KYBERNETIKA — VOLUME 47 (2011), NUMBER 3, PAGES 370–384

NONLINEAR BAYESIAN STATE FILTERING WITH MISSING MEASUREMENTS AND BOUNDED NOISE AND ITS APPLICATION TO VEHICLE POSITION ESTIMATION

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The paper deals with parameter and state estimation and focuses on two problems that frequently occur in many practical applications: (i) bounded uncertainty and (ii) missing measurement data. An algorithm for the state estimation of the discrete-time non-linear state space model whose uncertainties are bounded is proposed. The algorithm also copes with situations when some measurements are missing. It uses Bayesian approach and evaluates maximum a posteriori probability (MAP) estimates of states and parameters. As the model uncertainties are supposed to have a bounded support, the searched estimates lie within an area that is described by the system of inequalities. In consequence, the problem of MAP estimation becomes the problem of nonlinear mathematical programming (NLP). The estimation with missing data reduces to the omission of corresponding inequalities in NLP formulation. The proposed estimation algorithm is applied to the estimation of a moving vehicle position when incomplete data from global positioning system (GPS) together with complete data from vehicle sensors are at disposal.

Keywords: non-linear state space model, bounded uncertainty, missing measurements, state filtering, vehicle position estimation

Classification: 93E11, 93E12

1. INTRODUCTION

A state space model is frequently used for a description of real systems. Whenever some internal variables cannot be measured directly and some parameters are unknown the need for their estimation arises. These unknown parameters and unobserved states are estimated using measured data, i. e., system inputs and outputs, as well as modeled dependencies among particular quantities.

The uncertainties of state evolution as well as observation model are often supposed to have normal distribution and the problem is then solved by means of the Kalman filtering (KF) [8] and its variants and extensions.

However, the unbounded support of the Gaussian distribution can cause difficulties if the estimated quantity is physically bounded as, for instance, it may give unreasonable negative estimates of naturally non-negative variable. There are several ways how to deal with this drawback.

In the KF framework, the state estimates are projected onto the constraint surface via quadratic programming [4]. Use of truncated probability density functions (pdf) is another way of solving this problem [18]. Here, the constraints are incorporated by the cutting off that part the pdf describing the state estimate that violates the constraints. In both cases, the key drawback of KF – it works well only when noise covariances are well chosen – is enhanced.

Constraints on estimates can be respected by using truncated prior pdf, which ensures that the posterior also satisfies the constraints. These constraints can be well faced when using Monte-Carlo sampling, alias particle filtering. It suffices to respect them within the accept/reject steps of the algorithm [10]. The Monte-Carlo methods require, however, a huge amount of samples to obtain acceptable results.

Other techniques, dealing with unknown-but-bounded equation errors, are used [20]. A bounded set is constructed containing the unknown states or parameters. The complexity of this set is very high so approximation is needed to obtain recursively feasible solution. The approximation by ellipsoid is proposed in [14], by a union of non-overlapping boxes in [2]. A recursive Kalman-like algorithm for the state estimation of linear models with disturbances bounded by ellipsoids is proposed in [1]. These methods lack a stochastic interpretation of involved quantities and estimates.

Missing data represent another practical problem faced in real applications requiring a state estimation. Then, standard estimation methods cannot be used straightforwardly and specific approaches have to be developed. The following examples, biased towards the addressed problem of bounded uncertainties, indicate some of them.

The paper [17] considers the problem of missing data within the framework of a class of uncertain discrete-time systems with a deterministic description of noise and uncertainty. A recursive scheme for constructing an ellipsoidal bounding the set of possible states state, consistent with the available measured data and the considered noise and uncertainty description, is proposed.

In [19], KF with intermittent observations is considered. There, the existence of a critical value for the arrival rate of the observations is shown, beyond which a transition to an unbounded state error covariance occurs.

In [23], authors propose a robust filter for linear uncertain discrete-time stochastic systems. The parameter uncertainties are allowed to be norm-bounded. The system measurements may be missing at any sample time with a known probability of their occurrence. This filtering problem reduces to the solutions of a couple of algebraic Riccati-like inequalities or linear matrix inequalities.

The paper [5] reviews estimation problems with missing, or hidden data. The problem is formulated in the context of Markov models. Two interrelated issues are considered, namely, the state estimation given the measured data and model parameters, and the parameter estimation given the measured data alone. The measured data can be incomplete. Various combinations of discrete and continuous states and observations are considered.

In spite of the fact that bounds and missing measurement occur jointly in practice it seems that there is no established methodology coping practically with both bounded uncertainty and missing data. In [13], author proposes a simple algorithm for the estimation of discrete-time *linear* state space model with bounded uncertainty when some measurements are missing. The Bayesian approach is used and MAP estimates are evaluated. The current paper extends this approach to a state space model with nonlinear state equation. The proposed algorithm is applied to the same technical problem as in [13], i. e. vehicle position estimation. We aim to obtain more precise state estimates. The refined non-linear model opens this possibility as it utilizes better information contained in the available data.

