Information, Sentiment, and Price in a Fast Order-Driven Market

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The paper models an order-driven market in which many traders with heterogeneous private values and information submit limit and market orders simultaneously. Order execution is partially random. There may be a bias in the traders' prior beliefs (market sentiment). In this environment, although market buys and sells depend monotonically on the degree of bullish sentiment, market order flows are in a non-monotonous relationship with the proportion of high private value traders (bulls). Additionally, sentiment has a stronger effect on volume and net direction of trades leading to a given central price, than the actual distribution of private values.

Introduction

This paper deals with modeling brokered segments of a 'fast' security market, meaning a market characterized by three interrelated features:

- A large number of participants who place orders within a narrow time span;
- Very short-lived motives for trade (frequent changes of desired positions); and
- Uncertain terms of individual trade for any given participant due to a high concentration in time of other participants' actions.

The three named elements are typical for electronically brokered trading in many upper-end stocks, bonds and major currencies. (An electronic FX broker for a frequently traded currency pair is among the best examples of a fast market.) Electronic brokerage systems with fully or partially[1] observable books are now implemented by most exchanges around the world. A market for any top-tier security traded there would meet our criteria.[2] In addition, there exists a special category of 'high-frequency traders' who exploit short-lived arbitrage opportunities with the help of computerized trading, making the market for the involved instrument even faster. It is known that all but the biggest high-frequency traders operate

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1 Not all electronic brokerage systems display the whole book. Some of them only show the first- and second-best quotes-cum-quantities on both sides of the market. As we argue in the sequel, nominal degree of book openness is less important for trader decisions in a fast market than its effective openness. The latter goes down with increasing order arrival frequency.

2 From the evolutionary perspective, fastness of trading through any electronic brokerage is likely to rise endogenously, since its operator would normally prefer to attract frequently traded securities that generate a lot of fee income and discourage others.

through brokers, i.e., contribute to increased trade frequency in order-driven markets. Therefore, formalizing the physiology of a continuous double auction under fast market conditions without resorting to the zero-intelligence simplification should be of interest for practitioners. No less pertinent is this task for understanding asset price deviations from fundamentals, the phenomenon actively researched by financial economics, e.g., in the context of exchange rate (or risk-free bond yield) disconnect conundrum. Although, at present, the majority of transactions in this asset class happen in fast order-driven markets, the latter have been largely left unexplored by theoretical literature, at least that part of it which draws on individual rationality. Consequently, existing theories of continuous double auctions say very little about the properties of price discovery resulting from high frequency of orders and trades.

Fast order-driven markets create terms-of-trade uncertainty, which gives rise to a difference between the formal and the real/effective transparency, since the order book changes many times before an order is submitted. A Limit Order (LO) submitter does not know for sure with what other limit orders he will compete for execution, whereas a Market Order (MO) submitter may be unable to hit the observed best quote because others might attain service priority by pure luck, if their orders are registered a split-second earlier.

The terms-of-trade risk is reinforced by unobserved preferences, which are most relevant in markets for highly liquid securities with transaction and store-of-value roles, such as gilt bonds and FX. Differently from stocks, the economic information available on currencies and top-end bonds is usually hard to interpret in terms of conventional risk-return criteria, and the notions of fundamentals and private information in their case become fuzzy.[3] This is why our analysis does not involve an exogenous benchmark value.

Considering the quickly evolving environment in which fast market participants have to make decisions, the role of 'learning' in the usual sense is likely to be limited. There simply does not have to exist any stable parameter of the security to survive and be learned, a minute from now. This is why a one-shot game seems an appropriate approach to analyze brokerage in one of the named assets. Thus, our attention is focused on a static round of trading.[4] In a nutshell, we believe that the more 'dynamic' (i.e., fast) the market gets, the more important become those properties that transpire from a 'static' analysis at a point in time.

Our response to the above challenges is a model with many rational (although imperfectly informed) agents trading a single asset through an open LO book. These traders are risk neutral and have heterogeneous private values for the asset as well as heterogeneous

[3] Quite often, investors in those instruments are unable to agree on pricing implications of news traditionally tagged as 'fundamental'.

