Note on Construction of Probabilities on Many-valued Events via Schauder Bases and Inverse Limits

Tomáš Kroupa
Institute of Information Theory and Automation of the ASCR
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
Email: kroupa@utia.cas.cz
Department of Mathematics, Faculty of Electrical Engineering
Czech Technical University in Prague, Technická 2, 166 27 Prague

***Abstract*—Every probability on many-valued events (a state on a finitely-generated free MV-algebras) is uniquely represented by refining finitely-supported probabilities across all Schauder bases. This procedure enables reconstructing the state space as the inverse limit of an inverse system of finite-dimensional simplices.**

I. Introduction

States were introduced by Mundici [1] in order to model the notion of "average truth-value" of formulas in Łukasiewicz logic. Probability on MV-algebras that are algebraic counterparts of Łukasiewicz logic was further developed by Riečan and Mundici in [2], building on the previous work of the Slovak School—see [3] and the references there. It was proved in [4] and independently in [5] that the mathematical properties of states indeed fit the idea of averaging truth-values: namely, every state is the Lebesgue integral of (an equivalence class of) a formula with respect to a uniquely determined Borel probability measure on possible worlds. Panti's proof is even more general in relaxing the semisimplicity of an MV-algebra.

In this paper we pave the way to another proof of the integral representation theorem in a particular case of a free finitely-generated MV-algebra (Theorem 2). States and measures on such MV-algebras are of special importance. In particular, de Finetti coherence characterization of states on many-valued events (represented by formulas in infinite-valued Łukasiewicz propositional logic) is proved in [6]. Measures on free MV-algebras play a crucial role in developing coalition game theory with coalitions modeled by formulas [7]. The approach pursued in this article relies on theory of Schauder hats and bases [8, Section 9.1], which generalize partitions in Boolean algebras. The key fact is that any two Schauder bases admit a joint refinement. Refinement of bases is then used in the "compactness" argument that renders the representing Borel probability measure as the unique measure resulting from the refinement of representing finitely-supported measures across all the Schauder bases. The integral representation of states implies that the state space of a free MV-algebra is a so-called Bauer simplex (Corollary 1), which holds true even for all semisimple MV-algebras (see [4, Theorem 22], [1]). The state space of a free MV-algebra can be viewed as the inverse limit of finite-dimensional simplices (Proposition 4), which are constructed as "state spaces" restricted to Schauder bases. It is worth mentioning that this result follows from a much more general theorem [9, Theorem 12.45(b)], but the proof of the special case presented here is very simple and geometrically appealing. An inessential modification of the proofs in this article makes possible to extend all the results to any finitely-presented MV-algebra.

The article is structured as follows. Section II contains necessary definitions and results concerning Łukasiewicz infinite-valued propositional logic and its associated Lindenbaum algebra L_k of (equivalence classes of) formulas over k propositional variables. Section III contains the above mentioned results.

II. Basic Notions

The aim of this section is to provide a survey of Łukasiewicz infinite-valued propositional logic [8, Chapter 4] and its associated Lindenbaum algebra. Formulas $\varphi, \psi, \ldots$ are constructed from propositional variables $A_1, \ldots, A_k$ by applying the standard rules known in Boolean logic. The connectives are negation, disjunction and conjunction, which are denoted by $\neg$, $\oplus$ and $\odot$, respectively. This is already a complete set of connectives so that, for instance, the implication $\varphi \to \psi$ can be defined as $\neg\varphi \oplus \psi$. The set of all formulas in propositional variables $A_1, \ldots, A_k$ is denoted by $\mathrm{Form}(A_1, \ldots, A_k)$.

Semantics for connectives of Łukasiewicz logic is defined by operations in algebras called MV-algebras [8]. The algebra of truth degrees of Łukasiewicz logic is the *standard MV-algebra*, which is the unit interval $[0,1]$ endowed with the operations $\neg, \oplus, \odot$ defined as follows:

$$\begin{aligned} \neg a &= 1 - a \\ a \oplus b &= \min\{a+b, 1\} \\ a \odot b &= \max\{a+b-1, 0\} \end{aligned}$$

A *valuation* is a mapping $V : \mathrm{Form}(A_1, \ldots A_k) \to [0,1]$ such

0195-623X/10 $26.00 © 2010 IEEE
DOI 10.1109/ISMVL.2010.42

that

$$V(\neg\varphi) = 1 - V(\varphi)$$
$$V(\varphi \oplus \psi) = V(\varphi) \oplus V(\psi)$$
$$V(\varphi \odot \psi) = V(\varphi) \odot V(\psi)$$