2. PRELIMINARIES

2.1. Notation

Throughout the paper, the following notation is used:

$\equiv$	equality by definition
$\propto$	equality up to a constant factor (proportionality)
$\mathbf{z}^*$	a set of $\mathbf{z}$-values, $\mathbf{z} \in \mathbf{z}^*$, $\mathbf{z}$ is a column vector
$\mathbf{z}_t$	value of $\mathbf{z}$ in discrete time instant t; $t \in t^* \subset \{0, 1, 2, \dots T\}$, $T < \infty$
$\hat{\mathbf{z}}_t$	estimate of $\mathbf{z}_t$;
$\mathbf{z}^{k:l}$	the ordered sequence; $\mathbf{z}^{k:l} \equiv [\mathbf{z}'_k, \mathbf{z}'_{k+1}, \dots, \mathbf{z}'_l]'$, $0 \le k \le l$
$'$	transposition
$\underline{\mathbf{z}}$, $\overline{\mathbf{z}}$	lower and upper bound on $\mathbf{z}$, respectively; inequalities like $\mathbf{z} \ge \underline{\mathbf{z}}$ are meant entry-wise
$\mathbf{0}_{(\alpha)}$, $\mathbf{1}_{(\alpha)}$	column vector of zeros and ones of the indicated size, respectively; index can be omitted when the vector size is obvious from context
$\mathbf{0}_{(\alpha,\beta)}$	zero matrix of the indicated dimensions
$\mathbf{I}_{(\alpha)}$	square identity matrix of the order α
$\mathcal{R}^n$	n-dimensional real space
$f(\cdot\|\cdot)$	probability density functions (pdf); respective pdfs are distinguished by the argument names; no formal distinction is made between a random variable, its realization and pdf argument

Integrals used are always definite and multivariate ones. The integration domain coincides with the support of the pdf in its argument.

2.2. Calculus with pdfs

Let us consider the joint pdf $f(a, b, c)$. For any $(a, b, c) \in (a, b, c)^*$, the following relations between pdfs hold [9]:

Chain rule

$$f(a, b|c) = f(a|b, c)f(b|c) = f(b|a, c)f(a|c)$$

Marginalization

$$f(b|c) = \int f(a,b|c)\mathrm{d}a, f(a|c) = \int f(a,b|c)\mathrm{d}b$$

Bayes rule

$$f(b|a,c) = \frac{f(a|b,c)f(b|c)}{f(a|c)} = \frac{f(a|b,c)f(b|c)}{\int f(a|b,c)f(b|c)\,\mathrm{d}b} \propto f(a|b,c)f(b|c). \tag{1}$$

2.3. Basics of Bayesian learning

In Bayesian view [3, 9], the system is described by probability density functions (pdfs). The quantities describing the system consist generally of observable outputs $\mathbf{y}^{1:T}$, optional inputs $\mathbf{u}^{1:T}$ and internal quantities that are never observed directly. The internal quantities consist of system states $\mathbf{x}^{0:T}$ and a time invariant unknown parameters $\mathbf{\Theta}$. The collection of the outputs and inputs is called data and denoted $\mathbf{d}^{1:T}$, i.e., $\mathbf{d}_t = (\mathbf{y}_t, \mathbf{u}_t)$, $t \in t^* = \{1, \ldots, T\}$. The joint pdf

$$f(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \mathbf{\Theta})$$

describing both observed and internal quantities can be decomposed onto a product of the following elements:

- observation model
$$\{f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \mathbf{\Theta})\}_{t\in t^*} \tag{2}$$
- time evolution model
$$\left\{f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \mathbf{\Theta})\right\}_{t\in t^*} \tag{3}$$
- controller
$$\{f(\mathbf{u}_t|\mathbf{d}^{1:t-1}) \equiv f(\mathbf{u}_t|\mathbf{d}^{1:t-1}, \mathbf{x}^{0:t-1}, \mathbf{\Theta})\}_{t\in t^*} \tag{4}$$
here the validity of the natural conditions of control is supposed [9], i.e., $\mathbf{x}^{0:t-1}$ and $\mathbf{\Theta}$ are unknown to the controller
- prior pdf
$$f(\mathbf{x}_0, \mathbf{\Theta}). \tag{5}$$

Under (2) – (5), it holds

$$\begin{aligned} f(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \mathbf{\Theta}) &= f(\mathbf{x}_0, \mathbf{\Theta}) \prod_{t=1}^{T} f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \mathbf{\Theta}) \\ &\quad \times f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \mathbf{\Theta}) f(\mathbf{u}_t|\mathbf{d}^{1:t-1}) \\ &\propto f(\mathbf{x}_0, \mathbf{\Theta}) \prod_{t=1}^{T} f(\mathbf{y}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_t, \mathbf{\Theta}) f(\mathbf{x}_t|\mathbf{u}_t, \mathbf{d}^{1:t-1}, \mathbf{x}_{t-1}, \mathbf{\Theta}). \end{aligned} \tag{6}$$

As the controller does not depend on the internal quantities $\mathbf{x}_t$ and $\mathbf{\Theta}$, it plays no role in estimation. Therefore, the knowledge of the controller is not required. Only, the generated input values have to be known.

The Bayesian state and parameter estimation works with characteristics of the joint pdf (6). This pdf combines prior information in $f(\mathbf{x}_0, \mathbf{\Theta})$, theoretical knowledge described by both observation (2) and time evolution (3) models and observed data $\mathbf{d}^{1:T}$ by using deductive rules of the calculus with pdfs (1).

