Controllability of Non-square Linear Systems

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Introduction

Consider a linear, time-invariant system (E, A, B):

$$E\dot{x}(t) = Ax(t) + Bu(t), \ t \geq 0$$

where
- $E, A \in \mathbb{R}^{q \times n}$
- $B \in \mathbb{R}^{q \times m}, \ \operatorname{rank} B = m$

- arise in network modelling, Petri nets, composite systems...

Motivation

The approach there is adapted to different tasks and leads to different conditions.

Systems with the internal degree of freedom, for example:

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + Bu(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} x_0$$

Fr, Ge: such a system is controllable for any B,
Mal, Ish : the non-uniqueness and indefiniteness of a solution leads to its uncontrollability.

some state variables can be described by some free functions, say $\theta_i(t)$

Let $x_1 = \theta(t)$

$$x = \left[\begin{array}{l} \theta(t) \\ \int_0^t \theta(\tau) + \int_0^t Bu(\tau) + x_{02} \end{array} \right]$$

Basic definitions

- (E, A, B) is called *regular* if $\det(sE - A) \neq 0$.

$:= (E_{f+s}, A_{f+s}, B_{f+s})$

Applying **the state feedback**

$$u(t) = Fx(t) + v(t),$$

where • $F \in \mathbb{R}^{m \times n}$, and $v(t)$ is a new external input

gives the closed-loop system $(E, A + BF, B)$:

$$E\dot{x}(t) = (A + BF)x(t) + Bv(t), \quad t \geq 0$$

- (E, A, B) is called *regularizable* if $\exists$ a state feedback: $(E, A + BF, B)$ is regular .

Definitions: reachability, impulse controllability

a state trajectory - the **continuously differentiable** function $x : t \mapsto x(t)$;

$\mathbb{S}_u$: the set of piecewise, sufficiently many times differentiable functions

Reachability

A state x_f is said to be *reachable* from a state x_s if there exists an input $u(t) \in \mathbb{S}_u$ and $T > 0 : \exists$ a state trajectory $x(t)$ on $[0, T]$ with $x(0) = x_s$ and $x(T) = x_f$.

The system is said to be *reachable* if every state $z \in \mathbb{R}^n$ is reachable from zero.

Impulse controllability

The system is said to be *impulse controllable*, if for any state x_0 there exists an input $u(t) \in \mathbb{S}_u$ such that the (distributional) solution is impulse-free.

Conditions

Controllability is usually defined as reachability of any state from any state, Yip, Cobb.

reachability $\equiv$ controllability (in regular/regularizable systems).

A regular $(E_{f+s}, A_{f+s}, B_{f+s})$ consists of fast and slow subsystem.

The system $(E_{f+s}, A_{f+s}, B_{f+s})$ is controllable iff

- $R_f \oplus R_s = \mathbb{R}^f \oplus \mathbb{R}^s = \mathbb{R}^n,$
  $R_k := \mathrm{Im}[B_k \;\; E_k B_k \;\; \dots \;\; E_k^{k-1} B_k], \;\; k = f, s$
- $\mathrm{Im} E_{f+s} + \mathrm{Im} B_{f+s} = \mathbb{R}^n,$
  $\mathrm{Im}[sE_{f+s} - A_{f+s}] + \mathrm{Im} B_{f+s} = \mathbb{R}^n \;\; \forall s \in \mathbb{C}.$

The regularizability $\Rightarrow$
A regularizable system is controllable provided that there exists F : the closed-loop system is regular and controllable in the regular sense.

Conditions

The system $(E_{f+s}, A_{f+s}, B_{f+s})$ is impulse controllable iff

- $R_f + \mathrm{Ker} E_f = \mathbb{R}^f$;
- $\mathrm{Im} E_f + \mathrm{Im} B_f + \mathrm{Ker} E_f = \mathbb{R}^f$.

controllability of (E_f, A_f, B_f) implies the impulse controllability, but not vise versa.

$\mathrm{Ker} E_f$ increases to $\mathbb{R}^f$ – the non-dynamic variables, i.e. $0\dot{x}_{nd} - Ix_{nd} = Bu$

The main feature of a system containing the internal degree of freedom is the non-uniqueness of the solution.

- $\theta_i(t)$ are not constrained as well as are not guaranteed to be continuously differentiable ones $\implies$ differences in the approaches
- $\theta_i(t)$ could be either continuously differentiable, impulsive or trivial ones $\implies$ the impulse controllability is not equivalent to the existence of an impulse eliminating feedback (as in the regularizable systems)

The definition of reachability accepts that the system is reachable along the trajectory which is generated by the system itself (that is by its internal degree of freedom included).