[4] With this, the temporal uncertainty of the physical fast market reality is approximated by the spatial uncertainty of the model. Conversely, in a 'slow' market, one needs to identify long-lived underlying factors that drive sequentially arriving orders. Statistical properties of those factors are the natural object of learning. The pattern of trade and price discovery determinants would then have to be split into the static part linked to the most recent order and the dynamic part hidden behind permanent sources of all orders. In a fast market, the latter part can be naturally separated from the former and treated as exogenous.

information about the parameters of the private value distribution.[5] There are two periods. In the first period, investors each simultaneously submit a limit order for one asset unit. An individual limit price maximizes expected profit given a private value and beliefs about other investors' private values. In the second period, investors submit a market buy/sell for the same standard quantity if the expected execution price is below/above their private value. Market buys/sells then cross with a subset of the submitted limit sells/buys. Thanks to risk neutrality, limit and market order decisions are mutually independent and traders do not face a trade-off between them. The subjective uncertainties of an investor who cannot keep up with changing market conditions are formally embodied in two risks: execution uncertainty for LO and random transaction price assignment for MO. We consider two idealized cases of the latter.

In the case of Hidden Priority (HP), MO submitters observe the aggregate book after Period 1. Since they also know the model, everyone can reconstruct the full market state by inferring the distribution of private values. However, traders still do not know which LO their MO will be crossed with. As MO arrives quickly, competing submitters do not know how much of the initial book will be used up by the other MO that randomly arrives before theirs. We also introduce a stronger form of terms-of-trade uncertainty, called Hidden Liquidity (HL), in which the aggregate order book itself is not known when MOs are submitted. More precisely, even if the book is being displayed, the physical time is too scarce to make quantitative inferences: the LO book is formally 'open', but for time reasons the traders are unable to benefit from it.

An equilibrium pattern of trade for each LO-cum-MO-submission round is proved to exist in both the HL and HP case. We then report some comparative statics for numerical solutions of a parametric HP model. Preliminary numeric evidence indicates that the quantitative difference between HP and HL markets is likely to be small as long as the agents' (heterogeneous) information about the market state is unbiased on average[6]. Thus, based on the numerical results obtained so far, we conjecture that:

a. Even the less realistic, but computationally better tractable HP model is sufficiently representative of the general case.

b. The most prominent pattern-of-trade determinants in a fast market are the first moments of the private value and information distributions, as well as population-averaged informational biases, with higher moments only playing a secondary role.

The main outcome of the theoretical analysis is a characterization of the equilibrium map of informational parameters of the investor population into orders, trades and the publicly

[5] Thanks to this, private values are always transformed by the broker into a public price ('price', more exactly 'central' or 'mid'-price, means the mid-point between the inside bid and inside ask), i.e., the broker supports an equilibrium trading pattern even without noise traders (*cf.* Kyle, 1985, who needs noise traders). Trades contain price-relevant information by construction.

[6] The HP setting overstates the traders' effective knowledge of the book, whereas the HL setting understates it. Ideally, one would need an execution uncertainty definition somewhere halfway between HP and HL. However, due to analytic and numeric complexities of the HL case, its detailed study is left to future research.

observed central price.[7] Any common element in trader beliefs co-determines the price. We study the situation in which traders start with a common prior belief and then receive different signals about the true parameters of the private value distribution. The deviation of the mean prior belief from the true parameter value is called sentiment.[8] We identify some properties of the common sentiment that are novel in the context of price impact of trades:

A. For a fixed (specifically, zero) prior bias, the usual price statistics of the order book, such as the central and the average Market Buy (MB) and Market Sell (MS) execution prices, have an intuitive monotonic relation to the percentage of high private value (bull) individuals in the trader population. On the other hand, the MB and MS volumes' dependences on the same bull-bear imbalance are non-monotonic (see an extended discussion in Section 4) and cannot be easily predicted.

B. For a fixed (in particular, zero) bull-bear imbalance, a low/high central price, high/low MB volume and low/high MS volume are obtained under prior sentiment biased towards low/high transaction price expectations. That is, 'prejudice' of LO submitters is aligned with liquidity-taking behavior by MO submitters. In 'reduced form', one looks at a negative relation between net MB order flow and price (negative feedback trading). If the context is ignored, this may go against the spirit of most microstructural asset pricing models.[9]

C. The same central price can be a consequence of either a bull-bear imbalance under zero prior bias or a non-zero bias under equal bull and bear weights.[10] It turns out that the former 'no prejudice' case gives rise to a lower gross MO volume than the latter 'balanced values, prejudiced minds' case. The elevated trading activity due to prejudice can be regarded as the social cost of the latter. Extending the traditional message of microstructure finance, our model shows that, as the medium of price discovery, MO flow is vulnerable to severe distortions by biased information.

