Complete Fast Analytical Solution of the Optimal Odd Single-Phase Multilevel Problem

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***Abstract*—In this paper, we focus on the computation of optimal switching angles for general multilevel (ML) odd symmetry waveforms. We show that this problem is similar to (but more general than) the optimal pulsewidth modulation (PWM) problem, which is an established method of generating PWM waveforms with low baseband distortion. We introduce a new general modulation strategy for ML inverters, which takes an analytic form and is very fast, with a complexity of only $\mathcal{O}(n \log^2 n)$ arithmetic operations, where n is the number of controlled harmonics. This algorithm is based on a transformation of appropriate trigonometric equations for each controlled harmonics to a polynomial system of equations that is further transformed to a special system of composite sum of powers. The solution of this system is carried out by a modification of the Newton's identity via Padé approximation, formal orthogonal polynomials (FOPs) theory, and properties of symmetric polynomials. Finally, the optimal switching sequence is obtained by computing zeros of two FOP polynomials in one variable or, alternatively, by a special recurrence formula and eigenvalues computation.**

***Index Terms*—Composite sum of powers, formal orthogonal polynomials (FOPs), multilevel (ML) inverters, Newton's identities, optimal pulsewidth modulation (PWM) problem, Padé approximation, polynomial methods, selected harmonics elimination.**

I. Introduction

The optimal multilevel (ML) or pulsewidth modulation (PWM) problem, sometimes called the selected harmonic elimination (SHE) problem, is an established method for generating ML waveforms with low baseband distortion. The principal problem is to determine the switching times (angles) to produce the baseband and to not generate specific higher order harmonics. This way, it is possible to separate the undesirable highest harmonics.

The optimal ML problem offers several advantages compared to traditional modulation methods [1]–[4]. This approach allows better performance with low switching frequency, direct control over output waveform harmonics, and the ability to leave untouched harmonics divisible by three for three-phase systems.

Up to now, a lot of different perspectives were proposed. All the methods assume quarter symmetry, and all formulations result in the Fourier series representation for different waveforms. The principal problem lies in solving a multivariate trigonometric system of equations or, after substitution for Chebyshev polynomials, in solving a multivariate polynomial system of equations. There are several techniques of how to solve them.

The most effective method for single-phase quarter-symmetric inverter is described in [5]–[7]. This method is based on trigonometric identity for cosine function where the original trigonometric system is transformed to a polynomial system of specific structure leading to the polynomial system of sum of odd powers. The problem results in the construction of a special set of one variable polynomials and computation of their zeros. These polynomials are formal orthogonal (FOPs), and a recurrence formula is derived for them. The solution is based on diagonal Padé approximation. In the case of single-phase inverter for a given modulation index,[1] one or no solution exists. An exact algorithm with a small complexity $\mathcal{O}(n \log^2 n)$ was found. The main result of this paper is, in fact, a generalization of this work for general odd symmetry ML waveforms.

Three-phase inverter systems pose a very interesting topic with many industrial applications. In the three-phase connection, all harmonics divisible by three are ignored as they are automatically canceled in the electric system. This is a more complicated problem because a special structure of the system of equations is damaged. One unique, several different, or no solution exists for a given modulation index. From these, only one solution is selected—the one that minimizes other undesirable and uncontrolled higher harmonics. For more details, see [8]–[11]. These papers also rely on the conversion to a system of polynomials using trigonometric identities. This system of polynomials is solved by the Gröbner basis theory or by the elimination method based on computation of resultants [10]. In addition, a substitution for elementary symmetric polynomials or power sums is applied in [9] and [10]. Applicability of this method is restricted to say five odd harmonics because an appropriate system for a higher number of eliminated harmonics is too large, and its solution is extraordinarily time consuming. Nevertheless, fast analytical methods similar to the algorithms

Manuscript received April 24, 2009; revised July 20, 2009 and September 10, 2009; accepted October 5, 2009. Date of publication October 20, 2009; date of current version June 11, 2010. The work of P. Kujan was supported by the Grant Agency of the Czech Republic under Grant P103/10/P323. The work of M. Hromčík was supported by the Ministry of Education of the Czech Republic under Project 1M0567. The work of M. Šebek was supported by the Grant Agency of the Czech Republic under Grant 102/08/0186.

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Digital Object Identifier 10.1109/TIE.2009.2034677

[1]This is basically the ratio of the first harmonic to the amplitude of one level of an ML waveform.

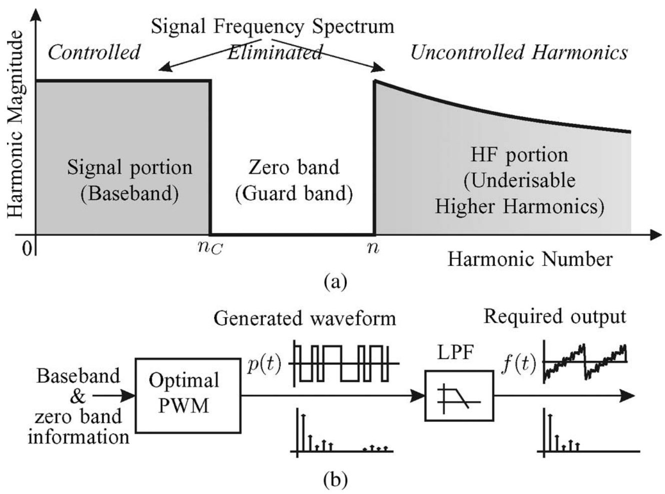

Fig. 1. (a) Frequency spectrum of a separated baseband signal. The baseband can be recovered by an LPF. (b) Principal scheme for the optimal PWM or ML problem.

for single-phase systems presented further in this paper seem to appear soon (see [12] for some first results).

Other methods presented in the literature dealing with the system of polynomial equations are numerical iterative routines [13], genetic algorithms [14], optimization theory [15]–[18], homotopy and continuation [19], or a predictive control algorithm [20].

Applications of the optimal ML or PWM problem cover the control of large electric drives, power electronics converters, active harmonic filters, control of (micro) electromechanical systems, or digital audio amplifier. Implementation of fast and efficient algorithms proposed in this paper on dedicated hardware, e.g., digital signal processors, opens a possibility of a more effective on-the-fly realizations and more accurate and faster solutions. It can result in increasing fuel or power efficiency and better performance (see [21]).

II. Optimal ML Problem

A key issue in the optimal ML problem is the determination of the switching times (angles) to produce the signal portion (baseband) and to not generate specific higher order harmonics (guard band or zero band). This spectral gap separates the baseband, which has to be identical to the required output waveform, from an uncontrolled higher frequency portion. The required output signal can be recovered by means of an analog low-pass filter (LPF) with a cutoff frequency in the guard band. The procedure is depicted in Fig. 1.

Methods described in this section are based on exploiting appropriate trigonometric transcendental equations that define the harmonic content of the generated periodic ML waveform $p(t)$, which is equal to the required finite frequency spectrum of $f(t)$. The main problem lies in solving these systems of equations.

The solution of the optimal ML problem is a sequence of switching times $\overline{\alpha}^\star = (\alpha_1, \ldots, \alpha_n)$. This sequence is obtained from the solution of the following system of equations:

$$a_{p_0}(\overline{\alpha}) = a_{f_0} \tag{1a}$$

$$\left.\begin{array}{l} a_{p_k}(\overline{\alpha}) = a_{f_k} \\ b_{p_k}(\overline{\alpha}) = b_{f_k} \end{array}\right\} \text{ for all } \quad k \in \mathcal{H}_C \tag{1b}$$

$$\left.\begin{array}{l} a_{p_k}(\overline{\alpha}) = 0 \\ b_{p_k}(\overline{\alpha}) = 0 \end{array}\right\} \text{ for all } \quad k \in \mathcal{H}_E \tag{1c}$$

$$\text{subject to} \quad 0 < \alpha_i < T \tag{1d}$$

where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are unknown variables, a_{p_0} and a_{p_k}, b_{p_k} are the zeroth and kth cosine and sine Fourier coefficients of the generated waveform $p(t)$, respectively, and a_{f_0} and a_{f_k}, b_{f_k} are the zeroth and kth cosine and sine Fourier coefficients of the required output waveform $f(t)$. $\mathcal{H}_C$ is the set of controlled harmonics, and the number of elements is n_C. $\mathcal{H}_E$ is the set of eliminated harmonics, and the number of elements is n_E. The number of equations is $n = 1 + 2(n_C + n_E)$.

If only one solution $\overline{\alpha}$ of (1) exists, then it is the optimal solution, and $\overline{\alpha}^\star = \overline{\alpha}$. If the solutions of (1) are $\overline{\alpha}_1, \ldots, \overline{\alpha}_m$, $m > 1$, then the optimal solution $\overline{\alpha}^\star$ is chosen as the minimizer of the total harmonic distortion (THD), i.e.,

$$\overline{\alpha}^\star = \arg \min_{\overline{\alpha} = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_m\}} \text{THD}(\overline{\alpha}) \tag{2}$$

where

$$\text{THD}(\overline{\alpha}) \text{ (in percent)} = 100 \sqrt{\frac{\sum_{i=n_c+1}^{n+N} \left(\frac{a_{p_i}(\overline{\alpha}) + b_{p_i}(\overline{\alpha})}{i}\right)^2}{\sum_{i=1}^{n_c} \left(\frac{a_{p_i}(\overline{\alpha}) + b_{p_i}(\overline{\alpha})}{i}\right)^2}}. \tag{3}$$

If no solution of (1) is found, then the optimal solution $\overline{\alpha}^\star$ is computed as a general minimization problem, i.e.,

$$\overline{\alpha}^\star = \arg \min_{\overline{\alpha}} \sqrt{\sum_{k \in \mathcal{H}_E} \left(a_{p_k}(\overline{\alpha}) + b_{p_k}(\overline{\alpha})\right)^2}$$

$$\text{subject to} \quad \text{(1a) and (1b).} \tag{4}$$

In the rest of this paper, we focus on single-phase odd ML and bilevel PWM waveforms, which lead to a special structure of (1), with only one solution satisfying the condition (1d). The solution of (1) is then found by an analytical procedure.

III. Switching Waveforms

We will show by analysis of different ML waveforms (general, odd, even, half-wave, quarter-wave and bilevel, three-level) that an effective (analytical) solution is possible for waveforms with odd and quarter-wave symmetry only. The Fourier series of these waveforms are odd and, therefore, contain sine coefficients only (the zeroth harmonic and cosine coefficients are equal to zero). The sine and cosine Fourier coefficients are included in other cases, and therefore, it is not possible to make simplifying arrangements for an effective solution.

The optimal PWM problem for a quarter-symmetric three-level inverter is solved in [5] and [6]. This waveform generates only odd sine harmonics, and only the first harmonic is controlled. In this paper, we present solutions for more general odd ML waveforms, which generate all (odd as well as even)

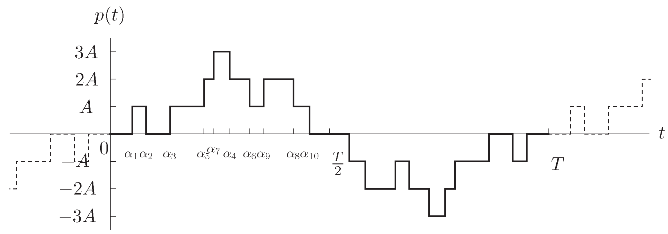

Fig. 2. General odd multilevel (seven-level) waveform.

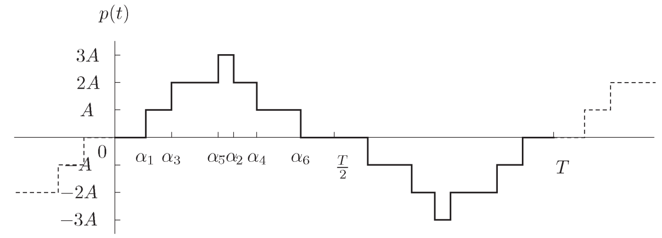

Fig. 3. Odd proper multilevel waveform.