Controllability

A state x_f is said to be *controllable* from a state x_s if there exists an input $u(t) \in \mathbb{S}_u$ and a final time T such that the state trajectory $x(t)$ is unique on $[0, T]$ with $x(0) = x_s$ and $x(T) = x_f$.

The system is said to be *controllable* if every state is controllable from any state.

Definitions

Zeros, irreducibility

The *i.d. zeros* of the system $(E_{f+s}, A_{f+s}, B_{f+s})$ are the (finite or infinite) zeros of $\left[sE_{f+s} - A_{f+s}, \ \ -B_{f+s}\right]$.

A regular system (3) is said to be *irreducible* if it has no (finite or infinite) i.d. zeros.

The irreducibility of (E_f, A_f, B_f) - controllability at infinity , and of $(E_{f+s}, A_{f+s}, B_{f+s})$ is called as strong controllability.

The relationship between the controllability and strong controllability:

A regular system $(E_{f+s}, A_{f+s}, B_{f+s})$ is strongly controllable/irreducible if and only if

- (E_s, A_s, B_s) is controllable/irreducible
- (E_f, A_f, B_f) is controllable at infinite/irreducible/impulse controllable

Definitions, conditions

The system $(E_{f+s}, A_{f+s}, B_{f+s})$ is irreducible iff

$$\mathrm{rank}\left[sE_{f+s} - A_{f+s}, \;\; -B_{f+s}\right] = q \quad \forall s \in \mathbb{C} \cup \infty$$

Note: $\left[sE_{\varepsilon} - A_{\varepsilon}, \;\; -B_{\varepsilon}\right]$ is always of full row rank $\forall s \in \mathbb{C} \cup \infty$.

It is natural to define the free-response modes influenced by the internal degree of freedom, as the non-fixed zeros.

Strong controllability

The system is said to be *strongly controllable* iff it has no (finite, infinite i.d. or non-fixed) zeros.

Feedback Canonical Form

The feedback group (P, Q, G, F)

- $P,\ Q,\ G$ are invertible matrices over $\mathbb{R}$
- $F\ \in \mathbb{R}^{m\times n}$

Feedback canonical form (FCF) :

$$(P, Q, G, F) \circ (E, A, B) = (PEQ, P(A+BF)Q, PBG) \quad =:$$

$$=: (E_C,\ A_C,\ B_C)$$

FCF: 6 types of blocks of $sE_C - A_C$

(1)
$$\underbrace{\overbrace{\begin{bmatrix} \boldsymbol{s} & \boldsymbol{-1} & & \\ & \ddots & \ddots & \\ & & \boldsymbol{s} & \boldsymbol{-1} \end{bmatrix}}^{\epsilon_i+1}}_{\epsilon_i \text{ rows}}$$

(2)
$$\overbrace{\begin{bmatrix} \boldsymbol{s} & \boldsymbol{-1} & & \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{-1} \\ & & & \boldsymbol{s} \end{bmatrix}}^{\sigma_i} \Bigg\} \sigma_i \qquad \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix}$$

(3)
$$\overbrace{\begin{bmatrix} \boldsymbol{-1} & & \\ \boldsymbol{s} & \ddots & \\ & \ddots & \boldsymbol{-1} \\ & & \boldsymbol{s} \end{bmatrix}}^{q_i} \Bigg\} q_i + 1 \qquad \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{matrix}$$

(4)
$$\overbrace{\begin{bmatrix} \boldsymbol{-1} & \boldsymbol{s} & & \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{s} \\ & & & \boldsymbol{-1} \end{bmatrix}}^{p_i+1} \Bigg\} p_i + 1$$

(5)
$$\overbrace{\begin{bmatrix} \boldsymbol{s} & \boldsymbol{-1} & & \\ & \ddots & \ddots & \\ & & \ddots & \boldsymbol{-1} \\ \boldsymbol{-a_{i0}} & \boldsymbol{-a_{i1}} & \cdots & \boldsymbol{s-a_{li}} \end{bmatrix}}^{l_i} \Bigg\} l_i$$

(6)
$$\overbrace{\begin{bmatrix} \boldsymbol{s} & & \\ \boldsymbol{-1} & \ddots & \\ & \ddots & \boldsymbol{s} \\ & & \boldsymbol{-1} \end{bmatrix}}^{\eta_i} \Bigg\} \eta_i + 1$$

