4. Combine numerical relaxation with a finite element minimization of the total energy in the system. The ultimate challenge is to determine the effective energy of the variational problem: Minimize

$$\mathcal{F}(u) = \int_\Omega W(Du)\,\mathrm{d}x$$

from a joint relaxation and minimization in the sense that one minimizes

$$\mathcal{F}_h(u) = \int_\Omega W_h^{\mathrm{rc}}(DU)\,\mathrm{d}x$$

where $U \in S^1$ is a finite element function (e.g. continuous and affine on the elements of a regular triangulation).

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Rate-independent damage at large strains

Tomáš Roubíček

(joint work with A. Mielke)

We consider damage in the context of nonlinear elasticity at large strains, which is certainly a relevant concept especially because damaged materials may allow indeed for very large deformations. On the other hand, only materials with quasiconvex stored energy of a polynomial growth $p > 3$, as Ogden's type materials, are analyzed. Moreover, we consider damage as a *rate-independent* process, as standardly applied to concrete, filled polymers, or filled rubbers. Being rate-independent, it is necessarily an *activated* process, i.e. to trigger a damage the mechanical stress must achieve a certain activation threshold. We consider the *isotropic damage* that can be described by a scalar parameter $z \in [0, 1]$ and neglect any other rate dependent processes like viscosity and inertia. In accord to some engineering literature and for mathematical reasons, our model involves also the *gradient of damage*, expressing certain nonlocality in the sense that damage of a particular spot is to some extent influenced by its surrounding.

At a fixed time, the state of the system is considered as $q = (u, \zeta)$ where $u : \Omega \to \mathbb{R}^3$ is the *deformation* considered on the reference body configuration

$\Omega \subset \mathbb{R}^3$, and $\zeta : \Omega \to [0,1]$ is a distribution of *damage*; $\zeta(x) = 1$ means 100% quality of the material, 0 means that the material is completely damaged at the current point $x \in \Omega$, and $0 < \zeta(x) < 1$ means that some portion of material is already damaged due to, e.g., microcracks or microvoids.

The *stored energy* density $\varphi(x, F, z)$ is then a function of deformation gradient $F = \nabla u$ and the damage variable z:

$$\varphi(F, z) := \varphi_0(F) + z\varphi_1(F). \tag{1}$$

Dissipative mechanisms are routinely described by a (pseudo)*potential of dissipative forces*, here denoted by R, as a function of the rate of $q = q(t)$. The only dissipation of energy we consider is due to the damage and, on the microscopical level, it is related with irreversible structural changes of the material starting with microcracks and ending by its complete disintegration. We describe it by a single phenomenological parameter $d > 0$ having the meaning of a specific energy (per volume, i.e. in physical units Jm^{-3} =Pa) needed for complete damage of the unit volume of the material, i.e. the energy needed to switch $\zeta(x)$ from 1 to 0.

The classical formulation of the quasi-static problem consists in the balance of Piola-Kirchoff stress and the activated evolution of the damage parameter described by a complementarity problem:

$$-\mathrm{div}\big(\varphi_0'(\nabla u) + \zeta\varphi_1'(\nabla u)\big) = 0, \tag{2a}$$

$$\frac{\partial \zeta}{\partial t} \leq 0, \tag{2b}$$

$$\zeta\varphi_1(\nabla u) - r_\zeta \leq d + \kappa\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big), \tag{2c}$$

$$\frac{\partial \zeta}{\partial t}\Big(d - \zeta\varphi_1(\nabla u) + \kappa\,\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) + r_\zeta\Big) = 0 \tag{2d}$$

on the reference domain Ω, here $\kappa > 0$ is a so-called factor of influence of damage and $r > 3$, and $r_\zeta \in \partial\chi_{[0,+\infty)}(\zeta)$ is an additional force balancing the natural constraint $\zeta \geq 0$; the notation $\chi_{[0,+\infty)}$ stands for the indicator function of $[0, +\infty)$.

This system is completed by time-dependent hard-device loading, i.e. time-dependent Dirichlet boundary conditions $u|_\Gamma = w_{\mathrm{D}}(t)$ are prescribed on some part Γ of the boundary $\partial\Omega$ while zero normal stress is considered on the rest. Due to the damage gradient term, some boundary conditions (here of Neumann's type) should be considered also for ζ.

The energetics involves the overall Gibbs' *stored energy*

$$G(t, u, \zeta) := \begin{cases} \displaystyle\int_\Omega \varphi\big(\nabla u(x), \zeta(x)\big) + \frac{\kappa}{r}|\nabla\zeta(x)|^r \,\mathrm{d}x & \text{if } u|_\Gamma = w_{\mathrm{D}}(t),\ \zeta \geq 0 \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases} \tag{3}$$

and the *dissipation* rate

$$R(\dot q) := \int_\Omega \varrho(\dot\zeta(x))\,\mathrm{d}x \quad \text{where} \quad \varrho(\dot z) := \begin{cases} -d\dot z & \text{if } \dot z \leq 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

here $\dot{q} = (\dot{y}; \dot{\zeta})$ stands for the rate of q. The *energetic solution* $q : [0,T] \to Q := W^{1,p}(\Omega;\mathbb{R}^3) \times L^1(\Omega)$ to (2) on a fixed time interval $[0,T]$ is required to satisfy the *stability* condition