2.4. Discrete-time state-space model with bounded innovations

We consider a discrete-time state space model that describes a given system with l-dimensional input $\mathbf{u}_t$, m-dimensional state $\mathbf{x}_t$ and n-dimensional output $\mathbf{y}_t$ by the following non-linear state (7) and linear output (8) equations in the discrete time instants $t \in t^* = 1, 2, \ldots, T$

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \mathbf{w}_t \tag{7}$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{e}_t \tag{8}$$

where g is a real vector function, $g : \mathcal{R}^{m+l} \rightarrow \mathcal{R}^m$;
$\mathbf{C}$ is known model matrix of an appropriate dimensions;
$\mathbf{w}_t$, $\mathbf{e}_t$ are vectors of the state and output noises respectively; they are zero mean, mutually conditionally independent and identically distributed.

Here, $\mathbf{w}_t$, $\mathbf{e}_t$ are assumed to have uniform distribution on a multivariate box with the center $\mathbf{0}$ and unknown half-widths of the support intervals $\mathbf{q}$ and $\mathbf{r}$, respectively, i.e.,

$$f(\mathbf{w}_t|\mathbf{q}) = \mathcal{U}(\mathbf{0}, \mathbf{q}), \quad f(\mathbf{e}_t|\mathbf{r}) = \mathcal{U}(\mathbf{0}, \mathbf{r}). \tag{9}$$

These assumptions complete the state and output equations into state evolution and observation models, respectively. Note that we make here no formal distinction of a random variable, its realization and pdf argument, see Section 2.1.

Further, we suppose that $\mathbf{x}_0$, $\mathbf{q}$, and $\mathbf{r}$ are a priori mutually independent and that it holds

$$\underline{\mathbf{x}}_0 \leq \mathbf{x}_0 \leq \overline{\mathbf{x}}_0,\ \mathbf{0} \leq \mathbf{q} \leq \overline{\mathbf{q}},\ \mathbf{0} \leq \mathbf{r} \leq \overline{\mathbf{r}},\ \{\underline{\mathbf{x}} \leq \mathbf{x}_t \leq \overline{\mathbf{x}}\}_{t \in t^*}. \tag{10}$$

Note that restrictions in (10) are defined by the user so that they reflect the reality. These known optional values specify user's prior information.

Equations (7) and (8) together with the assumptions (9) and (10), define the state uniform model (SU model).

We introduce the column vector $\mathbf{X}$ as follows

$$\mathbf{X} = \left[\ \left(\mathbf{x}^{0:T}\right)' \quad \mathbf{q}' \quad \mathbf{r}'\ \right]'. \tag{11}$$

The joint pdf (6) of data $\mathbf{d}^{1:T}$, the state trajectory $\mathbf{x}^{0:T}$, and unknown parameters $\mathbf{\Theta} = [\mathbf{q}', \mathbf{r}']'$ of the SU model takes the form

$$f\left(\mathbf{d}^{1:T}, \mathbf{x}^{0:T}, \Theta\right) \propto \left[\prod_{i=1}^{m} q_i \prod_{j=1}^{n} r_j\right]^{-T} \chi(\mathcal{S}) \tag{12}$$

where m, n are the lengths of the state and output vector, respectively, $\chi(\mathcal{S})$ is the indicator of the support $\mathcal{S}$ of this pdf.

The set $\mathcal{S}$ is a set of $\mathbf{X}$ (11) such that (for given realization $\mathbf{u}^{1:T}$, $\mathbf{y}^{1:T}$) the noise terms in (7) and (8) are inside multivariate box defined by (9) and (10), i. e.,

$$\mathcal{S} = \{\mathbf{X} \in \mathcal{S}0; \forall\, t \in t^* : |\mathbf{x}_t - g(\mathbf{x}_{t-1}, \mathbf{u}_t)| \leq \mathbf{q},\ |\mathbf{y}_t - \mathbf{C}\mathbf{x}_t| \leq \mathbf{r}\}\,. \tag{13}$$

where $\mathcal{S}0$ is the set of $\mathbf{X}$ that meet (10).

The linear version of SU model was introduced in [12] and algorithms for both state filtering and parameters estimations were designed. The extension considered in this paper uses a non-linear state equation and proposes an algorithm for the state estimation of this model applicable even when some measurements are missing.

3. ESTIMATION OF SU MODEL WITH MISSING DATA

3.1. One-shot state and parameter estimation

We suppose that the considered system is described by the SU model (12). We aim to estimate states $\mathbf{x}^{0:T}$ and the noise bounds $\mathbf{q}, \mathbf{r}$, i. e. vector $\mathbf{X}$ (11). We focus on a maximum a posteriori (MAP) estimation, see e. g. [3], that provides a point estimate of the internal quantity $\mathbf{X}$. The MAP estimate $\hat{\mathbf{X}}_{\mathrm{MAP}}$ of $\mathbf{X}$ with linearized logarithm of a posteriori pdf has the following form [12]

$$\hat{\mathbf{X}}_{\mathrm{MAP}} = \arg\,\min_{\mathbf{X}\in\mathcal{S}} \left(\sum_{i=1}^{m} q_i + \sum_{j=1}^{n} r_j \right), \tag{14}$$

where m, n are the lengths of the state and output vector, respectively, $\mathcal{S}$ is given by (13).