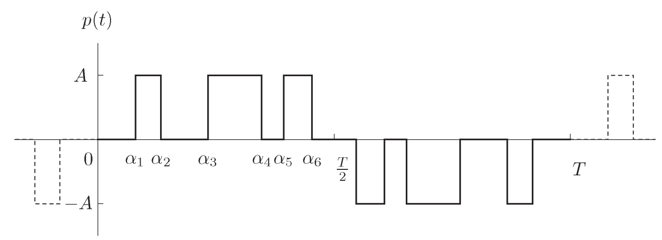

Fig. 4. Odd proper three-level waveform.

sine harmonics. Therefore, our approach covers the solution of the quarter-symmetric PWM and ML problem, and it is more general. Furthermore, the first few n_c harmonics are controlled.

A. General Odd ML Waveform

The Fourier series of a T periodic general odd ML waveform $p(t)$ with amplitude A (see Fig. 2) is sine, i.e.,

$$p(t) \sim \sum_{k=1}^{\infty} b_k \sin \omega k t \tag{5}$$

where

$$b_k = \frac{2A}{k\pi}\left((-1)^{k+1} o_n - \sum_{i=1}^{n} (-1)^i \cos \omega k \alpha_i\right), \quad k = 1, 2, 3, \ldots. \tag{6}$$

The unknown switching times $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are subject to $0 < \alpha_1 < \alpha_3 < \cdots < \alpha_{2\lceil n/2 \rceil - 1} < T/2$ ($\lceil n/2 \rceil$ rising edges) and $0 < \alpha_2 < \alpha_4 < \cdots < \alpha_{2\lfloor n/2 \rfloor} < T/2$ ($\lfloor n/2 \rfloor$ falling edges), and $\omega = 2\pi/T$ is the angular frequency. The integer n is the number of switching times in the half period, and o_n is the odd parity test described by

$$o_n = \frac{1 - (-1)^n}{2} = \begin{cases} 0, & \text{for even } n, \\ 1, & \text{for odd } n. \end{cases} \tag{7}$$

The number of levels is equal to

$$2 \max_{i=1,\ldots,n} |\Lambda_i| + 1 \tag{8}$$

where

$$\Lambda_1 = M(a_1) \quad \Lambda_{i+1} = \Lambda_i + M(a_{i+1}) \quad i=1,\ldots,n-1$$

$$(a_1, \ldots, a_n) = \text{sort}_<(\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{9}$$

$$M(a_i) = \begin{cases} 1, & a_i \in \alpha_{2j-1} \\ -1, & a_i \in \alpha_{2j}. \end{cases} \tag{10}$$

In the following, we describe some special cases:

1) *proper odd ML waveform:* $(2\lceil n/2 \rceil + 1)$-level waveform with n switching times in the half period, satisfying the condition $\alpha_{2\lceil n/2 \rceil - 1} < \alpha_2$ (see Fig. 3);
2) *proper three-level waveform:* only 0 and $+A$ levels in the half period, satisfying the condition $0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n$ (see Fig. 4);
3) *bilevel waveform:* has a slightly different Fourier series expansion and is therefore described separately in Section III-B.

For the sequel, we put $T = 2\pi$ and $\omega = 1$ for simplicity. Then, all solutions α_i are transformed back to the original period by a substitution $\alpha_i \mapsto \alpha_i T/(2\pi)$.

For further generalization and simplification of the notation, we introduce (6) and (27) for the bilevel waveform (see Section III-B) in the following form:

$$b_k(\overline{\alpha}) = A_k\left(B_k + C_k \sum_{i=1}^{n} (-1)^i \cos(k\alpha_i)\right), \quad k = 1, 2, \ldots. \tag{11}$$

The parameters for 2π periodic odd ML waveform are

$$A_k = \frac{2A}{k\pi} \quad B_k = (-1)^{k+1} o_n \quad C_k = -1. \tag{12}$$

According to the previous analysis of the optimal ML problem then, for a single-phase system, the controlled harmonics of the output ML waveform $p(t)$ are b_{p_k}, $k \in \mathcal{H}_C = \{1, 2, \ldots, n_C\}$, and the eliminated harmonics are b_{p_k}, $k \in \mathcal{H}_E = \{n_C + 1, n_C + 2, \ldots, n_C + n_E\}$. Thus, we have

$$b_{p_k}(\overline{\alpha}) = A_k\left(B_k + C_k \sum_{i=1}^{n} (-1)^i \cos(k\alpha_i)\right) = b_{f_k}, \quad k = 1, 2, \ldots, n_c \tag{13a}$$

$$b_{p_k}(\overline{\alpha}) = B_k + C_k \sum_{i=1}^{n} (-1)^i \cos(k\alpha_i) = 0, \quad k = n_c + 1, n_c + 2, \ldots, n \tag{13b}$$

subject to

$$0 < \alpha_1 < \alpha_3 < \cdots < \alpha_{2\lceil n/2 \rceil - 1} < \pi \tag{13c}$$

$$0 < \alpha_2 < \alpha_4 < \cdots < \alpha_{2\lfloor n/2 \rfloor} < \pi \tag{13d}$$

where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ are unknown variables (switching times), $n = n_C + n_E$, A_k, B_k, and C_k are set according to (12), and b_{f_k}, $k = 1, 2, \ldots, n_C$, on the right-hand side (RHS) of the equations are real numbers defining the required signal $f(t)$ (baseband frequency spectrum). The integer n_E defines the number of zero harmonics in the guard band.

1) Polynomial Equations: In this section, we convert the trigonometric equations in (13) to polynomial equations and simplify them. According to the trigonometric identity for multiple angles of cosine

$$\cos(k\alpha_i) = T_k(\cos\alpha_i). \tag{14}$$

We substitute by Chebyshev polynomial T_k of the first kind (see, e.g., [22, p.771] or [5]) and convert the kth harmonic of (11) to multivariate polynomials, i.e.,

$$b_{p_k}(\overline{x}) = A_k\left(B_k + C_k\sum_{i=1}^{n}(-1)^i T_k(x_i)\right) \tag{15}$$

in variables $(x_1, \ldots, x_n) = \overline{x}$. The dependence between x_i and α_i is given by

$$\alpha_i = \arccos x_i, \qquad i = 1, \ldots, n. \tag{16}$$

According to (13c) and (13d)

$$\begin{aligned} &-1 < x_n < \cdots < x_4 < x_2 < 1\\ &-1 < x_{n-1} < \cdots < x_3 < x_1 < 1. \end{aligned} \tag{17}$$

Thus, the trigonometric system (13) is transformed to a polynomial system, i.e.,

$$\begin{aligned} b_{p_k}(\overline{x}) &= A_k\left(B_k + C_k\sum_{i=1}^{n}(-1)^i T_k(x_i)\right) = b_{f_k},\\ &\quad k = 1, 2, \ldots, n_c \end{aligned} \tag{18a}$$

$$\begin{aligned} b_{p_k}(\overline{x}) &= B_k + C_k\sum_{i=1}^{n}(-1)^i T_k(x_i) = 0,\\ &\quad k = n_c + 1, n_c + 2, \ldots, n\\ &\text{subject to (17)} \end{aligned} \tag{18b}$$

where the variables are $(x_1, \ldots, x_n) = \overline{x}$. This polynomial system (18) can be re-solved using existing methods, such as the Gröbner basis approach, elimination based on resultants, and other algorithms (see [23] and [24]). Note that the polynomials in this system are partially symmetric. It means that we can arbitrarily permutate variables x_{2i} or x_{2i-1} and the function $b_{p_k}(\overline{x})$ is left unchanged.

However, the following steps show how the system of equations in (18) [respectively (15)] can be further simplified by conversion to a new linear system in new variables. These new variables are composite sums of powers and create new polynomial system of equations. We present new effective algorithm for this system, which is much more effective compared to direct application of standard polynomial methods to (18).

From (15), the expression $\sum_{i=1}^{n}(-1)^i T_k(x_i)$ for odd k reads

$$\begin{aligned} \sum_{i=1}^{n}(-1)^i T_k(x_i) &= -\sum_{j=1}^{\frac{k+1}{2}} t_{k,2j-1}\sum_{i=1}^{n}(-1)^{i+1}x_i^{2j-1}\\ &= -\sum_{j=1}^{\frac{k+1}{2}} t_{k,2j-1}p_{2j-1}, \quad k \text{ is odd} \end{aligned}$$

where $t_{k,2j-1}$ is the $(2j-1)$th coefficient of x^{2j-1} in the Chebyshev polynomial of degree k, and p_{2j-1} are composite sums of powers (new unknown variables) for which the following identity holds:

$$\begin{aligned} p_{2j-1} &= \sum_{i=1}^{n}(-1)^{i+1}x_i^{2j-1}\\ &= x_1^{2j-1} - x_2^{2j-1} + \cdots + (-1)^{n+1}x_n^{2j-1},\\ &\quad j = 1, 2, \ldots. \end{aligned} \tag{19}$$

Then, one can write (15) in the following form:

$$\begin{aligned} &b_{p_{2i-1}}(p_1, p_3, \ldots, p_{2i-1})\\ &\quad = A_{2i-1}\left(B_{2i-1} - C_{2i-1}\sum_{j=1}^{i} t_{2i-1,2j-1}p_{2j-1}\right),\\ &\quad i = 1, \ldots, \lceil n/2\rceil. \end{aligned} \tag{20}$$

Similarly, for even k, we have

$$\begin{aligned} &b_{p_{2i}}(p_2, p_4, \ldots, p_{2i})\\ &\quad = A_{2i}\left[B_{2i} - C_{2i}\left((-1)^i o_n + \sum_{j=1}^{i} t_{2i,2j}p_{2j}\right)\right],\\ &\quad i = 1, \ldots, \lfloor n/2\rfloor \end{aligned} \tag{21}$$

where

$$\begin{aligned} p_{2j} &= \sum_{i=1}^{n}(-1)^{i+1}x_i^{2j}\\ &= x_1^{2j} - x_2^{2j} + \cdots + (-1)^{n+1}x_n^{2j}, \qquad j = 1, 2, \ldots. \end{aligned} \tag{22}$$

Finally, we apply back substitution to (18) having the following polynomial system of equations:

$$\begin{aligned} b_{p_{2i-1}}(\overline{p}) &= A_{2i-1}\left(B_{2i-1} - C_{2i-1}\sum_{j=1}^{i} t_{2i-1,2j-1}p_{2j-1}\right)\\ &= b_{f_{2i-1}}, \qquad i = 1, 2, \ldots, \left\lceil\frac{n_c}{2}\right\rceil \end{aligned} \tag{23a}$$

$$\begin{aligned} b_{p_{2i}}(\overline{p}) &= A_{2i}\left[B_{2i} - C_{2i}\left((-1)^i o_n + \sum_{j=1}^{i} t_{2i,2j}p_{2j}\right)\right]\\ &= b_{f_{2i}}, \qquad i = 1, 2, \ldots, \left\lfloor\frac{n_c}{2}\right\rfloor \end{aligned} \tag{23b}$$

$$b_{p_{2i-1}}(\overline{p}) = B_{2i-1} - C_{2i-1}\sum_{j=1}^{i} t_{2i-1,2j-1}p_{2j-1} = 0,$$
$$i = \left\lceil \frac{n_c}{2} \right\rceil + 1, \ldots, \left\lceil \frac{n}{2} \right\rceil \qquad (23c)$$

$$b_{p_{2i}}(\overline{p}) = B_{2i} - C_{2i}\left((-1)^i o_n + \sum_{j=1}^{i} t_{2i,2j}p_{2j}\right) = 0,$$
$$i = \left\lfloor \frac{n_c}{2} \right\rfloor + 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \qquad (23d)$$

where $\overline{p} = (p_1, p_2, \ldots, p_n)$ are unknown variables. Because $\mathcal{H}_C = \{1, 2, \ldots, n_C\}$ and $\mathcal{H}_E = \{n_C + 1, n_C + 2, \ldots, n_C + n_E\}$, $n = n_C + n_E$, the previous system is linear and of n equations with n unknown variables $p_1, \ldots, p_n$. Now, if we separate unknowns $p_1, \ldots, p_n$ on the left-hand side (LHS) of (23), then the new RHS for b'_{f_i} are

$$-\frac{1}{C_{2i-1}}\left(\frac{b_{f_{2i-1}}}{A_{2i-1}} - B_{2i-1}\right)$$
$$-(-1)^i o_n - \frac{1}{C_{2i}}\left(\frac{b_{f_{2i}}}{A_{2i}} - B_{2i}\right) \quad \frac{B_{2i-1}}{C_{2i-1}}$$
$$-(-1)^i o_n + \frac{B_{2i}}{C_{2i}}.$$