The form of B_C, indices in FCF

Matrix $\boldsymbol{B_C} := \text{blockdiag}\{0, B_\sigma, B_q, 0, 0, 0\}$, where

$$B_\sigma := \text{blockdiag}\left\{[0 \ldots 0 \ 1]^T \in \mathbb{R}^{\sigma_i}\right\}$$

$$B_q := \text{blockdiag}\left\{[0 \ldots 0 \ 1]^T \in \mathbb{R}^{q_i+1}\right\}$$

The quantities describing the blocks:

1. the **nonproper** indices, $\epsilon_1 \geq \ldots \geq \epsilon_{k_\epsilon} \geq 0$;
2. the **proper** indices, $\sigma_1 \geq \ldots \geq \sigma_{k_\sigma} > 0$;
3. the **almost proper** indices, $q_1 \geq \ldots \geq q_{k_q} \geq 0$;
4. the **almost nonproper** indices, $p_1 \geq \ldots \geq p_{k_p} > 0$;
5. the **fixed invariant** polynomials $\alpha_1(s) \triangleright \alpha_2(s) \triangleright \cdots \triangleright \alpha_{k_l}(s)$, $\alpha_i(s) = s^{l_i} + a_{il_i} s^{l_i - 1} + \cdots + a_{i1} s + a_{i0}$;
6. the **row minimal** indices of $[sE_C - A_C \ \ -B_C]$, $\eta_1 \geq \ldots \geq \eta_{k_\eta} \geq 0$.

Main results

The system (E, A, B) is controllable iff:

- $$(k_q \geq k_\epsilon) \wedge (k_l = 0) \wedge (k_p = 0) \wedge$$
  $$\wedge (k_q = 0 \oplus q_i = 0) \wedge (k_\eta = 0 \oplus \eta_i = 0)$$

- $$\bar{q} \geq n$$
  $$\mathrm{Im}\bar{E} + \mathrm{Im}\bar{B} = \mathbb{R}^{\bar{q}}$$
  $$\mathrm{Im}[s\bar{E} - \bar{A}] + \mathrm{Im}\bar{B} = \mathbb{R}^{\bar{q}} \quad \forall s \in \mathbb{C}$$

where $[\bar{E}\ \bar{A}\ \bar{B}] \in \mathbb{R}^{\bar{q}\times n}$ is of full row rank matrix such that

$$[E \quad A \quad B] = \begin{bmatrix} I_{\bar{q}} \\ Y \end{bmatrix} [\bar{E} \quad \bar{A} \quad \bar{B}] \tag{1}$$

with $\bar{E} \in \mathbb{R}^{\bar{q}\times n}$, $\bar{A} \in \mathbb{R}^{\bar{q}\times n}$, $\bar{B} \in \mathbb{R}^{\bar{q}\times m}$, $Y \in \mathbb{R}^{(q-\bar{q})\times \bar{q}}$.

Main results

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The system (E, A, B) is strong controllable iff:

- $$(k_q \geq k_\epsilon) \wedge (k_l = 0) \wedge (k_p = 0 \oplus p_i = 0) \wedge$$
  $$\wedge (k_q = 0 \oplus q_i = 0) \wedge (k_\eta = 0 \oplus \eta_i = 0)$$

- $$\bar{q} \geq n$$
  $$\mathrm{Im}\bar{E} + \mathrm{Im}\bar{B} + \bar{A}\mathrm{Ker}\bar{E} = \mathbb{R}^{\bar{q}}$$
  $$\mathrm{Im}[s\bar{E} - \bar{A}] + \mathrm{Im}\bar{B} = \mathbb{R}^{\bar{q}} \quad \forall s \in \mathbb{C}$$

where $[\bar{E}\ \bar{A}\ \bar{B}] \in \mathbb{R}^{\bar{q} \times n}$ is of full row rank matrix such that (1).

Some remarks

$$\bar{q} \geq n$$

The system (E, A, B) is *strong controllable* iff:

$$\bar{q} \geq n, \quad \text{rank}\left[s\bar{E} - \bar{A}, \quad \bar{B}\right] = \bar{q} \quad \forall s \in \mathbb{C} \cup \infty$$

- the concepts of irreducibility and strong controllability are still equivalent
- in such systems the equivalence of the impulse controllability and the existence of an impulse eliminating feedback still remains valid.