$$\forall \tilde{q} \in Q: \qquad G\big(t, q(t)\big) \le G(t, \tilde{q}) + R\big(\tilde{q} - q(t)\big), \tag{5}$$

for all $0 \le t \le T$, and the *energy equality*

$$G(t,q(t)) + \mathrm{Var}_R(q; s,t) = G(s,q(s)) + \int_s^t P(\theta, q(\theta))\,\mathrm{d}\theta \tag{6}$$

$$\text{with} \quad P(t,q) \equiv P(t,u,\zeta) := \int_\Omega \varphi_F'\big(\nabla u(x), \zeta(x)\big) : \nabla \frac{\partial u_{\mathrm{D}}}{\partial t}(t,x)\,\mathrm{d}x.$$

for any $0 \le s < t \le T$ where the total variation $\mathrm{Var}_R(q;s,t) := \sup \sum_{i=1}^{j} R(q(t_i) - q(t_{i-1}))$ with the supremum taken over all $j \in \mathbb{N}$ and over all partitions of $[s,t]$ in the form $s = t_0 < t_1 < \ldots < t_{j-1} < t_j = t$, and eventually q is required also to satisfy a prescribed *initial condition* $q(0) = 0$.

Main assumptions are p-polynomial coercivity and growth both for φ_0 and φ_1 which are to be polyconvex, the p-growth for φ_0' and φ_1', and qualification of the Dirichlet loading $w_{\mathrm{D}} \in W^{1,1}(0,T; W^{1,\infty}(\Omega;\mathbb{R}^3))$. The coercivity of φ_0 means that only an incomplete damage is considered now.

Existence of an energetic solution $q \in \mathrm{B}([0,T]; W^{1,p}(\Omega;\mathbb{R}^3)) \times (\mathrm{BV}([0,T]; L^1(\Omega)) \cap L^\infty(W^{1,r}(\Omega)))$ with "B$(\cdot)$" and "BV$(\cdot)$"denoting the spaces of bounded and bounded-variation functions, respectively, is proved by a convergence of approximate solutions q_τ with $q_\tau|_{(\tau(k-1),\tau k]} = q_\tau^k$ solving the following recursive minimization problem

$$\left\{ \begin{array}{ll} \text{Minimize} & G(t_\tau^k, q) + R(q - q_\tau^{k-1}) \\ \text{subject to} & q \equiv (u,\zeta) \in Q; \end{array} \right. \tag{7}$$

existence of q_τ^k is by the direct method. Of course, we put $q_\tau^0 = q_0$ a given initial condition. This suggests, after a further spatial discretization, a constructive computational strategy.

A-priori estimates that can be obtained are the following:

$$\big\| u_\tau \big\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \le C_1, \quad \text{and} \tag{8a}$$

$$\big\| \zeta_\tau \big\|_{\mathrm{BV}([0,T];L^1(\Omega)) \,\cap\, L^\infty(0,T;W^{1,r}(\Omega))} \le C_2, \tag{8b}$$

$$\big\| t \mapsto G_\tau(t, q_\tau(t)) \big\|_{\mathrm{BV}([0,T])} \le C_3. \tag{8c}$$

with G_τ defined like in (3) but with a piecewise constant approximation of w_{D}. Moreover, a discrete stability and two-sided energy estimate can be derived.

Convergence can then be shown by the methodology developed in [1, 2], i.e. selecting a subsequence converging weakly* in the topologies indicated in (8) and, by Banach-space-valued Helly's selection principle, even pointwise in time for all quantities under the BV-estimates in (8). Then a limit passage in the discrete stability and two-sided energy estimate goes through, using various sophisticated techniques, e.g. Tikhonov's (non-sequential) compactness of a product of a countable number of copies of a (weakly compact) ball in $W^{1,p}(\Omega;\mathbb{R}^3)$ or an approximation of Lebesgue integrals by Riemann's sums.

The contribution is based on [3] where several generalizations are considered: ζ may act nonlinearly in (1), beside the hard-loading device also a prescribed-trajectory impact of an ideally rigid body is considered, and eventually some ideas are outlined for a complete damage.

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Analysis of damage models

Gilles A. Francfort

Rate independence is a shared feature of many constitutive behaviors for solids, from brittle fracture, to associated elasto-plasticity, damage or phase transformation. The material is assumed to be described by a free energy and a dissipation potential. In the case of brittle damage, the free energy $W(F,z)$ is a function whose first entry is the gradient of the deformation u, a $\mathbb{R}^N$-valued vector, and whose second entry is an internal variable (in $[0,1]$) that measures the state of damage in the material, so that $W \searrow$ with z. The dissipation potential $\mathcal{D}$, associated to the rate of change of z, is chosen such that $\mathcal{D}(\dot z(t)) \geq 0$, $\mathcal{D}$ is convex with $\mathcal{D}(0)=0$. This ensures the positivity of the mechanical dissipation. In all that follows we take

$$\mathcal{D}(s) = \begin{cases} ks, & s \geq 0 \\ \infty, & \text{else}, \end{cases}$$

the last condition translating the irreversibility of the process.

Consider a domain $\Omega \in \mathbb{R}^N$, occupied by such a material, clamped throughout its boundary, and submitted to, say, time dependent body loads $f(t)$. If we assume that inertia is negligible, then the material will follow a quasi-static evolution.