The itemized form of (23) for an ML waveform [parameters A_i, B_i, and C_i are (12)] for $n = 6$ and $n_C = 3$ reads

$$\begin{bmatrix} t_{1,1} & & & & & \\ 0 & t_{2,2} & & & & 0 \\ t_{3,1} & 0 & t_{3,3} & & & \\ 0 & t_{4,2} & 0 & t_{4,4} & & \\ t_{5,1} & 0 & t_{5,3} & 0 & t_{5,5} & \\ 0 & t_{6,2} & 0 & t_{6,4} & 0 & t_{6,6} \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} \frac{\pi b_{f_1}}{2A} \\ \frac{\pi b_{f_2}}{A} \\ \frac{3\pi b_{f_3}}{2A} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (24)$$

The system of equations in (23) is a special linear system where $t_{k,i}$ on the LHS are the ith coefficients of the k degree Chebyshev polynomial of the first kind, and on the RHS, there are b'_{f_i} and zeros. According to Gauss–Banachiewitz decomposition for orthogonal Chebyshev polynomials (for more details, see [25]), the solution of (23) for the general ML waveform is

$$p_{2i} = o_n + 2^{-2i+1}\frac{\pi}{A}\sum_{j=1}^{\underline{K}} \binom{2i}{i-j} j\, b_{f_{2j}}, \qquad (25a)$$

$$\underline{K} := \begin{cases} i, & \ldots i < \lfloor n_c/2 \rfloor \\ \lfloor n_c/2 \rfloor, & \ldots i \geq \lfloor n_c/2 \rfloor \end{cases};$$
$$i = 1, 2, \ldots, \lfloor n/2 \rfloor \qquad (25b)$$

$$p_{2i-1} = -o_n + 2^{-2i+1}\frac{\pi}{A}\sum_{j=1}^{\overline{K}} \binom{2i-1}{i-j} (2j-1)\, b_{f_{2j-1}}, \qquad (25c)$$

$$\overline{K} := \begin{cases} i, & \ldots i < \lceil n_c/2 \rceil \\ \lceil n_c/2 \rceil, & \ldots i \geq \lceil n_c/2 \rceil \end{cases};$$
$$i = 1, 2, \ldots, \lceil n/2 \rceil. \qquad (25d)$$

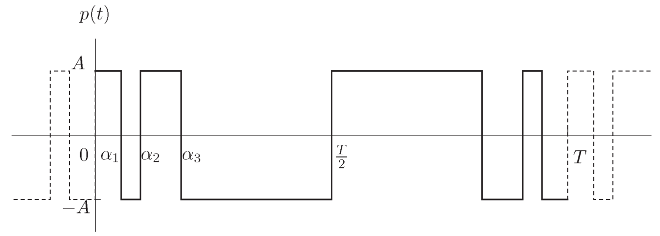

Fig. 5. Odd bilevel PWM waveform.

The number of operations is $\mathcal{O}(nn_C)$ only, instead of the standard recursive procedure for solution of triangular linear system (23), which takes $\mathcal{O}(n^2)$ operations, and it is moreover not necessary to generate and store in memory the coefficients of Chebyshev polynomials $t_{i,j}$. For example, in the converter problem, where $n_C = 1$ (only the first harmonic is controlled), the number of operations is linear compared to quadratic.

To sum up, the problem of optimal ML, namely, the solution of trigonometric system (13) or polynomial system (18), was converted to a more simple solution of system of composite sum of powers (19) and (22), which is in compact form, i.e.,

$$p_j = x_1^j - x_2^j + \cdots + (-1)^{n+1}x_n^j, \qquad j = 1, 2, \ldots, n$$
$$\text{subject to (17)} \qquad (26)$$

where p_j are easily solved according to (25), and unknowns are $\overline{x} = (x_1, x_2, \ldots, x_n)$. The effective solution for this special polynomial system of composite sum of powers is described in Section IV. The unknown switching times α_i are then obtained according to (16).

B. Odd Bilevel PWM Waveform

The Fourier series of T periodic odd bilevel PWM waveform $p(t)$ with amplitude A (see Fig. 5) is sine with the following coefficients:

$$b_k = \frac{4A}{k\pi}\left(o_{n+k} + \sum_{i=1}^{n}(-1)^i \cos(\omega k \alpha_i)\right),$$
$$k = 1, 2, \ldots, \qquad (27)$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < T/2$ are the unknown switching times.

The parameters according to (11) are

$$A_k = \frac{4A}{k\pi} \quad B_k = o_{n+k} \quad C_k = 1 \qquad (28)$$

and the composite sum of powers is

$$p_{2i} = o_n - 2^{-2i}\frac{\pi}{A}\sum_{j=1}^{\underline{K}} \binom{2i}{i-j} j\, b_{f_{2j}},$$
$$i = 1, 2, \ldots, \lfloor n/2 \rfloor \qquad (29a)$$

$$p_{2i-1} = o_{n+1} - 2^{-2i}\frac{\pi}{A}\sum_{j=1}^{\overline{K}} \binom{2i-1}{i-j} (2j-1)\, b_{f_{2j-1}},$$
$$i = 1, 2, \ldots, \lceil n/2 \rceil \qquad (29b)$$

where $\overline{K}$ and $\underline{K}$ are according to (25d) and (25b). The inequality condition for variables x_i is

$$-1 < x_n < x_{n-1} < \cdots < x_2 < x_1 < 1. \tag{30}$$

IV. Composite Sum of Powers

As shown in Section III-A1, the solution of the optimal odd ML problem depends only on computation of the composite sum of powers (26). The itemized form is

$$x_1 - x_2 + \cdots + (-1)^{n+1}x_n = p_1 \tag{31a}$$
$$x_1^2 - x_2^2 + \cdots + (-1)^{n+1}x_n^2 = p_2$$
$$\vdots$$
$$x_1^n - x_2^n + \cdots + (-1)^{n+1}x_n^n = p_n \tag{31b}$$

subject to (17) for optimal ML problem or

subject to (30) for optimal bilevel PWM problem (31c)

where the RHS are real numbers according to (25) for the general odd ML waveform, or (29) for the odd bilevel PWM waveform. Note that this system is very similar to standard power sums $\sum_{i=1}^{n} x_i^k = p_k$, $k = 1, \ldots, n$, that are easily solvable by the Newton's identity (see [24] and [26]).

For the following steps, it is better to focus on the following configuration of the power sums:

$$p_j(y_1, \ldots, y_n) = \sum_{i=1}^{k} y_i^j - \sum_{i=k+1}^{n} y_i^j, \qquad j = 1, \ldots, n \tag{32}$$

where $k \leq \lfloor n/2 \rfloor$. When $k > \lfloor n/2 \rfloor$, we can multiply the equation system in (32) by -1 and convert it to the case $k < \lfloor n/2 \rfloor$. This form in (32) can be obtained by resorting variables in (31). The polynomials $p_j(y_1, y_2, \ldots, y_n)$ in (32) are partially symmetric because the power sums $\sum_{i=1}^{k} y_i^j$ and $\sum_{i=k+1}^{n} y_i^j$ are symmetric polynomials (see [24]) in variables $\overline{y}^{\,+} = (y_1, \ldots, y_k)$ and $\overline{y}^{\,-} = (y_{k+1}, \ldots, y_n)$ separately. Then, we have

$$p_j(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) = p_j\left(y_{\pi_1(1)}, \ldots, y_{\pi_1(k)}, y_{\pi_2(k+1)}, \ldots, y_{\pi_2(n)}\right) \tag{33}$$

where $\left(y_{\pi_1(1)}, \ldots, y_{\pi_1(k)}\right)$ and $\left(y_{\pi_2(k+1)}, \ldots, y_{\pi_2(n)}\right)$ are arbitrary permutations of $\overline{y}^{\,+}$ and $\overline{y}^{\,-}$, respectively. Therefore, the total number of solutions is $k!(n-k)!$. All of them are combinations of two sets coming from permutations of elements of vectors $\overline{y}^{\,+}$ and $\overline{y}^{\,-}$.

Equation (31) is converted to (32) in the following way. If n is an even integer, then $n/2$ variables with positive sign and the same number with negative sign are in (31). Therefore, converting to (32) is accomplished by introducing the following new variables:

$$\overline{y}^{+} = (y_1, y_2, \ldots, y_k) = (x_1, x_3, \ldots, x_{2k-1}) \tag{34a}$$
$$\overline{y}^{-} = (y_{k+1}, y_{k+2}, \ldots, y_{2k}) = (x_2, x_4, \ldots, x_{2k}) \tag{34b}$$

where $k = n/2$. If n is odd, then $\lfloor n/2 \rfloor + 1$ variables with positive sign and $\lfloor n/2 \rfloor$ variables with negative sign are in (31). Therefore, conversion similar to the case with n even leads to $k > \lfloor n/2 \rfloor$, which is not in agreement with condition $k \leq \lfloor n/2 \rfloor$ of (32). Therefore, each equation in (31) must be multiplied by -1, and for that reason, the signs of RHS of (32) must be changed, i.e., $p_i \mapsto -p_i$. Then, the following substitution can be done:

$$\overline{y}^{\,+} = (y_1, y_2, \ldots, y_k) = (x_2, x_4, \ldots, x_{2k}) \tag{35a}$$
$$\overline{y}^{\,-} = (y_{k+1}, \ldots, y_{2k+1}) = (x_1, x_3, \ldots, x_{2k+1}) \tag{35b}$$

where $k = \lfloor n/2 \rfloor$.

The solution $x_1, \ldots, x_n$ of the optimal odd ML problem is obtained as follows. From all solutions of (32), only one is chosen—the one that is in agreement with (31c), which means that all elements $\overline{y}^{\,+}$ and $\overline{y}^{\,-}$ are real numbers strictly inside the interval $(-1, 1)$. When no such solution exists, then none of the switching sequences allows us to generate the required harmonics (e.g., this situation arises when we require high first harmonic for low amplitude of ML waveform for a given n). As all elements $\overline{y}^{\,+}$ and $\overline{y}^{\,-}$ can be permuted, the elements of $\overline{y}^{\,+}$ and $\overline{y}^{\,-}$ are reindexed so that for $\overline{y}^{\,+}$, $-1 < y_k < \cdots < y_1 < 1$ holds, and for y^{-}, $-1 < y_n < \cdots < y_{k+1} < 1$ holds. Therefore, according to (34) for even n and (35) for odd n, we have $(x_1, \ldots, x_n) = (y_1, y_{k+1}, y_2, y_{k+2}, \ldots, y_n, y_k)$ and $(x_1, \ldots, x_n) = (y_{k+1}, y_1, y_{k+2}, y_2, \ldots, y_k, y_n)$, respectively. Finally, the condition (31c) for $\overline{x}$ must hold.