Formulas $\varphi, \psi \in \mathrm{Form}(A_1, \ldots, A_k)$ are called *equivalent* when $V(\varphi) = V(\psi)$, for every valuation V. The *equivalence class* of φ is denoted $[\varphi]$. The set of all such equivalence classes is an MV-algebra L_k with the operations

$$\neg[\varphi] = [\neg\varphi]$$
$$[\varphi] \oplus [\psi] = [\varphi \oplus \psi]$$
$$[\varphi] \odot [\psi] = [\varphi \odot \psi]$$

for every $\varphi, \psi \in \mathrm{Form}(A_1, \ldots, A_k)$.

Since every valuation V is uniquely determined by its restriction to the propositional variables

$$V \mapsto V(A_1, \ldots, A_k) \in [0,1]^k,$$

every "possible world" V is matched with a unique point x_V from the k-dimensional unit cube $[0,1]^k$ and vice versa. Let V_x be the valuation corresponding to $x \in [0,1]^k$. Put $[\varphi](x) = V_x(\varphi)$, for every $x \in [0,1]^k$. Hence the equivalence class $[\varphi]$ of every $\varphi \in \mathrm{Form}(A_1, \ldots A_k)$ can be viewed as a function $[0,1]^k \to [0,1]$ and L_k is the algebra of all such functions endowed with the pointwise operations $\neg, \oplus, \odot$.

McNaughton theorem ([10]). *$(L_k, \oplus, \odot, \neg)$ is precisely the MV-algebra of all functions $[0,1]^k \to [0,1]$ that are continuous and piecewise linear, where each linear piece has integer coefficients.*

Let $f \vee g = \neg(\neg f \oplus g) \oplus g$, $f \wedge g = \neg(\neg f \vee \neg g)$. These operations are in fact the pointwise supremum and infimum of functions in L_k, respectively, and they make L_k into a distributive lattice.

Theory of Schauder hats and bases in L_k, which was developed for the purely geometrical proof of McNaughton theorem [8, Section 9.1], is briefly repeated in this paragraph. The basic familiarity with polyhedral geometry and topology is assumed, see [11], [12], for instance. A *polyhedral complex (in $[0,1]^k$)* is a finite set of polyhedra $\mathscr{R}$ such that: (i) each polyhedron of $\mathscr{R}$ is included in $[0,1]^k$ and all its vertices have rational coordinates; (ii) if $P \in \mathscr{R}$ and Q is a face of P, then $Q \in \mathscr{R}$; (iii) if $P, Q \in \mathscr{R}$, then $P \cap Q$ is a face of both P and Q. The set $\bigcup_{P \in \mathscr{R}} P$ is called a *support* of $\mathscr{R}$. When all the polyhedra of a polyhedral complex are simplices, then the polyhedral complex is said to be a *simplicial complex*. Alternatively, a simplicial complex with the support S is called a *triangulation* of S. The *denominator* $\mathrm{den}(q)$ of a point $q \in [0,1]^k$ with rational coordinates $(\frac{r_1}{s_1}, \ldots, \frac{r_k}{s_k})$, where $r_i \geq 0, s_i > 0$ are the uniquely determined relatively prime integers, is the least common multiple of $s_1, \ldots, s_k$. Passing to homogeneous coordinates in $\mathbb{R}^k$, put

$$\tilde{q} = \left(\tfrac{\mathrm{den}(q)}{s_1} r_1, \ldots, \tfrac{\mathrm{den}(q)}{s_k} r_k, \mathrm{den}(q)\right)$$