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[7] Our 'central' price, i.e., the mid-point between the inside ask and bid prices, is just a prominent summary statistic of the trade pattern (and also the only one accessible to outside observers). Since it does not enter the investor surplus from any of the orders, which is only a function of the private value and the price at which the order is executed, no investor cares for it per se. Although the central price can be far away from a representative private value (such as average or median), it would be wrong to make conclusions about, e.g., winner's curse, self-fulfillment of beliefs, the presence of sunspots, or a breakdown of rationality in the market, based on the central price value. This reasoning would be justified, though, if a particular central price (e.g., obtained precisely at noon or at the closing of daily trading) served as an input to some other, say a derivative, contract. However, this lies outside the scope of the paper.

[8] Notwithstanding the name, this prior bias can reflect attitudes with a fundamental background. For instance, long-term ingredients of the asset demand might be better captured by the prior belief, whereas instantaneous realizations of the market state may be contaminated by noisy inventory movements. *See* also the discussion at the end of Section 4.

[9] However, if it is not ignored, one finds that both negative and positive feedback trading are consistent with rationality, depending on the private value and information distributions. *See* more in Section 4.

[10] As well as a continuum of bull-bear imbalance/prior bias combinations in-between.

Our methodological contribution is three-fold. First, we may be the first to offer a fast order-driven market model with decision-theory foundations, in a simultaneous trade setting.[11] This is an order-driven counterpart to simultaneous trade dealership models. The latter are very economical with regard to the structure of liquidity providers. The question of order impact on price is usually reduced to the one about liquidity-consumers' impact on it. This is innocuous in the quote-driven, but could be seriously misleading in the order-driven market. In the latter, order originators are not identical with liquidity-consumers, so that the order-price causality needs to be refined.

Second, we propose a quantification of the 'soft' notion of market sentiment in a microstructural context.[12] By incorporating a formally defined sentiment factor into the traditional trade-price equivalence concept of microstructure finance, we provide a qualifier to the usual attribute of microstructure models, a map from trades into price. The broker indeed channels order flow into the price. However, the price discovery is sensitive to the incorrectly perceived terms of trade. The effect is non-negligible since, for a fixed price impact, the prior trader sentiment (positively related to the central price) has a stronger influence on the realized order volumes than the actual distribution of private asset values.[13]

Third, touching on empirical implications, our analysis suggests a possible classification of both price- and trading volume-volatility determinants in a fast order-driven market. Namely, the model identifies three categories of disturbances inherent in the pattern-of-trade formation: 'fundamental' (distribution of private values), informational imprecision-related (dispersion of private signals), and finally, those caused by 'prejudice' (movements of investor sentiment prior to the order submission). By seeing the quantitative importance of sentiment fluctuations, we might better understand its proper place in the price discovery.

Literature Review

Apart from a few empirical papers on FX brokers (e.g., Daníelsson and Payne, 2002; and Rime, 2003), most empirical work on LO books is based on stock exchange data (e.g., Biais *et al.*, 1995; Handa and Schwartz, 1996; Hollifield *et al.*, 2004 and 2006; and Hall and Hautsch, 2007[14]). Although a theory of book formation in a fast order-driven market would be useful, it is bound to be technically complex. This is probably why it is so hard to find theoretical literature on the subject. Most existing LO book models (Chakravarty and Holden, 1995; Parlour, 1998; Foucault, 1999; Handa *et al.*, 2003; and Foucault *et al.,* 2005) reflect the institutions for trading in equities that have gradually evolved over the course of time, starting with infrequent trading of a limited number of titles. All these authors employ a

[11] The widely known simultaneous trade models by Kyle (1985), Glosten (1989), or Evans and Lyons (2002), refer to quote-driven (dealership) markets.

[12] The few known models trying to give an exact meaning to the notion of investor sentiment are Walrasian in nature, do not involve specific trading mechanisms and, as a consequence, have to operate with ad hoc wrong models of the world employed by the agents (*see* e.g., Barberis *et al.,* 1998).