A. Solving Composite Sum of Powers

In this section, the algorithm for solving the composite sum of powers in (32) is described. The solution is inspired by [5] and [6], where a special case of quarter-symmetric three-level inverter problem is studied. The problem was also tackled in [27] and [28], the authors, however, did not use the Padé approximation and the theory of FOPs that play a crucial role in the analytical solution of the whole problem. The other applications of solving composite sum of powers are in coding theory and geometric optics. We will find the exact solution as the set of roots of the following two polynomials:

$$V_k(y) = \prod_{i=1}^{k}(y - y_i) = y^k + v_{k,k-1}y^{k-1} + \cdots + v_{k,0} \tag{36}$$

$$W_{n-k}(y) = \prod_{i=1}^{n-k}(y - y_{i+k}) = y^{n-k} + w_{n-k,n-k-1}y^{n-k-1} + \cdots + w_{n-k,0}. \tag{37}$$

Then, let us do a logarithmic derivative of

$$\frac{V_k(y)}{W_{n-k}(y)} = \frac{\prod_{i=1}^{k}(y - y_i)}{\prod_{i=1}^{n-k}(y - y_{i+k})} \tag{38}$$

to get

$$\frac{V_k'(y)}{V_k(y)} - \frac{W_{n-k}'(y)}{W_{n-k}(y)} = \sum_{i=1}^{k} \frac{1}{y-y_i} - \sum_{i=1}^{n-k} \frac{1}{y-y_{i+k}}. \quad (39)$$

The expansion of $1/(y-z)$ at $y=\infty$ is the series $\sum_{j=0}^{\infty} z^j/y^{j+1}$. Then, we have

$$\frac{V_k'(y)}{V_k(y)} - \frac{W_{n-k}'(y)}{W_{n-k}(y)} = \sum_{j=0}^{\infty} \frac{p_j^+}{y^{j+1}} - \sum_{j=0}^{\infty} \frac{p_j^-}{y^{j+1}}. \quad (40)$$

where $p_j^+ = \sum_{i=1}^{k} y_i^j$, $p_j^- = \sum_{i=1}^{n-k} y_{i+k}^j$ and $p_j = p_j^+ - p_j^-$. Thus, we get

$$\frac{V_k'(y)}{V_k(y)} - \frac{W_{n-k}'(y)}{W_{n-k}(y)} = \sum_{j=0}^{\infty} \frac{p_j}{y^{j+1}}. \quad (41)$$

By integrating, (41) we get

$$\frac{V_k(y)}{W_{n-k}(y)} = y^{2k-n} e^{\left(-\sum_{j=1}^{\infty} \frac{p_j}{j y^j}\right)} = f(y). \quad (42)$$

The series expansion of $f(y)$ leads to the Padé approximation.

B. Padé Approximation

In this section, we will find the unknown coefficients of polynomials $V_k(y)$ and $W_{n-k}(y)$ according to the theory of Padé approximation (for more details, see [29] and [30]). We rewrite (42) in the following way:

$$\begin{aligned} &\frac{V_k(y)}{W_{n-k}(y)} + O(y^{-n+k-2}) \\ &= \left(\frac{1}{y}\right)^{n-2k} \left(\mu_0 + \mu_1 \frac{1}{y} + \mu_2 \left(\frac{1}{y}\right)^2 + \cdots\right) \\ &= f(y), \qquad y \to \infty \end{aligned} \quad (43)$$

where the RHS of (43) is the series expansion of $f(y)$ at infinity. In this case, the expansion of function $f(y)$ contains the negative powers of y.

We consider the following form:

$$\begin{aligned} \frac{\widetilde{V}_k(y)}{\widetilde{W}_{n-k}(y)} + O(y^{n+1}) &= y^{2k-n} f(y^{-1}) \\ &= e^{\left(-\sum_{j=1}^{\infty} \frac{p_j}{j} y^j\right)} = F(y), \qquad y \to 0 \end{aligned} \quad (44)$$

where $\widetilde{V}_k(y) = y^k V_k(y^{-1})$ and $\widetilde{W}_{n-k}(y) = y^{n-k} V_{n-k}(y^{-1})$ (this is only reversion of polynomial coefficients). Therefore, we solve (44) [instead of solving (43)] as the problem of Padé approximation with the following notation:

$$[k/n-k]_F(y) = \frac{\widetilde{V}_k(y)}{\widetilde{W}_{n-k}(y)} = \frac{\widetilde{V}_k^{[k,n-k]}(y)}{\widetilde{W}_{n-k}^{[k,n-k]}(y)} \quad (45)$$

of the function

$$F(y) = e^{\left(-\sum_{j=1}^{\infty} \frac{p_j}{j} y^j\right)} = e^{\sum_{j=1}^{\infty} c_j y^j} \quad \text{at } y \to 0, \quad \text{where } c_j = -\frac{p_j}{j}. \quad (46)$$

The solution of the original problem in (43) is then obtained by reversing the coefficients of polynomials $\widetilde{V}_k(y)$ and $\widetilde{W}_{n-k}(y)$.

Now, it is necessary to solve the series expansion of the function $F(y)$ at $y=0$ in the form

$$F(y) = \sum_{i=0}^{\infty} \mu_i y^i = \mu_0 + \mu_1 y + \mu_2 y^2 + \cdots. \quad (47)$$

The direct solution is carried out according to [31, Ch. 4.7, exercise 4] and reads

$$\mu_0 = 1, \mu_k = -\frac{1}{k} \sum_{j=1}^{k} p_j \mu_{k-j}, \qquad k = 1, 2, \ldots. \quad (48)$$

In the case of the optimal odd ML problem (or odd bi-level PWM problem), two eventualities can occur (see Section IV). The first is for odd n and $k = \lfloor n/2 \rfloor$, and the second is for even n and $k = n/2$. Both cases will be described separately.

Equation (44), after cross multiplication, gives

$$\widetilde{V}_k^{[k,n-k]}(y) = \widetilde{W}_{n-k}^{[k,n-k]}(y) F(y) + O(y^{n+1}) \quad (49)$$

and a detailed form of the previous equation, considering (47), leads to

$$\begin{aligned} &(\widetilde{v}_{k,k} y^k + \widetilde{v}_{k,k-1} y^{k-1} + \cdots + \widetilde{v}_{k,0}) - O(y^{n+1}) \\ &= (\widetilde{w}_{n-k,n-k} y^{n-k} + \widetilde{w}_{n-k,n-k-1} y^{n-k-1} + \cdots + \widetilde{w}_{n-k,0}) \\ &\quad \times (\mu_0 + \mu_1 y + \mu_2 y^2 + \cdots). \end{aligned} \quad (50)$$

First, let us consider the following cases.

n Is an Odd Number and $k = \lfloor n/2 \rfloor$: The problem of the shifted diagonal Padé approximation, i.e.,

$$[k, k+1]_F(y) = \frac{\widetilde{V}_k^{[k,k+1]}(y)}{\widetilde{W}_{k+1}^{[k,k+1]}(y)} \quad (51)$$

is solved. Equating the coefficients of $y^{k+1}, \ldots, y^{2(k+1)+1}$ in (50) leads to the following linear system:

$$\left[\begin{array}{cccc|c} \mu_0 & \mu_1 & \cdots & & \mu_{k+1} \\ \mu_1 & & \iddots & & \vdots \\ \vdots & \iddots & & & \mu_{2k+1} \\ \hline \mu_{k+1} & \cdots & \mu_{2k+1} & & \mu_{2(k+1)} \end{array}\right] \cdot \left[\begin{array}{c} \widetilde{w}_{k+1,k+1} \\ \vdots \\ \widetilde{w}_{k+1,1} \\ \hline \widetilde{w}_{k+1,0} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline \widetilde{K}_k \end{array}\right] \quad (52)$$

where $\widetilde{w}_{k+1,0}$ is coefficient of y^0 of polynomial $\widetilde{W}_{k+1}^{[k,k+1]}(y)$, and due to definiteness and the condition that $w_{k+1,k+1} = 1$, we put $\widetilde{w}_{k+1,0} = 1$, and $\widetilde{K}_k$ will be a nonzero constant. The last equation of the system in (52) is reduced. Therefore, we solve the linear system with a Toeplitz structure (Hankel matrix) of size $(k+1)\times(k+1)$ as follows:

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_k \\ \mu_1 & & \iddots & \vdots \\ \vdots & \iddots & & \mu_{2k-1} \\ \mu_k & \cdots & \mu_{2k-1} & \mu_{2k} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w}_{k+1,k+1} \\ \widetilde{w}_{k+1,k} \\ \vdots \\ \widetilde{w}_{k+1,1} \end{bmatrix} = \begin{bmatrix} -\mu_{k+1} \\ -\mu_{k+2} \\ \vdots \\ -\mu_{2k+1} \end{bmatrix}. \tag{53}$$

From the found solution $\widetilde{W}_{k+1}^{[k,k+1]}(y)$, the polynomial $W_{k+1}^{[k,k+1]}(y)$ is recovered by reversing the coefficients. Alternatively, the solution can be obtained as the solution of the following linear system:

$$\left[\begin{array}{cccc|c} \mu_0 & \mu_1 & \cdots & & \mu_{k+1} \\ \mu_1 & & \iddots & & \vdots \\ \vdots & \iddots & & & \mu_{2k+1} \\ \hline \mu_{k+1} & \cdots & \mu_{2k+1} & & \mu_{2(k+1)} \end{array}\right] \cdot \left[\begin{array}{c} \widetilde{w}_{k+1,0} \\ \vdots \\ \widetilde{w}_{k+1,k} \\ \hline \widetilde{w}_{k+1,k+1} \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline K_k \end{array}\right] \tag{54}$$

where $w_{k+1,k+1}$ is equal to 1.

Unknown polynomial coefficients of $V_k^{[k,k+1]}(y)$ are obtained from the known polynomial coefficients of $\widetilde{W}_{k+1}^{[k,k+1]}(y)$ as follows:

$$\begin{bmatrix} \widetilde{v}_{k,0} \\ \widetilde{v}_{k,1} \\ \vdots \\ \widetilde{v}_{k,k} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mu_0 \\ \vdots & \vdots & \iddots & \iddots & \mu_1 \\ 0 & 0 & \iddots & \iddots & \vdots \\ 0 & \mu_0 & \mu_1 & \cdots & \mu_k \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w}_{k+1,k+1} \\ \widetilde{w}_{k+1,k} \\ \vdots \\ \widetilde{w}_{k+1,1} \\ \widetilde{w}_{k+1,0} \end{bmatrix} \tag{55}$$

equating coefficients of $x^0, x^1, \ldots, x^k$ in (50). Obviously, $\widetilde{w}_{k+1,0} = 1$, $\mu_0 = 1$, and $\widetilde{v}_{k,0} = 1$. Therefore, the previous matrix equation is simplified to

$$\begin{bmatrix} \widetilde{v}_{k,1} \\ \widetilde{v}_{k,2} \\ \vdots \\ \widetilde{v}_{k,k} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \mu_0 \\ \vdots & \iddots & \iddots & \mu_1 \\ 0 & \iddots & \iddots & \vdots \\ \mu_0 & \mu_1 & \cdots & \mu_{k-1} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w}_{k+1,k} \\ \widetilde{w}_{k+1,k-1} \\ \vdots \\ \widetilde{w}_{k+1,1} \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}. \tag{56}$$

The polynomial $V_k^{[k,k+1]}(y)$ can be constructed analogously from the found solution $\widetilde{V}_k^{[k,k+1]}(y)$ by reversing coefficients or by the following linear system:

$$\begin{bmatrix} v_{k,k-1} \\ v_{k,k-2} \\ \vdots \\ v_{k,0} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \mu_0 \\ \vdots & \iddots & \iddots & \mu_1 \\ 0 & \iddots & \iddots & \vdots \\ \mu_0 & \mu_1 & \cdots & \mu_{k-1} \end{bmatrix} \cdot \begin{bmatrix} w_{k+1,1} \\ w_{k+1,2} \\ \vdots \\ w_{k+1,k} \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}. \tag{57}$$

n Is an Even Number and $k = n/2$: The procedure is similar to the previous case. The diagonal Padé approximation, i.e.,

$$[k,k]_F(y) = \frac{\widetilde{V}_k^{[k,k]}(y)}{\widetilde{W}_k^{[k,k]}(y)} \tag{58}$$

is solved. The coefficients of $\widetilde{W}_k^{[k,k]}(y)$ are due to

$$\begin{bmatrix} \mu_1 & \mu_2 & \cdots & \mu_k \\ \mu_2 & & \iddots & \vdots \\ \vdots & \iddots & & \mu_{2k-2} \\ \mu_k & \cdots & \mu_{2k-2} & \mu_{2k-1} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w}_{k,k} \\ \widetilde{w}_{k,k-1} \\ \vdots \\ \widetilde{w}_{k,1} \end{bmatrix} = \begin{bmatrix} -\mu_{k+1} \\ -\mu_{k+2} \\ \vdots \\ -\mu_{2k} \end{bmatrix} \tag{59}$$

equating the coefficients of $y^{k+1}, y^{k+2}, \ldots, y^{2k+1}$ in (50). The coefficients of $\widetilde{V}_k^{[k,k]}(y)$ are obtained as follows:

$$\begin{bmatrix} \widetilde{v}_{k,1} \\ \widetilde{v}_{k,2} \\ \vdots \\ \widetilde{v}_{k,k} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & \mu_0 \\ \vdots & \iddots & \iddots & \mu_1 \\ 0 & \iddots & \iddots & \vdots \\ \mu_0 & \mu_1 & \cdots & \mu_{k-1} \end{bmatrix} \cdot \begin{bmatrix} \widetilde{w}_{k,k} \\ \widetilde{w}_{k,k-1} \\ \vdots \\ \widetilde{w}_{k,1} \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}. \tag{60}$$

C. Family of FOPs

According to the theory of Padé approximation, $V(y)$ and $W(y)$ are FOPs, and therefore, related formulas and theorems can be applied (see, e.g., [30], [32], and [33] for references).