and note that $\tilde{q} \in \mathbb{Z}^{k+1}$. A k-simplex with vertices $v^0, \ldots, v^k$ is *unimodular* if $\{\tilde{v}^0, \ldots, \tilde{v}^k\}$ is a basis of the free Abelian group $\mathbb{Z}^{k+1}$. An n-simplex with $n < k$ is *unimodular* when it is a face of some unimodular k-simplex. We say that a triangulation Σ is *unimodular* if each simplex of Σ is unimodular. If $\mathscr{R}$ is a polyhedral complex, $\mathrm{V}_{\mathscr{R}}$ denotes the set of all the vertices of $\mathscr{R}$. Let Σ be a unimodular triangulation with the support $S \subseteq [0,1]^k$. For each $x \in \mathrm{V}_\Sigma$, the *Schauder hat (at x over Σ)* is the uniquely determined continuous piecewise linear function $h_x : S \to [0,1]$ which attains the value $\frac{1}{\mathrm{den}(x)}$ at x, vanishes at each vertex from $\mathrm{V}_\Sigma \setminus \{x\}$, and is a linear function on each simplex of Σ. The *basis* H_Σ (over Σ) is the set $\{h_x \mid x \in \mathrm{V}_\Sigma\}$. For each $x \in \mathrm{V}_\Sigma$, the *normalized Schauder hat (at x over Σ)* is the function $\hat{h}_x = \mathrm{den}(x) h_x$. Every (normalized) Schauder hat belongs to L_k. The *normalized basis* $\hat{H}_\Sigma$ (over Σ) is the set $\{\hat{h}_x \mid x \in \mathrm{V}_\Sigma\}$. The following properties of normalized bases follow immediately.

Proposition 1. *If $\hat{H}_\Sigma$ is a normalized basis, then $\hat{h}_x \odot \hat{h}_{x'} = 0$, for each $x, x' \in \mathrm{V}_\Sigma$ with $x \neq x'$, and $\sum_{x \in \mathrm{V}_\Sigma} \hat{h}_x = 1$.*

In the sequel $\mathfrak{T}$ denotes the collection of all unimodular triangulations of $[0,1]^k$. Let $\mathcal{H} = \{H_\Sigma \mid \Sigma \in \mathfrak{T}\}$. For any pair $H_{\Sigma_1}, H_{\Sigma_2} \in \mathcal{H}$, we will say that H_{Σ_2} *refines* H_{Σ_1} and write $H_{\Sigma_1} \preceq H_{\Sigma_2}$, if for each $h \in H_{\Sigma_1}$, there exist nonnegative integers β_x, $x \in \mathrm{V}_{\Sigma_2}$, such that $h = \sum_{x \in \mathrm{V}_{\Sigma_2}} \beta_x h_x$.

Theorem 1 ([13], [14]). *The set $\mathcal{H}$ is an up-directed partially ordered by $\preceq$.*

III. Main Results

States on MV-algebras are many-valued analogues of probabilities on Boolean algebras. The disjointness of functions in L_k is captured by the relation $f \odot g = 0$, for $f, g \in L_k$. This condition is equivalent to $f \oplus g = f + g$.

Definition 1. *A* state *s on L_k is a mapping $s : L_k \to [0,1]$ such that $s(1) = 1$ and $s(f \oplus g) = s(f) + s(g)$, for every $f, g \in L_k$ with $f \odot g = 0$.*

Theorem 2 ([4], [5]). *If s is a state on L_k, then there exists a uniquely determined Borel probability measure μ on $[0,1]^k$ such that $s(f) = \int f \, \mathrm{d}\mu$, for each $f \in L_k$.*

Note that the Borel probability measure μ from Theorem 2 is necessarily regular since $[0,1]^k$ is a compact metric space.

A main aim of this paper is to give an alternative geometrical proof of Theorem 2. For any integer $n \geq 0$ we use the notation

$$\Delta_n = \left\{ a \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} a_i = 1, a_i \geq 0, i = 1, \ldots, n+1 \right\}.$$

Proposition 2. *Let $\hat{H}_\Sigma$ be a normalized basis and*

$$a : \hat{H}_\Sigma \to [0,1]$$

be a function such that $(a(\hat{h}_x))_{x \in \mathrm{V}_\Sigma} \in \Delta_{|\hat{H}_\Sigma|-1}$. Then there exists a (finitely-supported) Borel probability measure δ on $[0,1]^k$ with $a(\hat{h}_x) = \int \hat{h}_x \, \mathrm{d}\delta$, for each $x \in \mathrm{V}_\Sigma$.