Note that the log-likelihood to be maximized by the MAP estimate is identical to that for linear case in [12]. The maximization differs because of the involved nonlinearity of the state evolution, see Section 3.3.

3.2. On-line state and parameter estimation

The real-time (on-line) estimation provides the state and parameter estimates in each time step. We use a moving horizon estimator principle [16] and perform the Bayesian estimation on a sliding window of the length $\delta \geq 1$, i. e., for estimating the states $\mathbf{x}^{t-\delta:t}$, $t = \delta+1, \ldots, T$ and parameters $\mathbf{q}$, $\mathbf{r}$, we use the data $\mathbf{d}^{t-\delta:t}$ and prior information on $\mathbf{x}_{t-\delta-1}$, $\mathbf{q}$, $\mathbf{r}$.

We denote an estimated quantity as $\mathbf{X}_t$, $t \in t^*$. It has the following form, cf. $\mathbf{X}$ in (11),

$$\mathbf{X}_t = \left[\ \left(\mathbf{x}^{t-\delta-1:t}\right)' \quad \mathbf{q}' \quad \mathbf{r}'\ \right]'. \tag{15}$$

The superfluous state $\mathbf{x}_{t-\delta-1}$ and data item $\mathbf{d}_{t-\delta-1}$ from the previous estimation step are integrated out from the posterior pdf in every time step t. This integration induces non-uniform term in the posterior pdf. In the time instant $t \in t^*$, this term is described by a piecewise polynomial function containing t powers of this state. With increasing t, the estimation becomes intractable because of increasing complexity of the support of the posterior pdf. An approximation of the non-uniform term in each step is applied [12]. It consists in the replacing of the oldest state by its point estimate from the previous step. The MAP estimate $\hat{\mathbf{X}}_t$ of $\mathbf{X}_t$ with linearized logarithm of a posteriori pdf has then the following form

$$\hat{\mathbf{X}}_t = \arg\min_{\mathbf{X}_t \in \tilde{\mathcal{S}}_t} \left(\sum_{i=1}^{m} q_i + \sum_{j=1}^{n} r_j \right) \tag{16}$$

where $\tilde{\mathcal{S}}_t$ is constructed similarly to (13).

For $\tau^* \equiv \{t-\delta, \dots, t\}$, $t^* \equiv \{\delta+1, \dots, T\}$, $1 < \delta \le T$, $\tilde{\mathcal{S}}_t$ is a set of $\mathbf{X}_t$ (15) such that (for given realization $\mathbf{u}^{t-\delta:t}$, $\mathbf{y}^{t-\delta:t}$ in the time instant t) the noise terms in (7) and (8) are inside multivariate box defined by (9) and (18), i. e.,

$$\tilde{\mathcal{S}}_t = \left\{ \mathbf{X}_t \in \tilde{\mathcal{S}}0_t; \forall\, \tau \in \tau^* : |\mathbf{x}_\tau - g(\mathbf{x}_{\tau-1}, \mathbf{u}_\tau)| \le \mathbf{q},\ |\mathbf{y}_\tau - \mathbf{C}\mathbf{x}_\tau| \le \mathbf{r} \right\}. \tag{17}$$

For $t \in t^*$, $\tilde{\mathcal{S}}0_t$ is a set of $\mathbf{X}_t$ (15) that meet requirements

$$\mathbf{x}_{t-\delta-1} = \hat{\mathbf{x}}_{t-\delta-1},\ \mathbf{0} \le \mathbf{q} \le \overline{\mathbf{q}},\ \mathbf{0} \le \mathbf{r} \le \overline{\mathbf{r}}, \{\underline{\mathbf{x}} \le \mathbf{x}_\tau \le \overline{\mathbf{x}}\}_{\tau \in \tau^*}, \tag{18}$$

where $\hat{\mathbf{x}}_{t-\delta-1}$ is the point estimate of $\mathbf{x}_{t-\delta-1}$ from the previous step.

Again, the maximized log-likelihood is identical to the one for linear case in [12] but the optimization discussed in the following section differs because of the involved nonlinearity.

Note that in the time step t we are interested only in the newest state estimate, i. e. $\hat{\mathbf{x}}_t$, from the whole estimate $\hat{\mathbf{X}}_t$.

3.3. MAP estimation as a problem of mathematical programming

Here, we present a solution of an on-line MAP estimate (16). The solution of one-shot estimation (14) can be performed as a special case of the on-line version for the window of length $\delta = T - 1$. In this case an on-line version is identical to the off-line version with $\mathbf{x}_0 \equiv \hat{\mathbf{x}}_{T-\delta-1}$.