1) Three-Term Recurrence Formula for $W(y)$ and $V(y)$:

n is an odd number and $k = \lfloor n/2 \rfloor$: According to [32, p. 101] with (51) we have

$$[k/k+1]_F(y) = \frac{\widetilde{V}_k^{[k,k+1]}(y)}{\widetilde{W}_{k+1}^{[k,k+1]}(y)} = \frac{\widetilde{Q}_{k+1}^{(0)}(y)}{\widetilde{P}_{k+1}^{(0)}(y)} \tag{61}$$

where $\widetilde{P}_{k+1}^{(0)}(y) = y^{k+1}P_{k+1}^{(0)}(y^{-1})$, and $\widetilde{Q}_{k+1}^{(0)}(y) = y^k Q_{k+1}^{(0)}(y^{-1})$. The polynomial $P_{k+1}^{(0)}(y)$ is an FOP of the first kind with respect to the linear functional $\mathcal{L}^{(0)}[y^i] = \mu_i$, where μ_i is generated according to (48). The polynomial $Q_{k+1}^{(0)}(y)$ is the associated FOP (sometimes called the polynomial of the second kind) to $P_{k+1}^{(0)}(y)$. Thus, according to (61), $W_{k+1}^{[k,k+1]}(y) = P_{k+1}^{(0)}(y), V_k^{[k,k+1]}(y) = Q_{k+1}^{(0)}(y)$, and we can write the following three-term recurrence formulas:

$$W_{-1}^{[-2,-1]}(y) = 0 \quad W_0^{[-1,0]}(y) = 1 \tag{62a}$$

$$W_i^{[i-1,i]}(y) = (y+B_i)W_{i-1}^{[i-2,i-1]}(y) - C_i W_{i-2}^{[i-3,i-2]}(y)$$
$$i = 1, 2, \ldots, k+1, \ldots \tag{62b}$$

where

$$B_i = -\frac{\mathcal{L}^{(0)}\left[y\left(W_{i-1}^{[i-2,i-1]}(y)\right)^2\right]}{K_{i-1}} \quad C_i = \frac{K_{i-1}}{K_{i-2}} \tag{63a}$$

$$K_i = \sum_{j=0}^{i} \mu_{i+j} w_{i,j}. \tag{63b}$$

The linear moment functional $\mathcal{L}^{(0)}[\cdot]$ in (63a) of arbitrary polynomial $Z(y) = \sum_{i=0}^n z_i y^i$ is solved according to $\mathcal{L}^{(0)}[Z(y)] = \sum_{i=0}^n z_i \mu_i$, where $\mathcal{L}^{(0)}[y^i] = \mu_i$ and $w_{i,j}$ are the coefficients of

$W_i^{[i-1,i]}(y) = \sum_{j=0}^{i} w_{i,j} y^j$. Note that the constant K_i in (63b) is the same as the constant in (54).

The polynomial $V_k^{[k,k+1]}(y)$ is associated FOP to $W_{k+1}^{[k,k+1]}(y)$, and therefore, we have

$$V_{-1}^{[-1,0]}(y) = -1 \quad V_0^{[0,1]}(y) = 0 \tag{64a}$$

$$V_i^{[i,i+1]}(y) = (y + B_i)V_{i-1}^{[i-1,i]}(y) - C_i V_{i-2}^{[i-2,i-1]}(y), \quad i = 1, 2, \ldots, k, \ldots \tag{64b}$$

where B_i and C_i are identical to (63).

n is an even number and $k = n/2$: Similarly as above, we have the following equation for (58):

$$[k/k]_f(y) = \frac{\widetilde{V}_k^{[k,k]}(y)}{\widetilde{W}_k^{[k,k]}(y)} = \mu_0 + y\frac{\widetilde{Q}_k^{(1)}(y)}{\widetilde{P}_k^{(1)}(y)} \tag{65}$$

where $\widetilde{P}_k^{(1)}(y) = y^k P_k^{(1)}(y^{-1})$, and $\widetilde{Q}_k^{(1)}(y) = y^{k-1} Q_k^{(1)}(y^{-1})$. The polynomial $P_k^{(1)}(y)$ is the adjacent FOP of the first kind with respect to the linear functional $\mathcal{L}^{(1)}[y^i] = \mathcal{L}^{(0)}[y^{i+1}] = \mu_{i+1}$, where μ_i is generated according to (48). The polynomial $Q_k^{(1)}(y)$ is the associated adjacent FOP to $P_k^{(1)}(y)$. Thus, according to (65) $W_k^{[k,k]}(y) = P_k^{(1)}(y)$, $\widetilde{V}_k^{[k,k]}(y) = \mu_0 \widetilde{P}_k^{(1)}(y) + y\widetilde{Q}_k^{(1)}(y)$, and therefore, we can write the following three-term recurrence formula for $W_i^{[i,i]}(y)$:

$$W_{-1}^{[-1,-1]}(y) = 0 \quad W_0^{[0,0]}(y) = 1 \tag{66a}$$

$$W_i^{[i,i]}(y) = (y + B_i)W_{i-1}^{[i-1,i-1]}(y) - C_i W_{i-2}^{[i-2,i-2]}(y), \quad i = 1, 2, \ldots, k, \ldots \tag{66b}$$

where

$$B_i = -\frac{\mathcal{L}^{(1)}\left[y\left(W_{i-1}^{[i-1,i-1]}(y)\right)^2\right]}{K_{i-1}} \quad C_i = \frac{K_{i-1}}{K_{i-2}} \tag{67a}$$

$$K_i = \sum_{j=0}^{i} \mu_{i+j+1} w_{i,j} \tag{67b}$$

where the linear moment functional $\mathcal{L}^{(1)}[\cdot]$ in (67a) of arbitrary polynomial $Z(y) = \sum_{i=0}^{n} z_i y^i$ is solved according to $\mathcal{L}^{(1)}[Z(y)] = \sum_{i=0}^{n} z_i \mu_{i+1}$, and $w_{i,j}$ are the coefficients of $W_i^{[i,i]}(y) = \sum_{j=0}^{i} w_{i,j} y^j$.

Finding a recurrent formula for the polynomial $V_k(y)$ is more difficult due to the fact that $V_k^{[k,k]}(y)$ is not an associated FOP to $W_k^{[k,k]}(y)$. From (65), we know, however, that $\widetilde{V}_k(y) = \mu_0 \widetilde{P}_k^{(1)}(y) + y\widetilde{Q}_k^{(1)}(y)$. We apply "tilde notation" (reversion of coefficients) on both sides of the equation $\widetilde{\widetilde{V}}_k^{[k,k]}(y) = \mu_0 \widetilde{\widetilde{P}}_k^{(1)}(y) + \widetilde{y\widetilde{Q}_k}^{(1)}(y)$ and get $V_k^{[k,k]}(y) = \mu_0 P_k^{(1)}(y) + Q_k^{(1)}(y)$. Thus, the recursion for $V_k^{[k,k]}(y)$ is a composition of $P_k^{(1)}(y)$ and $Q_k^{(1)}(y)$, where $Q_k^{(1)}(y)$ is the associated FOP to $P_k^{(1)}(y)$, with the following three-term recurrence formula:

$$Q_{-1}^{(1)}(y) = -1 \quad Q_0^{(1)}(y) = 0 \tag{68a}$$

$$Q_i^{(1)}(y) = (y + B_i)Q_{i-1}^{(1)}(y) - C_i Q_{i-2}^{(1)}(y), \quad i = 1, 2, \ldots, k, \ldots \tag{68b}$$

where B_i and C_i are due to (67). The recurrence formula for $P_k^{(1)}(y)$ is given by (66), where $P_k^{(1)}(y) = W_k^{[k,k]}(y)$. Therefore, we have

$$\begin{aligned} V_i^{[i,i]}(y) &= \mu_0\left((y + B_i)P_{i-1}^{(1)}(y) - C_i P_{i-2}^{(1)}(y)\right) \\ &\quad + (y + B_i)Q_{i-1}^{(1)}(y) - C_i Q_{i-2}^{(1)}(y) \\ &= (y + B_i)\left(\mu_0 P_{i-1}^{(1)}(y) + Q_{i-1}^{(1)}(y)\right) \\ &\quad - C_i\left(\mu_0 P_{i-2}^{(1)}(y) + Q_{i-2}^{(1)}(y)\right) \\ &= (y + B_i)V_{i-1}^{[i-1,i-1]}(y) - C_i V_{i-2}^{[i-2,i-2]}(y), \\ &\quad i = 1, 2, \ldots, k, \ldots \end{aligned} \tag{69}$$

where B_i and C_i are according to (67), and the initial conditions are

$$V_{-1}^{[-1,-1]}(y) = \mu_0 P_{-1}^{(1)}(y) + Q_{-1}^{(1)}(y) = 1 \cdot 0 + (-1) = -1$$
$$V_0^{[0,0]}(y) = \mu_0 P_0^{(1)}(y) + Q_0(y)^{(1)} = 1 \cdot 1 + 0 = 1.$$

2) Determinantal Formulas for $W(y)$ and $V(y)$: According to [32, Ch. 2], one can write the following determinantal formulas for polynomials $W(y)$ and $V(y)$.

n is an odd number and $k = \lfloor n/2 \rfloor$: We have

$$W_{k+1}^{[k,k+1]}(y) = D_{w_{k+1}} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_k & \mu_{k+1} \\ \mu_1 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \mu_{k+2} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \vdots \\ \mu_k & \mu_{k+1} & \cdots & \mu_{2k-1} & \mu_{2k} \\ 1 & y & \cdots & y^k & y^{k+1} \end{bmatrix} \tag{70}$$

$$V_k^{[k,k+1]}(y) = D_{v_k} \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{k-1} & \mu_k \\ \mu_1 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \mu_{k+1} \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \vdots \\ \mu_{k-1} & \mu_k & \cdots & \mu_{2k-2} & \mu_{2k-1} \\ 0 & 1 & \cdots & \sum_{i=0}^{k-1}\mu_i y^{k-i-1} & \sum_{i=0}^{k}\mu_i y^{k-i} \end{bmatrix} \tag{71}$$

where $D_{w_{k+1}}$ and D_{v_k} are normalization factors so that $W_{k+1}^{[k,k+1]}(y)$ and $V_k^{[k,k+1]}(y)$ are monomials, and the moments μ_i are generated according to (48).

n is an even number and $k = n/2$: We have (72)–(73), shown at the bottom of the next page, where D_{w_k} and D_{v_k} are normalization factors so that $W_k^{[k,k]}(y)$ and $V_k^{[k,k]}(y)$ are monomials, and the moments μ_i are generated according to (48).