[13] The above observations contribute to the 'flow-centrist' versus 'fundamentalist' debate in the finance literature (*cf.* a summary with the stress on FX markets in Froot and Ramadorai, 2005).

[14] With regard to liquidity-taking behavior as well as the order book information, our model is broadly consistent with empirical conclusions of Hall and Hautsch (2007), who find that traders extract information from the book and also, that they tend to take liquidity when its supply is high.

sequential trade approach. So, a theoretical coverage of fast LO markets is missing, whereas the LO book literature is captured by the sequential trade/external liquidation value paradigm (this is the case of Parlour, 1998; Foucault, 1999; Handa *et al.*, 2003, and other papers based on their models; Hollifield *et al.*, 2004, infer empirical restrictions on order strategies from the model by Foucault, 1999, and test them with mixed results). An exogenous liquidation value is also present in an earlier influential paper by Glosten (1994). On the other hand, Glosten's agents, similarly to the ones in our model, have heterogeneous information about the liquidation value. In addition, we share with this paper the focus on equilibrium at a point in time. Glosten (1994), models simultaneous trade in a continuous price and quantity space of orders. Market orders are executed according to a discriminatory pricing rule (they 'walk the book'). The resulting inside bid and ask prices are endogenous outcomes of limit order decisions that rationally anticipate market orders, in an adverse selection environment. Glosten's MO submitter and LO submitter sets are disjoint; the former consists of one representative risk-averse agent, and the latter of many risk neutral agents. LO users have no private information. Most importantly, the LO competition is reduced to a zero profit constraint. The LO traders' decision sets do not leave space for competitor undercutting/overbidding. Consequently, the Pareto equilibrium derived by Glosten corresponds to a representative liquidity supplier, who generates the order book single-handedly under the zero profit condition. In our model, heterogeneously informed traders compete against each other by taking decisions based on LO execution probabilities. That is, our equilibrium order book is undercut- and overbid-proof.

Beside the sequential trade and exogenous asset value assumptions, the other three mentioned models of LO trading (Chakravarty and Holden, 1995; Parlour, 1998; Foucault, 1999; and others derived from them) differ from the present one by full terms of trade observability. The same holds for Handa *et al.* (2003), who extend the model of Parlour (1998). In addition, Foucault (1999) requires from every trader a choice between a market and a limit order, whereas Parlour (1998) pre-assigns a buyer- or seller-type to every arriving trader. In contrast, Chakravarty and Holden (1995), exploit the consequences of joint limit- and market-order submission strategies, but their model does not analyze the price formation either, since the traders are already given the exogenous market-maker's bid-ask interval. Their model offers a rationale for the order-driven market segment coexistence with the specialist. As to Parlour, (1998) and Foucault (1999), their models address trade timing (or the book resiliency, as in Foucault *et al.*, 2005) and costs rather than price formation. What we share with all of the above is the assumption of private value distribution as a primitive of the model.

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The complexity and poor analytical tractability of broker models have resulted in development of numeric algorithms of equilibrium computation (Goettler *et al.*, 2005). These authors, after generalizing the model by Parlour (1998), compute an equilibrium on a discrete price grid under continuous order sizes. In view of the consensus asset value existence, price discovery is not an issue. We also resort to numerical solution of the equilibrium trade and price equations. Nevertheless, in our paper, although implicit, all the necessary analytical

expressions for equilibrium prices, the order book, etc., and their dependence on parameters are available, so that numerical methods are only required to get to explicit numbers. In this paper, we use numerical results primarily to evaluate the relative roles of actual private values versus sentiment in the behavior of order-driven markets.

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Continuous double auctions received attention in the bounded rationality literature (Luckock, 2003; Smith *et al.*, 2003; and references therein). Although some realities of markets with high-order arrival frequencies are accounted for, the focus there is on statistical order book properties under a stationary distribution of external uncertainties. Decision-theory considerations are absent since statistics of incoming orders are generated by zero-intelligence agents. Naturally, under this approach it is impossible to explain patterns of trade in terms of preferences and information. Therefore, our approach can be seen as a (near-) perfect complement to theirs.

The rest of the paper is organized as follows: Section 2 presents the main attributes of the fast brokered trading model in full generality, applicable to a wide range of asset value distributions. Then the equilibrium is defined for a model with a simplified parameterization of the market state space. Section 3, completes the parameterization of the model, by fixing functional forms for asset value and information parameter distributions. Section 4 discusses implications of the model as regards the role of prior information distribution in order placement and price discovery. Finally, the conclusion is offered. Proofs of technical statements are presented in the Appendix.