The MAP estimation can be given the following nonlinear mathematical programming (NLP) form [4]

$$\begin{aligned} &\text{Find a vector } \hat{\mathbf{X}}_t \text{ such that } \mathbf{J}'\hat{\mathbf{X}}_t = \sum\nolimits_{i=1}^{m} q_i + \sum\nolimits_{j=1}^{n} r_j \ \to \min \\ &\text{while } \mathcal{A}\hat{\mathbf{X}}_t \le \mathbf{b}_t,\ \mathcal{C}(\hat{\mathbf{X}}_t) \le \mathbf{0},\ \underline{\mathbf{X}} \le \hat{\mathbf{X}}_t \le \overline{\mathbf{X}}, t \in t^*, \\ &\text{and with given realizations of } \mathbf{u}^{1:T},\ \mathbf{y}^{1:T} \end{aligned} \tag{19}$$

where

(i) $$\mathbf{J}' \equiv [\mathbf{0}'_{((\delta+2)m)}, \mathbf{1}'_{(m+n)}]$$

(ii) $\mathcal{A}$ and $\mathbf{b}_t$ are known matrix and vector, respectively; they result from the linear part of inequalities describing the set $\tilde{\mathcal{S}}_t$ (17) and have the following form

$$\mathcal{A}\hat{\mathbf{X}}_t \le \mathbf{b}_t \quad \text{with} \quad \mathcal{A} = \left[\ \mathcal{A}1 \quad \mathcal{A}2\ \right], \tag{20}$$

with

$$\begin{aligned}
\mathcal{A}1 &= \mathcal{R}_m(\mathbf{I}_{(\delta+1)} \otimes \mathbf{K} \otimes \mathbf{C}),\\
\mathcal{A}2 &= -\mathbf{1}_{(2(\delta+1))} \otimes \mathcal{L}_m(\mathbf{I}_{(n)}),\\
\mathbf{b}_t &= \left[\mathbf{I}_{(\delta+1)} \otimes \mathbf{K} \otimes \mathbf{I}_{(n)}\right] \mathbf{y}^{t-\delta:t},
\end{aligned}$$

where $\otimes$ denotes Kronecker product;
$\mathcal{R}_{col}(\mathbf{M})$ and $\mathcal{L}_{col}(\mathbf{M})$ are operators adding *col* zero columns to the matrix $\mathbf{M}$ from the right and left, respectively; $\mathbf{K} \equiv [1 \ -1]'$; $\mathbf{C}$ is model matrix defined in (8).
(iii) $\mathcal{C}$ is a real vector function that corresponds to the nonlinear part of inequalities describing $\tilde{\mathcal{S}}_t$ (17), i. e.,

$$\begin{array}{rrl}
\mathbf{x}_{\tau-\delta} - g(\mathbf{x}_{\tau-\delta-1},\, \mathbf{u}_{\tau-\delta}) & -\mathbf{q} & \leq \mathbf{0}\\
-\mathbf{x}_{\tau-\delta} + g(\mathbf{x}_{\tau-\delta-1},\, \mathbf{u}_{\tau-\delta}) & -\mathbf{q} & \leq \mathbf{0}\\
\vdots & & \\
\mathbf{x}_{\tau} - g(\mathbf{x}_{\tau-1},\, \mathbf{u}_{\tau}) & -\mathbf{q} & \leq \mathbf{0}\\
-\mathbf{x}_{\tau} + g(\mathbf{x}_{\tau-1},\, \mathbf{u}_{\tau}) & -\mathbf{q} & \leq \mathbf{0}
\end{array}$$

(iv) $\underline{\mathbf{X}}$, $\overline{\mathbf{X}}$ are known vectors; they stem from the set $\tilde{\mathcal{S}}0_t$ (18) and have the following form

$$\underline{\mathbf{X}} = \begin{bmatrix} \hat{\mathbf{x}}_{t-\delta-1} \\ \mathbf{1}_{(\delta+1)} \otimes \underline{\mathbf{x}} \\ \mathbf{0}_{(m)} \\ \mathbf{0}_{(n)} \end{bmatrix}, \quad \overline{\mathbf{X}} = \begin{bmatrix} \hat{\mathbf{x}}_{t-\delta-1} \\ \mathbf{1}_{(\delta+1)} \otimes \overline{\mathbf{x}} \\ \overline{\mathbf{q}} \\ \overline{\mathbf{r}} \end{bmatrix}. \tag{21}$$

To solve (19), we use the MATLAB[1] function "fmincon" from optimization toolbox.

3.4. State and parameter estimation with missing data

The problem of missing measurement data can be easily incorporated into the SU model estimation algorithm. The missing measurement causes that output equality (8) is missing in given time instant. This fact influences matrices $\mathcal{A}1$, $\mathcal{A}2$ and vector $\mathbf{b}_t$ in (19). If N measurements are missing, then $2N$ rows in each above mentioned matrices and vector are omitted.

To prevent the state estimates divergence in the case of measurement outage, we choose the length sliding window δ so that data are present both at the beginning and end of estimated time interval. If no outages are present, the on-line state estimation runs with user defined δ. As soon as a measurement outage occurs, then δ is increasing up to δ_{OUT} until new measurements are coming. Then, state estimation with δ_{OUT} is performed. After it, we continue again in the on-line estimation with the sliding window of the length δ till the next data outage.

The introduced delay in supplying the state estimates causes no harm in the considered application.

[1]MATLAB is a high-level technical computing language and interactive environment for algorithm development, data visualization, data analysis, and numeric computation, see www.mathworks.com.

4. ESTIMATION OF A MOVING VEHICLE POSITION

Today, an importance of the precise tracking of moving vehicle is growing. The knowledge of the vehicle position is required in many practical application, e. g. in a navigation or in a vehicle trajectory reconstruction. There are many methods how to solve this task using various data sources and their combination, for illustration see eg. [6, 7, 11].

Frequently, a moving vehicle position is determined by means of global positioning system (GPS). The GPS provides the position directly in the Cartesian coordinates but signal outages can occur in the data that are caused e. g. by big trees or buildings that inhibit the signal receiving.