TABLE I

PARTIAL RESULTS FOR AN ILLUSTRATIVE EXAMPLE WHERE $n_C = 3, n_E = 13, A = 2.3$, AND $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}
-1.3659	0.3415	-0.5122	0.3415	-0.2134	0.3201	-0.0747	0.2988	0.	0.2801	0.044	0.2641	0.0715	0.2504	0.0894	0.2384	0.1013
μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	μ_{13}	μ_{14}	μ_{15}	μ_{16}	μ_{17}
1.3659	0.7621	0.3623	0.1482	0.0432	-0.0158	-0.0406	-0.0552	-0.0578	-0.0594	-0.0561	-0.0541	-0.0497	-0.047	-0.0428	-0.0403	-0.0367
$w_{8,0}$	$w_{8,1}$	$w_{8,2}$	$w_{8,3}$	$w_{8,4}$	$w_{8,5}$	$w_{8,6}$	$w_{8,7}$		$v_{8,0}$	$v_{8,1}$	$v_{8,2}$	$v_{8,3}$	$v_{8,4}$	$v_{8,5}$	$v_{8,6}$	$v_{8,7}$
0.001	-0.0512	-0.2215	0.3118	1.2376	-0.4483	-2.0023	0.1779		-0.0023	0.0209	0.1091	-0.5804	0.1302	1.6544	-1.1417	-1.188
y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_9	y_{10}	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}
-0.9222	-0.7985	-0.6458	-0.2468	0.0178	0.5245	0.9095	0.9836		-0.9109	-0.7287	-0.1693	0.0865	0.4478	0.6023	0.8841	0.9762
α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}	
0.1813	0.2186	0.4286	0.4863	1.0187	0.9244	1.553	1.1065	1.8202	1.4842	2.2729	1.7409	2.4956	2.3873	2.7446	2.7162	
b_{p_1}	b_{p_2}	b_{p_3}	b_{p_4}	...	$b_{p_{16}}$	$b_{p_{17}}$	$b_{p_{18}}$	$b_{p_{19}}$	$b_{p_{20}}$	$b_{p_{21}}$	$b_{p_{22}}$	$b_{p_{23}}$	$b_{p_{24}}$	$b_{p_{25}}$		THD
-2	0.5	1	0	...	0	0.2171	-0.0469	0.0158	0.3334	-0.3591	-0.2791	-0.0791	-0.0003	0.1343		1.81%

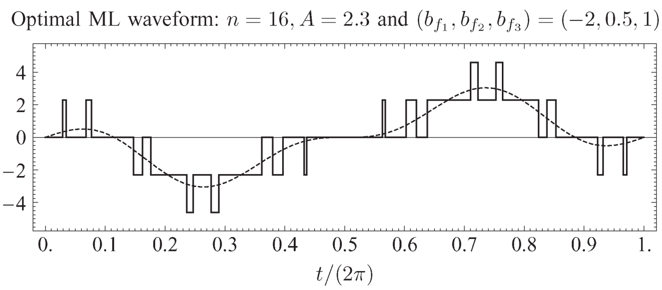

Fig. 6. Solution of an illustrative example.

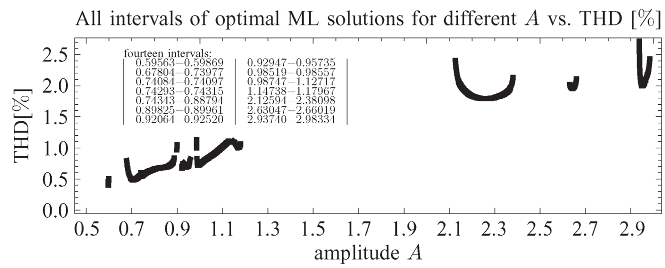

Fig. 7. All intervals of optimal ML solutions for increasing A versus THD (in percent): $n = 16$ and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

3) Eigenvalues Formulation: The solution of composite sum of powers is the set of zeros of polynomials $W(y)$ and $V(y)$. As these are FOPs, it is possible to obtain these zeros as eigenvalues of a special matrix (see [32, p. 79]) by

$$J_{k+1} = \begin{bmatrix} -B_1 & 1 & 0 & \dots & 0 \\ C_2 & -B_2 & 1 & \ddots & \vdots \\ 0 & C_3 & -B_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & C_{k+1} & -B_{k+1} \end{bmatrix} \tag{74}$$

where B_i and C_i are computed according to (63). Thus, for odd n, we have

$$\begin{aligned} W_{k+1}^{[k,k+1]}(y) &= \det(yI_{k+1} - J_{k+1}) \\ V_k^{[k,k+1]}(y) &= \det(yI_k - J'_k) \end{aligned} \tag{75}$$

where J'_k is the matrix obtained by suppressing the first row and the first column of J_{k+1}. Therefore, the zeros of $W_{k+1}^{[k,k+1]}(y)$ are the eigenvalues of J_{k+1}, and the zeros of $V_k^{[k,k+1]}(y)$ are the eigenvalues of J'_k.

4) Other Orthogonal Properties—The Zeros: The position of zeros of (classical) orthogonal polynomials has very important properties. Each n-degree polynomial in an orthogonal sequence has all n of its roots real from interval (a, b), distinct, and strictly inside the interval of orthogonality. The roots of each polynomial lie strictly between the roots of the next higher degree polynomial in the sequence. This interesting property can be partially employed in a numerical iterative search algorithms for the zeros in recurrence algorithm—for the choice of the initial iteration in Newton's method.

Not all nice properties extend to FOPs nevertheless. In particular, the zeros of FOPs need not be simple or even real. For

$$W_k^{[k,k]}(y) = D_{w_k} \det \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_k & \mu_{k+1} \\ \mu_2 & & \iddots & \iddots & \mu_{k+2} \\ \vdots & \iddots & \iddots & & \vdots \\ \mu_{k-1} & \mu_{k+1} & \dots & \mu_{2k-1} & \mu_{2k} \\ 1 & y & \dots & y^{k-1} & y^k \end{bmatrix} \tag{72}$$

$$V_k^{[k,k]}(y) = D_{v_k} \det \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_k & \mu_{k+1} \\ \mu_2 & & \iddots & \iddots & \mu_{k+2} \\ \vdots & \iddots & \iddots & & \vdots \\ \mu_{k-1} & \mu_{k+1} & \dots & \mu_{2k-1} & \mu_{2k} \\ 1 & y + \mu_1 & \dots & \sum_{i=1}^{k-1} \mu_i y^{k-i-1} & \sum_{i=1}^{k} \mu_i y^{k-i} \end{bmatrix} \tag{73}$$

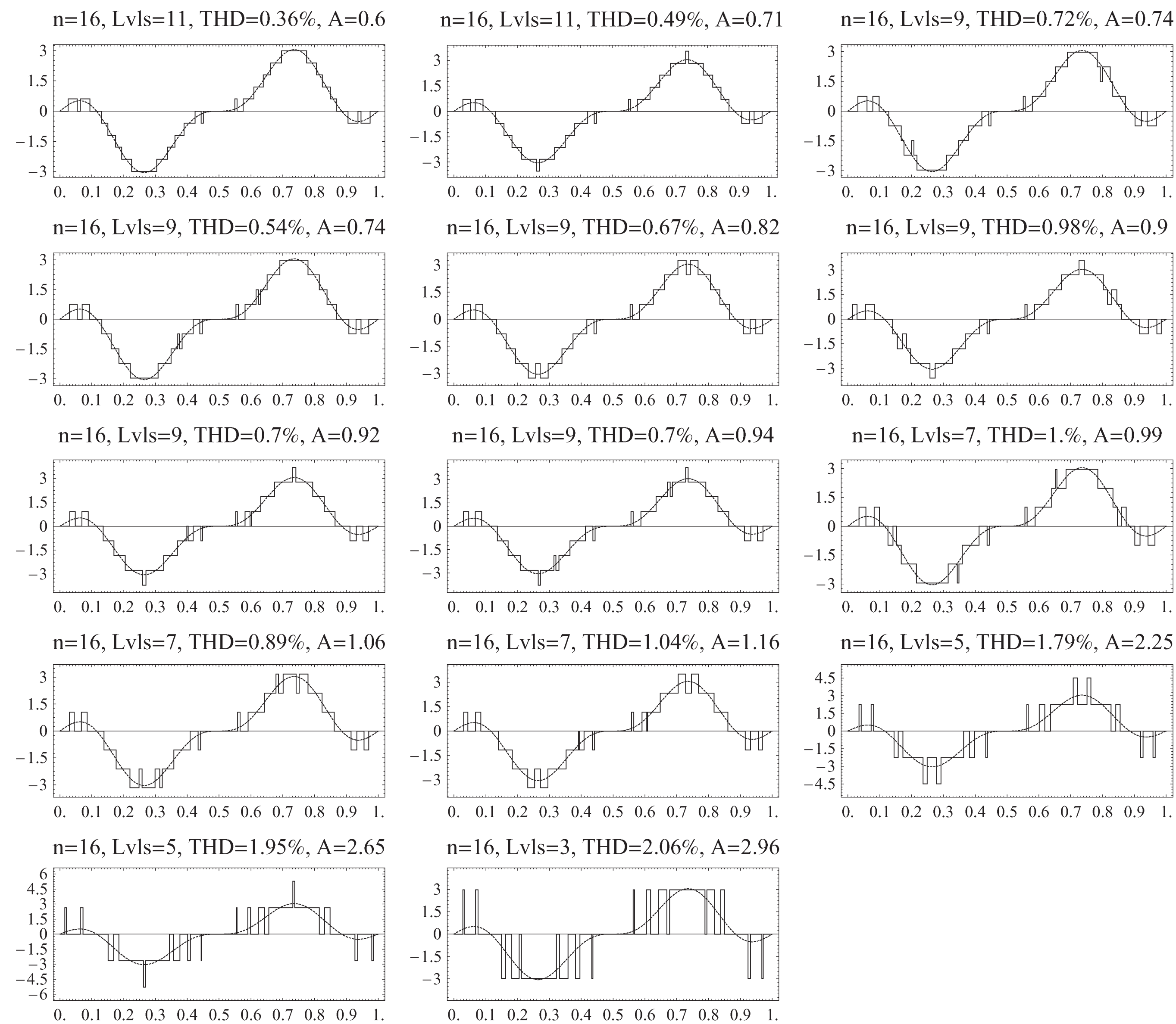

Fig. 8. All possible configurations of optimal ML waveforms for increasing amplitude: $A, n = 16$ and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

FOPs, the following holds nevertheless: if $\mathcal{L}[\cdot]$ is defined, then for all $k \geq 0$, 1) P_k and P_{k+1} have no common zeros, 2) Q_k and Q_{k+1} have no common zeros, and 3) P_k and Q_k have no common zeros.

V. Illustrative Numerical Example

Let us consider the optimal ML problem with controlled harmonics $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$, fixed $n = 16$, and amplitude $A = 2.3$. The partial results of computation for this specific n and A are shown in Table I (the line 2: power sums p_i, 4: moments μ_i, 6: the coefficients of FOPs W and V, 8: the zeros W and V, 10: result—switching times α_i, 12: test—the required frequency spectrum of the ML waveform b_{p_i} computed from α_i and THD). Fig. 6 depicts the obtained solution for the ML problem.

The following figures illustrate complete solution of ML problem where n and A are varying. Fig. 7 depicts increasing amplitude A (in steps of 10^{-4}) and fixed $n = 16$ versus THD (in percent) ($N = 20$). The solution is in 14 intervals for the amplitude A, where the ML problem has a solution (no other amplitude A solves this ML problem for $n = 16$), and Fig. 8 shows all switching configurations for all these intervals.