Proof: Let δ_x denotes the Dirac measure concentrated at a point $x \in [0,1]^k$. Put

$$\delta = \sum_{x \in \mathrm{V}_\Sigma} a(\hat{h}_x)\delta_x.$$

For each vertex $x \in \mathrm{V}_\Sigma$, we get

$$\int \hat{h}_x \,\mathrm{d}\delta = \sum_{x' \in \mathrm{V}_\Sigma} \int a(\hat{h}_{x'})\hat{h}_x \,\mathrm{d}\delta_{x'} = \sum_{x' \in \mathrm{V}_\Sigma} a(\hat{h}_{x'})\hat{h}_x(x')$$
$$= a(\hat{h}_x)\hat{h}_x(x) = a(\hat{h}_x).$$

∎

Given a Borel probability measure μ on $[0,1]^k$, let $s_\mu(f) = \int f \,\mathrm{d}\mu,\ f \in L_k$.

Proposition 3. *If μ, ν are Borel probability measures on $[0,1]^k$ with $\mu \neq \nu$, then the states s_μ, s_ν on L_k satisfy $s_\mu \neq s_\nu$.*

Proof: Evidently, both functions s_μ, s_ν are states on L_k. By way of contradiction, suppose $s_\mu = s_\nu$. The Borel subsets of $[0,1]^k$ are generated by the collection of all open (in the subspace Euclidean topology of $[0,1]^k$) hyperrectangles with rational vertices: indeed, every open subset of $[0,1]^k$ can be written as a countable union of such rectangles. As a consequence, [15, Theorem 3.3] yields the existence of an open rectangle $R \subseteq [0,1]^k$ with rational vertices and $\mu(R) \neq \nu(R)$.

Let $\mathscr{R}$ be the polyhedral complex consisting of all the faces of the closure $\overline{R}$ of R. Taking an arbitrary point $r \in R$ with rational coordinates, consider the stellar subdivision $\mathscr{R}'$ of $\mathscr{R}$ (see [12, p.15]). The polyhedral complex $\mathscr{R}'$ can be triangulated without introducing any new vertices [12, Proposition 2.9]. In turn, the resulting simplicial complex can be subdivided into a unimodular triangulation Σ of $\overline{R}$ with a possible introduction of new vertices (see [6, Claim 2], for example).

For each $v \in \mathrm{V}_\Sigma \cap R$, let h_v be the Schauder hat at v over Σ, and define a function $f_v : [0,1]^k \to [0,1]$ by

$$f_v(x) = \begin{cases} h_v(x), & x \in \overline{R}, \\ 0, & \text{otherwise.} \end{cases}$$

If $f = \bigoplus_{v \in \mathrm{V}_\Sigma \cap R} f_v$, then it follows directly from unimodularity of Σ and the definition of f_v that $f \in L_k$. In particular, note that $f(x)$ vanishes iff $x \in [0,1]^k \setminus R$ and thus

$$\sup_{m \in \mathbb{N}} \bigoplus_{i=1}^{m} f = \chi_R, \tag{1}$$

where χ_R is the characteristic function of R. For every $m \in \mathbb{N}$, the function $\bigoplus_{i=1}^m f$ is a k-variable McNaughton function, and (1) together with the Lebesgue's dominated convergence theorem leads to the equality

$$\mu(R) = \sup_{m \in \mathbb{N}} \int \bigoplus_{i=1}^{m} f \,\mathrm{d}\mu = \sup_{m \in \mathbb{N}} s_\mu\Big(\bigoplus_{i=1}^{m} f\Big) = \sup_{m \in \mathbb{N}} s_\nu\Big(\bigoplus_{i=1}^{m} f\Big)$$
$$= \sup_{m \in \mathbb{N}} \int \bigoplus_{i=1}^{m} f d\nu = \nu(R),$$

which is a contradiction. ∎

By $\mathcal{M}^1$ we denote the convex set of all Borel probability measures on $[0,1]^k$, which is a compact metric space in the subspace w^*-topology of $C^*([0,1]^k$ (see [16]). For every sequence (μ_n) in $\mathcal{M}^1$,

$$\mu_n \longrightarrow \mu \text{ iff } \textstyle\int f d\mu_n \longrightarrow \int f d\mu,$$

for every continuous function $f : [0,1]^k \to \mathbb{R}$.