Inertial navigation system (INS) is often an alternative data source. INS is a navigation aid that uses a computer, motion sensors (accelerometers) and rotation sensors (gyroscopes) to continuously calculate via dead reckoning the position, orientation, and velocity (direction and speed of movement) of a moving object without the need for external references. The starting location and the initial orientation (azimuth) have to be known. But the data is relative and therefore, the estimation error has a cumulative character.

Many estimation algorithms combining the GPS/INS data were developed. GPS corrects the errors in INS, INS supplies data during GPS signal outages. For estimation purposes, a model is usually constructed that uses kinematics laws, i. e., it is not concerned with the causing forces. The estimation algorithms mostly use a various modification of Kalman filter, see e. g. [15, 21, 22]. Nevertheless, as already mentioned above, KF is very sensitive to tuning covariances.

This demanding real-life problem is an ideal test case of the underlying general theory. It has oriented us to estimation based on SU models. In [13], a promising but still insufficiently precise solution for *linear* SU model is described. Here, we make a further step and propose a method for a reconstruction of the moving vehicle position based on estimation of the *nonlinear* SU model (12). Comparing to [13], it allows us to use more adequate selection of inputs and states. We focus on the precise estimation of the position of a moving ground vehicle. We aim to propose a simple algorithm that uses a readily available data sources.

As additional data source during GPS data outages, we use data from vehicle sensors. These data are provided by controller area network (CAN) that is a bus network that connects devices, sensors and actuators in a vehicle for control applications. The CAN data include a complete information about vehicle velocity, yaw rate and lateral acceleration. The CAN measurements are at disposal during the whole driving time. These data are an alternative to above mentioned INS data.

4.1. Non-linear SU model of a moving vehicle

Here, we construct the state (7) and output (8) equations that relate the vehicle movements to GPS/CAN data. In the model, we neglect altitude changes and use the Cartesian coordinate system. The list of available data is summarized in Table 1. Azimuth φ is the horizontal angular distance of vehicle moving from a northern direction, measured clockwise.

Table 1. GPS/CAN data – usage in the model.

quantity (unit)	notation	source	in model as
true position (m)	$(p_x,\, p_y)$	—	state
true azimuth (rad)	φ	—	state
measured position (m)	$(\tilde{p}_x,\, \tilde{p}_y)$	GPS	output
measured azimuth (rad)	$\tilde{\varphi}$	GPS	output
velocity (m/s^2)	v	CAN	input
yaw rate [rad/s]	ω	CAN	input

Using available data, a vehicle movement is characterized by a time evolution of the position vector $\mathbf{p} = (p_x, p_y)$, whereas the following relationships hold

$$\dot{p}_x = v \sin\varphi, \;\; \dot{p}_y = v \cos\varphi, \;\; \dot{\varphi} = \omega,$$

where $\dot{z}$ means a time derivative of z.

The above given relations cannot be used directly in the discrete-time equations (7) and (8). Therefore, we approximate the differential equations by the difference ones. Then, using assignment from the Table 1, the non-linear state equation (7) with $\mathbf{x}_t = [p_{x;t}\; p_{y;t}\; \varphi_t]'$, $\mathbf{u}_t = [v_t\; \omega_t]'$, $\mathbf{w}_t = [w_{x;t}\; w_{y;t}\; w_{\varphi;t}]'$ takes the following form

$$\begin{array}{llll} p_{x;t} & = p_{x;t-1} & + h v_t \sin\varphi_{t-1} & + w_{x;t} \\ p_{y;t} & = p_{y;t-1} & + h v_t \cos\varphi_{t-1} & + w_{y;t} \\ \varphi_t & = \varphi_{t-1} & - h\, \omega_t & + w_{\varphi;t} \end{array} \tag{22}$$

where h is the length of the time step, i. e., the time difference between two subsequent time instants of measurement, labelled by t and $t-1$.

The linear output equation (8) with $\mathbf{y}_t = [\tilde{p}_{x;t}\; \tilde{p}_{y;t}\; \tilde{\varphi}_t]'$, $\mathbf{C}$ equal to the unit matrix of size 3, $\mathbf{e}_t = [e_{x;t}\; e_{y;t}\; e_{\varphi;t}]'$ is as follows

$$\mathcal{I}_t \begin{bmatrix} \tilde{p}_{x;t} \\ \tilde{p}_{y;t} \\ \tilde{\varphi}_t \end{bmatrix} = \mathcal{I}_t \begin{bmatrix} p_{x;t} & + e_{x;t} \\ p_{y;t} & + e_{y;t} \\ \varphi_t & + e_{\varphi;t} \end{bmatrix} \tag{23}$$

where $\mathcal{I}_t$ is the measurement indicator: $\mathcal{I}_t = \mathbf{I}_{(n)}$ if the GPS data are available, $\mathcal{I}_t = \mathbf{0}_{(n,n)}$ otherwise; n is the length of the output vector.