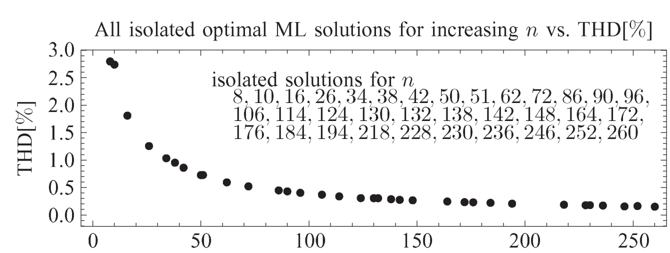

Fig. 9. All isolated optimal ML (five-level) solutions for increasing n versus THD (in percent): $A = 2.3$ and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

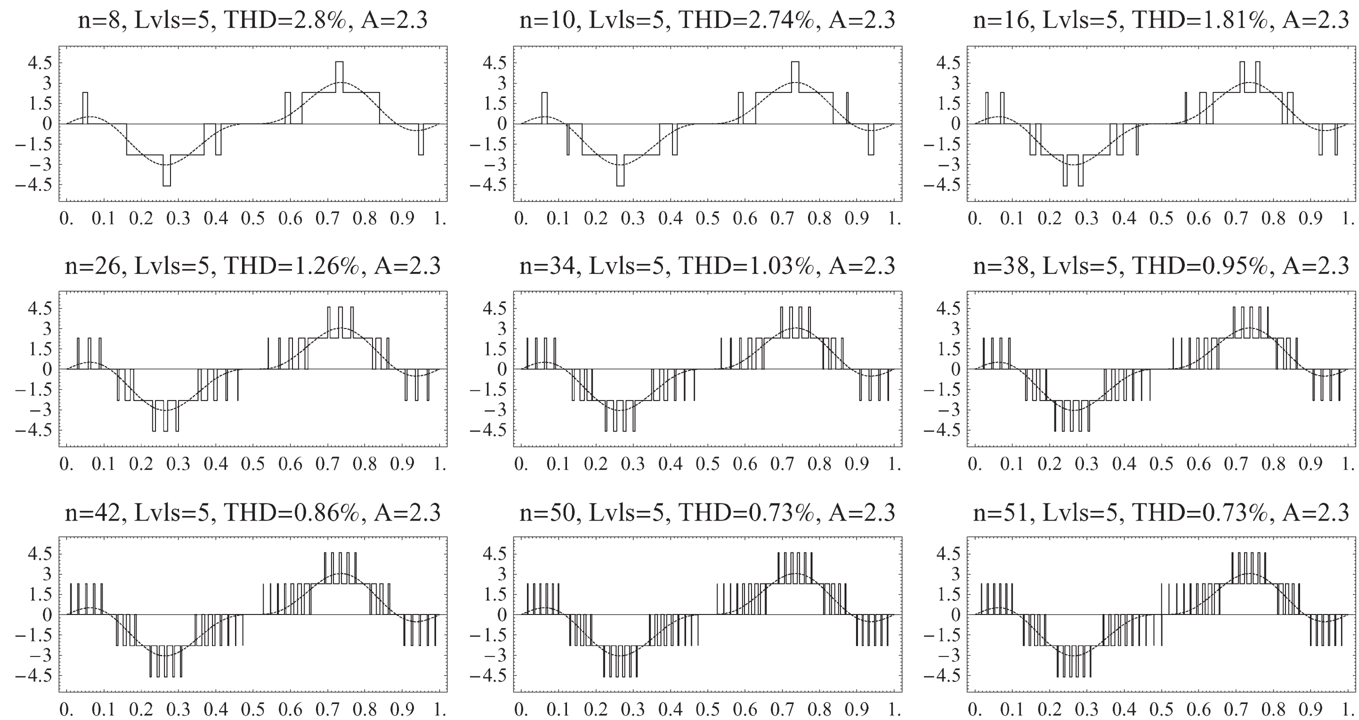

Fig. 10. All possible configurations of optimal ML waveforms for different n : $A = 2.3$ and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

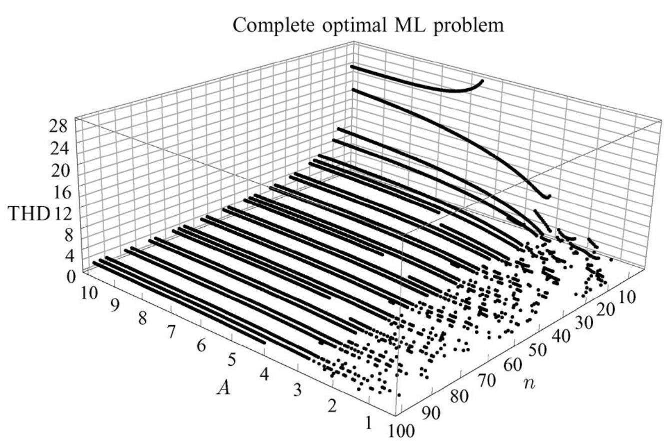

Fig. 11. Complete optimal ML solutions $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$ for varying n and A versus THD (in percent).

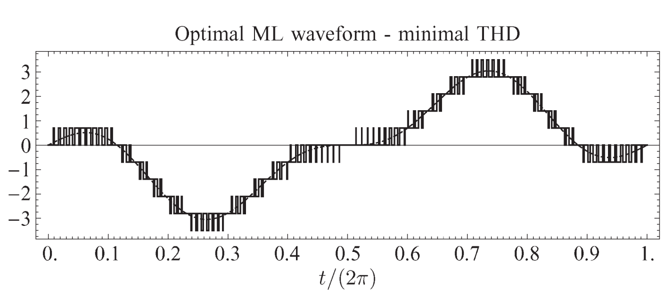

Fig. 12. Optimal ML with minimal THD (in percent): $A = 0.7$, $n = 96$, number of levels = 11, THD = 0.125%, and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

Fig. 9 depicts increasing number of switching n and fixed $A = 2.3$ versus THD. The first nine isolated solutions are given in Fig. 10.

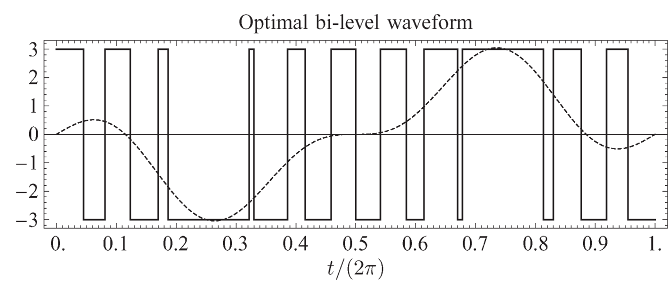

Fig. 13. Optimal bilevel waveform: $A = 3$, $n = 10$, THD = 11.96%, and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

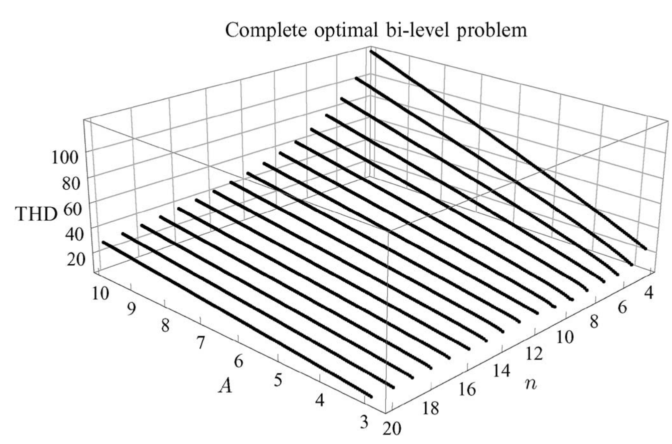

Fig. 14. Complete optimal bilevel solutions, varying n and A versus THD (in percent) and $(b_{f_1}, b_{f_2}, b_{f_3}) = (-2, 0.5, 1)$.

The complete solution (n is from 4 to 100, and A is from 0.05 to 10, with step 0.05) is visualized in Fig. 11, where a varying amplitude A and number of switching n versus THD are visualized. Fig. 12 show the ML signal with minimal THD.

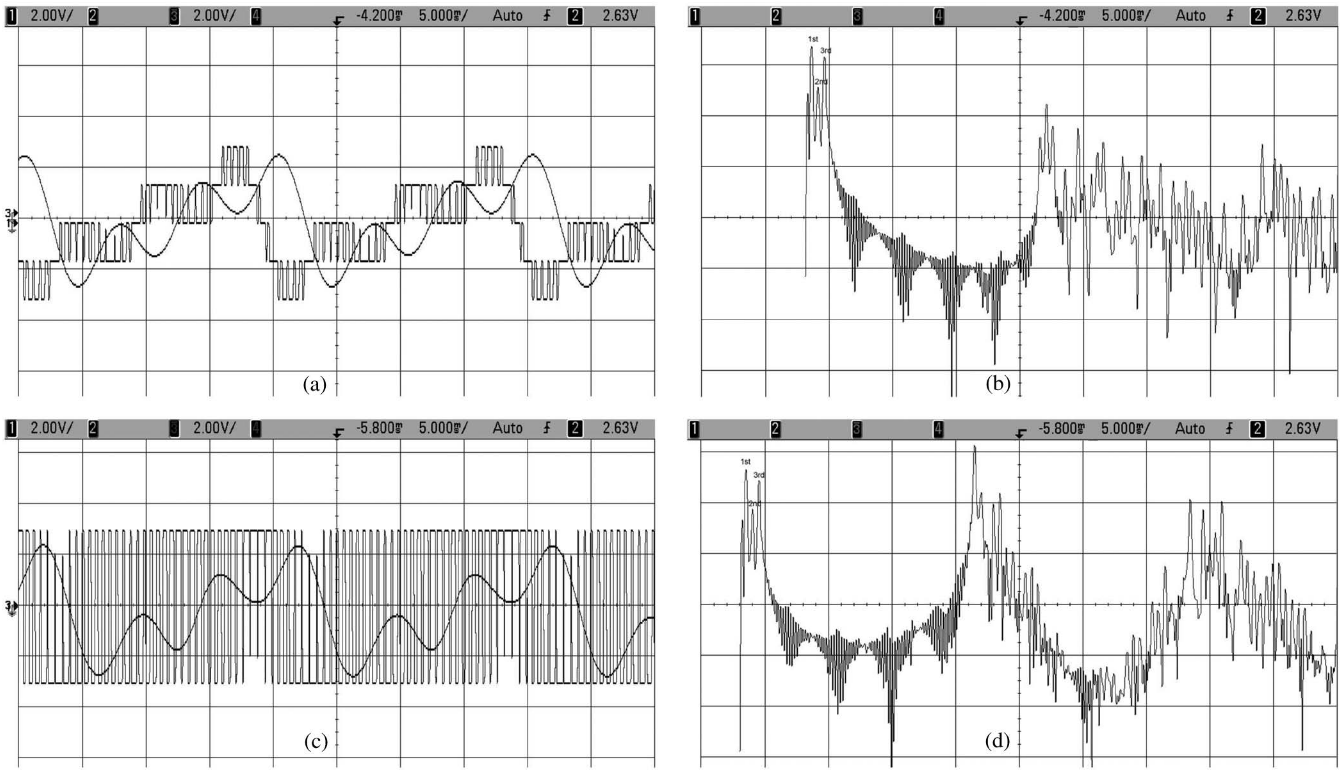

Fig. 15. Experimental results: output voltage (switching waveforms and filtered waveforms) and its spectrum. Baseband portion is $b_{f_{1,2,3}} = (1.5, -0.6, 1.2)$ and eliminated harmonics (zero band) are 4, 5, 6, . . ., 36. (a) and (b) Five-level waveform. (c) and (d) Bilevel waveform.

The results for the bilevel waveform (there is different solving procedure, see Section III-B) are depicted in Fig. 13, and the complete solution is in Fig. 14. We can see that there exist solutions in all cases (unlike ML), but THD is much worse in the ML case.

The *Mathematica*[2] package (all algorithms described in this paper) with other simulations and demo examples can be downloaded from the authors' webpages [35].

VI. Experimental Results

To verify the performance of the proposed algorithms, an experimental setup was built in the laboratory. It is composed of the Agilent 33120A waveform generator with related software Agilent IntuiLink WaveForm Editor installed on a laboratory personal computer.