Proof of Theorem 2: Let $\hat{H}_\Sigma$ be a normalized basis. Put

$$\mathcal{M}_\Sigma = \{\mu \mid s(\hat{h}_x) = \textstyle\int \hat{h}_x d\mu, \text{ for each } x \in \mathrm{V}_\Sigma\}$$

and note that $\mathcal{M}_\Sigma \neq \emptyset$ by Proposition 2. It follows directly from the definition of topology on $\mathcal{M}^1$ that $\mathcal{M}_\Sigma$ is closed. We are going to show that $\bigcap_{\Sigma \in \mathfrak{T}} \mathcal{M}_\Sigma \neq \emptyset$. The compactness of $\mathcal{M}^1$ means that it suffices to prove $\bigcap_{\Sigma' \in \mathfrak{T}'} \mathcal{M}_{\Sigma'} \neq \emptyset$ for every finite subset $\mathfrak{T}' \subseteq \mathfrak{T}$. Due to Theorem 1, there exists a basis H_Σ with $H_{\Sigma'} \preceq H_\Sigma$, for every $\Sigma' \in \mathfrak{T}'$. This means that for each normalized hat $\hat{h}_x \in \hat{H}_{\Sigma'}$, where $\Sigma' \in \mathfrak{T}'$ and $x \in \mathrm{V}_{\Sigma'}$, there exist (uniquely determined) nonnegative integers $(\beta_y)_{y \in \mathrm{V}_\Sigma}$ such that $\hat{h}_x = \sum_{y \in \mathrm{V}_\Sigma} \beta_y h_y$. Let $\delta = \sum_{y \in \mathrm{V}_\Sigma} s(\hat{h}_y)\delta_y$. Linearity of Lebesgue integral gives

$$\int \hat{h}_x \mathrm{d}\delta = \sum_{y' \in \mathrm{V}_\Sigma} s(\hat{h}_{y'}) \int \sum_{y \in \mathrm{V}_\Sigma} \frac{\beta_y}{\mathrm{den}(y)} \hat{h}_y \,\mathrm{d}\delta_{y'}. \tag{2}$$

Due to additivity of states and since for every $y \in \mathrm{V}_\Sigma$,

$$\frac{s(\hat{h}_y)}{\mathrm{den}(y)} = \frac{s(\mathrm{den}(y)h_y)}{\mathrm{den}(y)} = s(h_y),$$

the right-hand side of (2) can be expressed as

$$\sum_{y' \in \mathrm{V}_\Sigma} s(\hat{h}_{y'}) \sum_{y \in \mathrm{V}_\Sigma} \frac{\beta_y}{\mathrm{den}(y)} \hat{h}_y(y')$$
$$\sum_{y' \in \mathrm{V}_\Sigma} s(\hat{h}_{y'}) \frac{\beta_{y'}}{\mathrm{den}(y')} = \sum_{y' \in \mathrm{V}_\Sigma} s\Big(\frac{\beta_{y'}}{\mathrm{den}(y')} \hat{h}_{y'}\Big)$$
$$s\Big(\sum_{y \in \mathrm{V}_\Sigma} \beta_y h_y\Big) = s(\hat{h}_x).$$

Thus $\delta \in \mathcal{M}_{\Sigma'}$ for each $\Sigma' \in \mathfrak{T}'$, which leads to the conclusion $\bigcap_{\Sigma \in \mathfrak{T}} \mathcal{M}_\Sigma \neq \emptyset$. Every probability measure $\mu \in \bigcap_{\Sigma \in \mathfrak{T}} \mathcal{M}_\Sigma$ represents the state s. Indeed, given a McNaughton function $f \in L_k$, find $\Sigma^* \in \mathfrak{T}$ and the basis H_{Σ^*} such that $f = \sum_{x \in \mathrm{V}_{\Sigma^*}} a_x h_x$, for uniquely determined nonnegative integers a_x [8, Theorem 9.1.5]. It results that

$$s(f) = s\Big(\sum_{x \in \mathrm{V}_{\Sigma^*}} a_x h_x\Big) = \sum_{x \in \mathrm{V}_{\Sigma^*}} a_x s(h_x)$$
$$= \sum_{x \in \mathrm{V}_{\Sigma^*}} a_x \int h_x \,\mathrm{d}\mu = \int \sum_{x \in \mathrm{V}_{\Sigma^*}} a_x h_x \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu.$$

The set $\bigcap_{\Sigma \in \mathfrak{T}} \mathcal{M}_\Sigma$ is a singleton by Proposition 3. ∎

The integral representation enables us to easily derive a characterization of the state space $S(L_k)$ of L_k. A *Bauer simplex* [17] in a locally convex space is its nonempty compact convex subset K, which is affinely isomorphic to a base for a lattice cone in some linear space, and whose extreme boundary is closed. Bauer simplices are the compact convex sets in an infinite-dimensional space that are closest to finite-dimensional simplices.