This model is estimated using the technique described in Sections 3.3 and 3.4. To successfully run NLP, a starting point $\hat{\mathbf{X}}_0$ of the optimization has to be set appropriately. Improper setting of $\hat{\mathbf{X}}_0$ causes numerical instability. In experiments, we set the $\hat{\mathbf{X}}_0$ in the following way. We estimated the vehicle position with the simplified linear SU model proposed in [13] first and obtain $\hat{\mathbf{X}}_{\mathrm{LIN}}$. Here, the original linear SU model has been extended by one linear equation for an azimuth estimation. Then we set $\hat{\mathbf{X}}_0 = \hat{\mathbf{X}}_{\mathrm{LIN}}$ and run NLP.

5. EXPERIMENTS

5.1. Data description

For experiments, we use real data obtained from Škoda-Auto Inc. according to the Table 1. GPS data are provided with frequency 10 Hz and centimeter accuracy. CAN data are provided with the frequency 50 Hz. Here, we use only these CAN data entries that are in time with GPS data entries. During the experiments, we have a complete GPS data set at disposal. We simulate data outages by an artificial omission of some GPS data items. Afterwards, we run an estimation. The estimation is performed off-line on such part of complete data set that contains simulated outage with some data entries both before beginning and after end of outage. Finally, we compare the estimated states with the complete measured data.

The experiments run in the Matlab environment using its optimization toolbox. It suffices for the considered experimental off-line processing. Experience indicates that real-time version will be feasible with specialized software.

5.2. Evaluation of experiments

The estimation of the vehicle position is performed using the proposed algorithm described in Sections 3.3 and 3.4. The resulting estimates are compared with the actual values – we check whether the estimated values are within the given tolerance area. To evaluate the quality of the estimates, the absolute error of estimates Δ_{z_t} is used. It is defined as the difference between the measured vehicle position entry $\tilde{z}_t$ and estimated vehicle position entry $\hat{z}_t$; $\tilde{z}_t, \hat{z}_t \in \{p_x, p_y\}$, i. e.,

$$\Delta_{z_t} \equiv |\tilde{z}_t - \hat{z}_t|, \; t \in t^*. \tag{24}$$

The maximum entry of the sequence $\{\Delta_{z_t}\}_{t \in t^*}$ is denoted by $\max(\Delta_z)$. The mean of this sequence is denoted by $\mathrm{mean}(\Delta_z)$ and the median is denoted by $\mathrm{med}(\Delta_z)$.

5.3. Results

In experiments, the estimation of moving vehicle position was performed with the same lengths of simulated data outages that were placed on different parts of the vehicle trajectory. The length of the outages is $\delta = 5\,\mathrm{s}$, which is a typical length of data outage in real situations.

Figure 1 shows a whole vehicle trajectory where the star denotes a start of the vehicle movement (left) and a course of particular entries $p_{x;t}$, $p_{y;t}$, $t \in t^*$ (right).

Figure 2 shows the course of state estimation errors $\Delta_{p_{x;t}}$, $\Delta_{p_{y;t}}$ for one selected data outage with satisfactory estimation result (left) and respective part of trajectory (right).

Figure 3 shows the course of state estimation errors $\Delta_{p_{x;t}}$, $\Delta_{p_{y;t}}$ for one selected data outage with unsatisfactory estimation result (left) and respective part of trajectory (right).

Table 2 summarizes the obtained estimation results. Each row corresponds to one experiment. It contains information on the maximum of absolute estimation error, mean and median of absolute estimation error relating to the estimates of

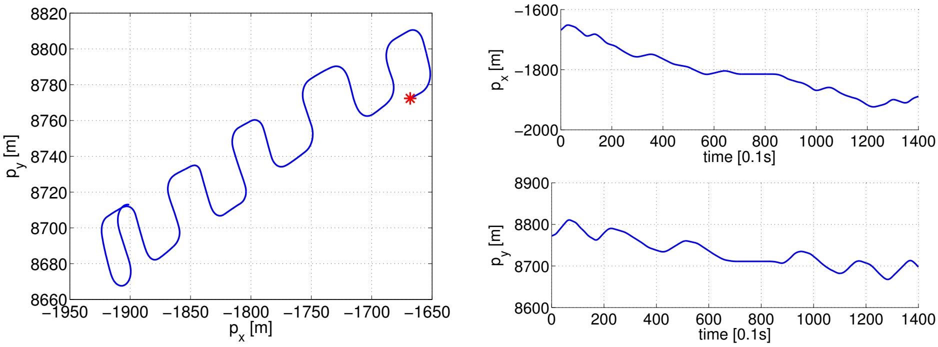

Fig. 1. The whole trajectory of a moving vehicle in cartesian coordinates where the star denotes a start of the vehicle movement (left) and the course of corresponding parts p_x, p_y.

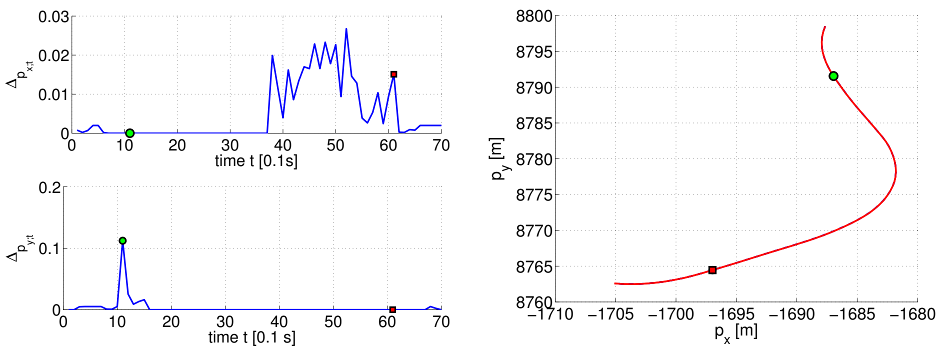

Fig. 2. Courses of absolute estimation errors $\Delta_{p_{x;t}}$ and $\Delta_{p_{y;t}}$ for an experiment with satisfactory results (left) and respective part of trajectory (right) with marked beginning (circle) and end (square) of the data outage.

