mathcal{H} : [0, T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathcal{R} \to \mathbb{R}$ so that

$$\begin{aligned}\mathcal{H}(t, x, a, y, z, r) = {} & b'(t, x, a)y + trace(\sigma'(t, x, a)z) + f(t, x, a) \\ & + \int_{\mathbb{R}^n} \Big[\sum_{j=1}^{l} \big(\sum_{i=1}^{n} \eta_{ij}(t, x, a, z) r_{ij}(t, z) \big) \\ & + \big(\eta'(t, x, a, z)p + x'r(t, z)\big)\big(I - diag(\chi)\big)\Big] d\lambda(z),\end{aligned} \tag{17}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}^{n+1} \to \mathbb{R}^{n\times l}$ such that the integral in (17) converges. Again, we suppose that $\mathcal{H}$ is differentiable w.r.t. x.

Then the corresponding BSDE is of the form

$$\begin{aligned}-dY_t &= \nabla_x \mathcal{H}(t, X_t, \alpha_t, Y_t, Z_t, r(t, \cdot))dt + Z_t dW_t + \int_{\mathbb{R}^n} r(t_-, z)\bar{N}(dt, dz) \\ Y_T &= \nabla_x g(X_T).\end{aligned} \tag{18}$$

The assertion of the stochastic Pontryagin's maximum principle for this Lévy case is analogous to Theorem 4, see [5].

3.3 Infinite time horizon stochastic control problem

When considering infinite time stochastic control problem, it is useful to stress that, in fact, we are looking for a stationary optimal control α^*, that is we do not consider time dependence of functions b, σ and f.

Then the functional to maximize is

$$J(x, \alpha) = \mathbf{E}\Big[\int_0^{+\infty} e^{-\beta s} f(X_s^x, \alpha_s)ds\Big] \tag{19}$$

with the associated cost function

$$v(x) = \sup_{\alpha\in\mathcal{A}(x)} J(x, \alpha). \tag{20}$$

Again, the set of admissible controls $\mathcal{A}(x)$ is such that for all $\alpha \in \mathcal{A}(x)$ there exist a unique solution to (9) and the integral in (19) converges.

The question is, how the generalized Hamiltonian will look like when introducing also jumps in the model (by using Lévy processes) and what assumptions are needed to prove the associated Pontryagin's maximum principle. This is the goal of my current research.

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