Corollary 1. *The set of all states $S(L_k)$ on L_k is a metrizable Bauer simplex.*

Proof: It suffices to establish the existence of an affine homeomorphism $\mathcal{M}^1 \to S(L_k)$, since $\mathcal{M}^1$ is known to be a metrizable Bauer simplex. Consider the mapping $\mu \mapsto s_\mu$. This mapping is continuous, it is one-to-one by Proposition 3 together with Theorem 2, and it is affine by linearity of Lebesgue integral. Thus $\mu \mapsto s_\mu$ is the affine homeomorphism. ■

Let $s|_{\hat{H}_\Sigma}$ be the restriction of $s \in S(L_k)$ to $\hat{H}_\Sigma$, and $S(\hat{H}_\Sigma) = \{s|_{\hat{H}_\Sigma} \mid s \in S(L_k)\}$, for any $\Sigma \in \mathfrak{X}$. Given $\Sigma, \Sigma' \in \mathfrak{X}$ with $H_\Sigma \preceq H_{\Sigma'}$, define $p_{\Sigma\Sigma'} : S(\hat{H}_{\Sigma'}) \to S(\hat{H}_\Sigma)$ by

$$p_{\Sigma\Sigma'}\big(s|_{\hat{H}_{\Sigma'}}\big) = s|_{\hat{H}_\Sigma}, \quad s|_{\hat{H}_{\Sigma'}} \in S(\hat{H}_{\Sigma'}).$$

Proposition 4. *The family $\big(S(\hat{H}_\Sigma), p_{\Sigma\Sigma'}\big)_{\Sigma,\Sigma'\in\mathfrak{X}}$ is an inverse system of finite-dimensional simplices and*

$$\varprojlim \big(S(\hat{H}_\Sigma), p_{\Sigma\Sigma'}\big)_{\Sigma,\Sigma'\in\mathfrak{X}} = S(L_k).$$

Proof: For every $\Sigma \in \mathfrak{X}$, the set $S(\hat{H}_\Sigma)$ is a finite-dimensional simplex since it is affinely homeomorphic to $\Delta_{|\hat{H}_\Sigma|-1}$. Indeed, consider a mapping

$$a : s|_{\hat{H}_\Sigma} \mapsto (s(\hat{h}_x))_{x\in \mathrm{V}_\Sigma}.$$

It can be easily checked that a is into $\Delta_{|\hat{H}_\Sigma|-1}$, affine, continuous, injective, and surjective by Proposition 2.

The family $\big(S(\hat{H}_\Sigma), p_{\Sigma\Sigma'}\big)_{\Sigma,\Sigma'\in\mathfrak{X}}$ is an inverse system in the category of compact convex sets. Precisely, $\mathfrak{X}$ is up-directed by the reversed inclusion ($\Sigma \supseteq \Sigma'$ iff $H_\Sigma \preceq H_{\Sigma'}$), the mapping $p_{\Sigma\Sigma'}$ is a continuous affine surjection for every $\Sigma, \Sigma' \in \mathfrak{X}$ with $H_\Sigma \preceq H_{\Sigma'}$, it is identity for $\Sigma = \Sigma'$, and $p_{\Sigma\Sigma'} \circ p_{\Sigma'\Sigma''} = p_{\Sigma\Sigma''}$, whenever $H_\Sigma \preceq H_{\Sigma'} \preceq H_{\Sigma''}$.

It remains to show that $S(L_k)$ is affinely homeomorphic to the inverse limit $\varprojlim\big(S(\hat{H}_\Sigma), p_{\Sigma\Sigma'}\big)_{\Sigma,\Sigma'\in\mathfrak{X}}$ that is equal to

$$\left\{ \big(s|_{\hat{H}_\Sigma}\big)_{\Sigma\in\mathfrak{X}} \in \prod_{\Sigma\in\mathfrak{X}} S(\hat{H}_\Sigma) \mid p_{\Sigma\Sigma'}(s|_{\hat{H}_{\Sigma'}}) = s|_{\hat{H}_\Sigma}, H_\Sigma \preceq H_{\Sigma'} \right\}.$$