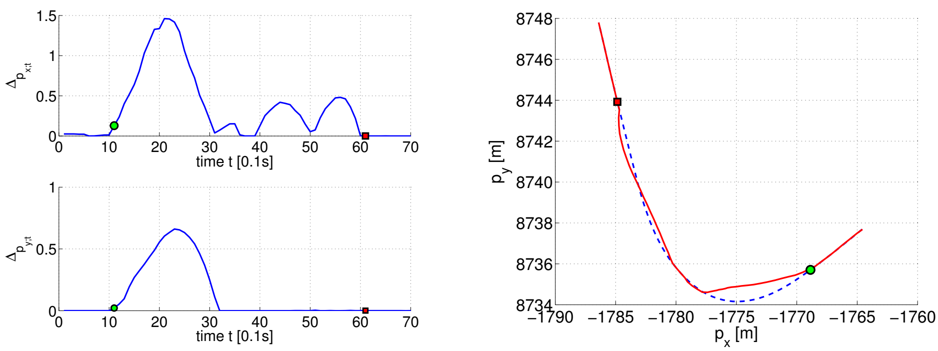

Fig. 3. Courses of absolute estimation errors $\Delta_{p_{x;t}}$ and $\Delta_{p_{y;t}}$ for an experiment with unsatisfactory results (left) and respective part of trajectory (right) with marked beginning (circle) and end (square) of the data outage where full line is the estimated trajectory, dashed line is the true trajectory.

Table 2. Maximum, mean and median of absolute estimation error Δ_{z_t} (24) for $z \in \{p_x, p_y\}$ in meters.

No.	$\max(\Delta_{p_x})$	$\mathrm{mean}(\Delta_{p_x})$	$\mathrm{med}(\Delta_{p_x})$	$\max(\Delta_{p_y})$	$\mathrm{mean}(\Delta_{p_y})$	$\mathrm{med}(\Delta_{p_y})$
1	0.0248	0.0022	0.0005	0.0527	0.0028	0
2	0.0268	0.0048	0.0007	0.1121	0.0030	0
3	0.0227	0.0015	0	0.2161	0.0456	0
4	0.0654	0.0122	0.0070	0.1255	0.0334	0.0134
5	1.4605	0.3348	0.1515	0.6608	0.1174	0.0004
6	0.0225	0.0021	0	0.0101	0.0001	0
7	0.5356	0.0570	0.0068	0.0466	0.0054	0.0004
8	0.0405	0.0019	0	0.3432	0.0181	0
9	0.0256	0.0021	0	0.0482	0.0055	0.0020
10	0.0673	0.0031	0	0.0321	0.0014	0
11	0.0151	0.0008	0	1.6401	0.3349	0.0030
12	0.1070	0.0145	0	1.1502	0.1318	0
13	0.3243	0.0318	0	0.1087	0.0227	0.0061
14	0.5726	0.0543	0	0.3542	0.0215	0.0004
15	0.2911	0.0370	0	2.0917	0.3801	0
16	0.7180	0.0993	0	0.5726	0.0823	0

particular entries p_x, p_y of respective experiment. An entry 0 means the value lower than 0.0001.

5.4. Discussion

In most of the experiments, we have obtained satisfactory results. It means that the maximal absolute estimation error was lower than 0.5m which is a requirement of the industrial partner.

We observe that the quality of estimation results depends strongly on the trajectory shape. The unsatisfactory results occur mostly when an outage is placed on the sharp turns of the trajectory. The used experimental data were from a training drive. A common drive is usually more quiet without such sharp turns.

Further, the estimation quality depends on the amount of available measurements before (NB) and after (NA) data outage. In our experiments, $NA = NB = 10$ was appropriate for 50 missing measurements.

The necessity of properly chosen starting point $\hat{\mathbf{X}}_0$ for NLP was discussed in Section 4.1.

6. CONCLUSIONS

This paper presents an algorithm for the estimation of nonlinear SU model when some measurements are missing. It generalizes previously proposed linear SU model [13] and refines on achieved results. The proposed algorithm is simple to perform, need no demanding initial setting and uses readily available data.

An algorithm performance was successfully tested on the problem of a moving vehicle position estimation.

Future research aims a further improvement of the state estimates. We will focus on the cases when NLP gives unsatisfactory results. We will try to prevent such situations by imposing boundaries on state values using more CAN data as a steering angle or a distance moved.

Another way how to improve the estimates quality is a redefining of the model so that all involved quantities will be considered either as outputs or states, i. e. no input will be considered. In such a way, we prevent the unprecise data to deteriorate the estimation results.

ACKNOWLEDGEMENT

This work was partially supported by the Research Centre DAR – project of the Ministry of Education, Youth and Sports of the Czech Republic No. 1M0572.

(Received July 31, 2010)

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