In the experimental example, we solve the optimal five-level and bilevel problems for $b_{f_{1,2,3}} = (1.5, -0.6, 1.2)$ and $n = 36$ with a frequency of 50 Hz and $A = 1.5$ V and $A = 3$ V, respectively. According to proposed algorithms, we obtain the switching times $\overline{\alpha} = (0.000373, 0.000533, \ldots, 0.009668, 0.009784)$ and $\overline{\alpha} = (0.000279, 0.000502, \ldots, 0.009533, 0.009725)$, respectively. The offline fast Fourier transform (FFT) analysis of the experimental data shows that the THDs are 1.25% and 5.43.%, respectively, which are slightly larger than the theoretical values of 1.08% and 5.21.%, respectively, for given A. The solution is depicted in Fig. 15. Subsequently, the switching output waveform is filtered by the low-pass Butterworth filter (switched capacitor filter Maxim MAX291, eighth order), and the filtered output corresponds to the required baseband.

VII. Case Study: Active Filters

The main goal of active filters is the cancellation of noise or distortion of harmonic signals. These undesirable effects are consequences of disturbances or nonlinearities of load (see [36] and [37] for more details).

Let us consider the simplified principal scheme according to Fig. 16(a). The basic principle of active filters is based on generating harmonic signals with an amplitude opposite that of the undesirable harmonics so that they are canceled in total. This suitable signal is then generated as a filtered PWM or ML waveform that is easily and efficiently realizable.

Active filters are installed in a wide range of industrial and nonindustrial applications (pulp and paper facilities, chemical plants, steel plants, car industry, and banks or telecommunication centers due to the large number of computers and Uninterruptible Power Supply (UPS) systems).

Numerical Example

Let us consider electrical power grid $f = 50$ Hz and compensate the harmonic distortion caused by a set of drives. The fundamental harmonic in a power grid is deviated strongly by the odd[3] saw signal and in addition amplified tenth and fifteenth harmonics. The signal, which

[2]The *Mathematica* Web pages are in [34].

[3]If the analyzed signal is not odd, we can make odd extension and use our approach.

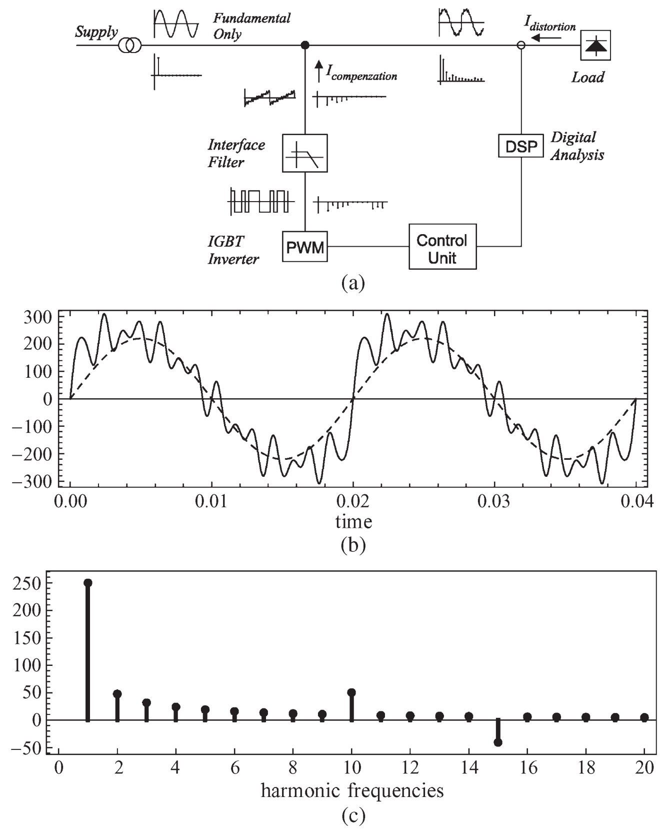

Fig. 16. (a) Diagram illustrating components of the connected active filter with waveforms showing cancellation of harmonics from load. (b) Fundamental harmonic and deviated fund. Harmonic in a power grid. (c) Spectrum of a deviated fundamental waveform.

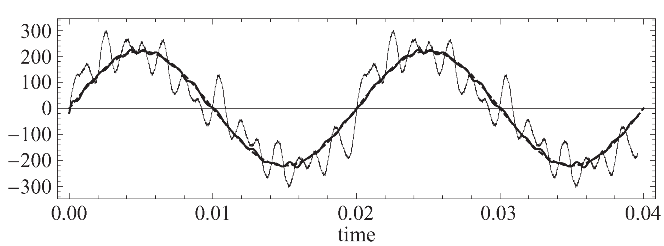

Fig. 17. Restored fundamental harmonic, filtered optimal odd, and filtered quarter-symmetric PWM waveform.

is biased, is depicted in Fig. 16(b). Its frequency amplitude spectrum $a_1, \ldots, a_{20}$ is depicted in Fig. 16(c), and it is $(250.1, 47.7, 31.8, 23.9, 19.1, 15.9, 13.6, 11.9, 10.6, 50.3, 8.7, 8.0, 7.3, 6.8, -40.6, 6.0, 5.6, 5.3, 5.0, 4.8, \ldots)$.

It is desirable to suggest appropriate switching $(\alpha_1, \ldots, \alpha_{220})$ of the odd bilevel PWM waveform so that after its filtration, we get harmonic signal with reverse amplitude spectrum $(b_1, b_2, \ldots, b_{20}, b_{21}, \ldots, b_{220}) = (-30.1, -47.7, \ldots, -4.8, 0, \ldots, 0)$. In this operation, we restrict the first 20 harmonics only, and the following 200 harmonics are zeroed. The nullity of higher harmonics is given because of consequent filtering (we use the Chebyshev filter of the fourth order with cutoff frequency $f_c = 23f$) of the odd bilevel waveform. The solution is depicted in Fig. 17. The solution obtained by a numerical algorithm for quarter-wave signals (see [5]) is also displayed in the figure. Apparently, the improvement in quality of filtration due to the results for odd harmonics presented in this paper is considerable compared to [5], where only quarter-symmetric waveforms are studied. The THD of odd symmetric waveform is 2.75% compared to the quarter symmetric 18.3% (the even harmonics are uncontrolled).

VIII. Complexity of the Optimal Odd ML Problem

The complexity analysis of the optimal odd ML problem follows. Solving the RHS of the system of composite sum of powers p_i [see (23)] takes $\mathcal{O}(nn_C)$ number of operations. The moments μ_i are computed in $\mathcal{O}(n^2)$ operations according to (47), but a significantly faster algorithm can be found. We can, for instance, use the fast Newton iteration method that takes only $\mathcal{O}(n \log n)$ operations (this method employs an FFT technique for polynomial multiplication) (see [38] and [39]). The computation of Hankel linear system takes $\mathcal{O}(n \log^2 n)$ number of operations (superfast algorithm; see [40] and [41]) or we can use the well-known Levinson–Durbin algorithm with complexity $\mathcal{O}(n^2)$ operations. The calculation of matrix equation with a triangular Hankel matrix takes $\mathcal{O}(n \log n)$ operations (see [40]). It is somewhat more intricate to establish the complexity for computations of the zeros of polynomials $V(y)$ and $W(y)$ because many algorithms of different complexity are available. For example, the algorithm based on computing the eigenvalues of the companion matrix takes $\mathcal{O}(n^3)$ operations. In contrast, the combination of three-term recurrence algorithm (which takes $\mathcal{O}(n^2)$ operations), employing the property of interlacing the zeros (if it is possible, but this property is not always guaranteed for FOPs), and the iterative Newton algorithm leads to a linear number of operations—we easily compute the zeroes in every step. Hence, the highest possible number of operations is considered during the computation of the recurrence formula.

It is important to mention that the solution of the Hankel system is ill-conditioned for high n, which restricts the computation in double precision real arithmetic. Therefore, either of the polynomials $V(y)$ and $W(y)$ is also ill-conditioned, and computation of their roots is difficult from numerical point of view. By using extended precision arithmetic, the range of n can be enlarged. However, we show that the solution can also be expressed as the solution to a Padé approximation problem and, consequently, introduce FOPs. Numerically stable algorithms using properties of FOPs should therefore exist and are subject to research now.

For a special case of the quarter-symmetric waveforms [5], it is possible to adopt these results and devise the solution of system of sums of odd powers that is needed for the solution of this problem. It is sufficient to put the odd harmonics equal to zero and compute the polynomial $W(y)$ only. Such a solution was described in [5] and [6], and our procedures cover their solution for $n_C = 1$ as a special case.

IX. Conclusion

Efficient algorithms for the optimal odd ML problem in the single-phase connection have been developed and studied in this paper. In Section III, we revealed that an efficient analytical solution can be found only for odd and quarter-wave symmetric waveforms with arbitrary number of levels. The quarter-wave symmetric case is solved in [5] and [6]. Therefore, we concentrated on more general odd symmetry waveforms, including all harmonics.

Both cases lead to the solution of special systems of composite sum of powers that are derived from generalization of the Newton's identity. We formulated and solved the problem via Padé approximation. The optimal switching times are the zeros of shifted diagonal Padé approximation polynomials $[k, k+1]_F(y) = V_k^{[k,k+1]}(y)/W_{k+1}^{[k,k+1]}(y)$ for an odd number of switching n and diagonal Padé approximation $[k,k]_F(y) = V_k^{[k,k]}(y)/W_k^{[k,k]}(y)$ for an even n. Due to the connection between the theory of Padé approximation and FOPs, we demonstrated that $V(y)$ and $W(y)$ are FOPs, and we formulated other methods for the solution of the optimal odd ML problem. Namely, we derived an appropriate three-term recurrence formula, a determinantal formula, and a formulation via eigenvalue computation. The obtained polynomials are FOPs.

The results are summarized as follow.

1) After variable transformations, the solution of the optimal odd ML problem is given by the zeros of two polynomials $W(y)$ and $V(y)$ that are suitably sorted.
2) The polynomials $W(y)$ and $V(y)$ are given by the shifted diagonal Padé approximation

$$[k, k+1]_f(y) = V_k^{[k,k+1]}(y)/W_{k+1}^{[k,k+1]}(y) = \exp\left(-\sum_{j=1}^{\infty}\frac{p_j}{j}y^j\right) = F(y) \qquad (76)$$

for odd n and by the diagonal Padé approximation

$$[k,k]_f(y) = V_k^{[k,k]}(y)/W_k^{[k,k]}(y) = F(y)$$

for even n, where $p_j = \sum_{i=1}^{k} y_i^j - \sum_{i=k+1}^{n} y_i^j,\ j = 1, \ldots, n$, is computed according to (25) for ML and (29) for the bilevel odd waveform.
3) The polynomials $V(y)$ and $W(y)$ also give the solution of a Padé approximation and therefore constitute a set of FOPs, where the polynomial $V_k^{[k,k+1]}(y)$ is the associated polynomial (or polynomial of the second kind) to $W_{k+1}^{[k,k+1]}(y)$ (polynomial of the first kind) for odd n. In the case of even n, the polynomials $V_k^{[k,k]}(y)$ and $W_k^{[k,k]}(y)$ are deduced from the adjacent family of FOPs $V_k^{(1)[k,k+1]}(y)$ and $W_{k+1}^{(1)[k,k+1]}(y)$.
4) The solution to the optimal ML problem can be obtained through the following:
   a) the Hankel system in (53) and (56) for odd n and in (59) and (60) for even n: the complexity of a fast algorithm being $\mathcal{O}(n \log n^2)$;
   b) the simple three-term recurrence relationship in (62) and (64) for odd n and in (66) and (69): the complexity being $\mathcal{O}(n^2)$ operations;
   c) the determinants of special polynomial matrices in (70) and (71) for odd n and in (72) and (73) for even n;
   d) the eigenvalues of special matrices in (74) and (75) for odd n.

It is also important to stress that our solution is consistent with the solution of [5] in the case of waveforms with quarter symmetry.

At the end of this paper, a numerical example and experimental verification results are presented. The numerical example illustrates a complete solution of the ML and bilevel PWM problem and the presented exact results could not be obtained without our fast analytical methods. Experimental results verified our expected behavior of optimal ML and bilevel PWM problem. An active filter case study then illustrates an advantage of our approach compared to an existing analytical scheme for quarter-symmetric waveforms.

Acknowledgment

The authors would like to thank M. Špiller, Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic, for his help and support with the experimental setup.

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