The routine verification yields that the mapping

$$s \in S(L_k) \mapsto \big(s|_{\hat{H}_\Sigma}\big)_{\Sigma\in\mathfrak{X}}$$

is the sought affine homeomorphism. ■

The author is indebted to one of the reviewers for the MV-algebraic interpretation of the limit construction appearing in Proposition 4. Namely, consider a finite quotient F_Σ of L_k given by $F_\Sigma = \{f|_{\mathrm{V}_\Sigma} \mid f \in L_k\}$, where $\Sigma \in \mathfrak{X}$ (see [8, Chapter 3] for details on ideals in free MV-algebras). The state space of each F_Σ is precisely the finite-dimensional simplex $\Delta_{|\hat{H}_\Sigma|-1}$. Since L_k is in fact an inverse limit of all the F_Σ's, Proposition 4 then shows that the state space $S(L_k)$ is the inverse limit of the state spaces of the finite quotients F_Σ.

Acknowledgement

The influence of Pták's paper [18] on the methods used by the author in this article cannot be overestimated.

The work of the author was supported by the grant GA ČR 201/09/1891 and by the grant No.1M0572 of the Ministry of Education, Youth and Sports of the Czech Republic.

References

[1] D. Mundici, "Averaging the truth-value in Łukasiewicz logic," *Studia Logica*, vol. 55, no. 1, pp. 113–127, 1995.

[2] B. Riečan and D. Mundici, "Probability on MV-algebras," in *Handbook of Measure Theory, Vol. I, II*. Amsterdam: North-Holland, 2002, pp. 869–909.

[3] B. Riečan and T. Neubrunn, *Integral, measure, and ordering*, ser. Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers, 1997, vol. 411.

[4] T. Kroupa, "Every state on semisimple MV-algebra is integral," *Fuzzy Sets and Systems*, vol. 157, no. 20, pp. 2771–2782, 2006.

[5] G. Panti, "Invariant measures in free MV-algebras," *Communications in Algebra*, vol. 36, no. 8, pp. 2849–2861, 2008.

[6] D. Mundici, "Bookmaking over infinite-valued events," *Internat. J. Approx. Reason.*, vol. 43, no. 3, pp. 223–240, 2006.

[7] T. Kroupa, "Core of coalition games on mv-algebras," *Journal of Logic and Computation*, 2009, doi: 10.1093/logcom/exp015.

[8] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici, *Algebraic foundations of many-valued reasoning*, ser. Trends in Logic—Studia Logica Library. Dordrecht: Kluwer Academic Publishers, 2000, vol. 7.

[9] J. Lukeš, J. Malý, I. Netuka, and J. Spurný, *Integral Representation Theory (Applications to Convexity, Banach Spaces and Potential Theory)*, ser. de Gruyter Studies in Mathematics 35. de Gruyter, 2009.

[10] R. McNaughton, "A theorem about infinite-valued sentential logic," *J. Symbolic Logic*, vol. 16, pp. 1–13, 1951.

[11] G. Ziegler, *Lectures on polytopes*, ser. Graduate Texts in Mathematics. New York: Springer-Verlag, 1995, vol. 152.

[12] C. P. Rourke and B. J. Sanderson, *Introduction to piecewise-linear topology*, ser. Springer Study Edition. Berlin: Springer-Verlag, 1982.

[13] C. Manara, "Relating the theory of non-boolean partitions to the design of interpretable fuzzy systems," Ph.D. dissertation, Università degli studi di Milano, 2003.

[14] D. Mundici, "Farey stellar subdivisions, ultrasimplicial groups and K_0 of AF $\mathcal{C}^*$ algebras," *Advances in Mathematics*, vol. 68, pp. 23–39, 1988.

[15] P. Billingsley, *Probability and measure*, 3rd ed., ser. Wiley Series in Probability and Mathematical Statistics. New York: John Wiley & Sons Inc., 1995, a Wiley-Interscience Publication.

[16] W. Rudin, *Functional analysis*. New York: McGraw-Hill Book Co., 1973, mcGraw-Hill Series in Higher Mathematics.

[17] E. M. Alfsen, *Compact convex sets and boundary integrals*. New York: Springer-Verlag, 1971, ergebnisse der Mathematik und ihrer Grenzgebiete, Band 57.

[18] P. Pták, "On extension of states on logics," *Bull. Polish Acad. Sciences-Mathematics*, vol. 33, no. 9-10, pp. 493–497, 1985.