2. The Model

2.1 General Setup

There is a large population of small risk-neutral and liquidity-unconstrained traders with mass normalized to unity. They have access to a broker with whom they place orders for one asset unit. Each trading round consists of two periods. In the first period, traders can post limit orders. After the order book has been formed, in the second period they can submit market orders to be executed against the book. The round ends with MO execution. (In a fast market, as per our definition from the **Introduction**, due to unspecified exogenous disturbances investors enter each subsequent round of trading with changed private values, so that deciding about new orders must be started from scratch.) Since traders are assumed to be liquidity-unconstrained and risk-neutral, their surpluses from the four possible types of orders are mutually independent and can be evaluated separately.[15] So, in our setting there is no difference between a single population of investors who trade through both LO and MO, and two identical populations, of which one only uses LO and the other MO.

[15] In principle, one could allow for the MO and LO user populations to be statistically different. We do not do this since it is both unimportant for our analytic objective and goes against informal evidence collected from actual electronic brokerage users. Those claim to practice switching between LO and MO even within short time intervals.

For a given trader, let x be the private asset value logarithm and y – a parameter characterizing his information (to be specified later). The mass of traders who have private value x and information y is equal to $z(x, y; c) \geq 0$. Here, c is a vector of parameters that determine density z. We will call z the 'trader histogram' and c the 'market state'. In equilibrium, market state can be put in a one-to-one relation with a vector of summary statistics that fully characterize the LO book and MO quantities (*cf.* Section 3). The traders are aware of this one-to-one correspondence. However, at the start of the trading round the exact value c_0 of the market state is unknown to them. For a trader with information y, c is a random variable with continuous non-atomic p.d.f. $c \mapsto f(y; c)$. The latter is a Bayesian update of a prior density which we assume to be common for simplicity. Trader beliefs can be biased/prejudiced in the sense that the prior distribution of c can have a mean c_p different from c_0.

Beside the bias $c_p - c_0$, the aggregate quality of information in the present model can be evaluated by the distance of $f(y; .)$ from the atomic density $\delta_{c_0}(.)$ in an appropriately chosen metric. For instance, in the parametric version of the model to be introduced in Section 3, this distance depends on the standard deviation parameter η: the higher is the precision value $1/\eta$, the better the trader population as a whole is informed about its value distribution. However, the model does not possess a meaningful 'full information' limit when $\eta \to +0$. Indeed, at such a limit, if it existed, traders would know the last executable limit prices exactly, which in fact would induce them to place their orders exactly there. Expected MO execution prices would then coincide with these two values for all traders. However, this outcome cannot constitute an equilibrium since, for a subset of traders with a positive mass, undercutting or overbidding would be a best response to such a quoting rule by others—a contradiction. This is a specific example of the general no-trade outcome in markets with full information, well known, at least, since Grossman and Stiglitz (1980).

Throughout the paper, we will use the logarithmic scale for prices, mostly omitting the 'log-' prefix when talking about price values. The buyer and the seller market sides will be distinguished by superscripts B and S based on the MO direction, even when the LO variables are involved. So, superscript B will also service the LS parameters, whereas superscript S the LB parameters. Fixing a market state c, let $Q^B(c, p)$ be the mass of LS with prices not exceeding p, and $Q^S(c, p)$ the mass of LB with prices not lower than p. $M^B(c)$ will denote the mass of MB and $M^S(c)$ that of MS under market state c. The pair of functions $[Q^B(c, .), Q^S(c, .)]$ from the real log-price line into cumulative LS and LB quantities will be called the order book and the quartet $[Q^B(c, .), Q^S(c, .), M^B(c), M^S(c)]$, the (instantaneous) trade pattern under market state c.

In this subsection, in order to avoid notational complexities following from specific parametric choices, we will set the scene in the most general available terms. We first discuss the decision problem of an LO-submitter faced with execution uncertainty and then the problem of a MO submitter facing uncertain assignment of transaction price. Finally, we explain what are the consistency restrictions connecting the MB and MS sides, the LO and the MO decisions.

2.1.1. Limit Order Submission

A trader endowed with information y and submitting a limit order faces subjective execution uncertainty depending on the information endowment and the LO price. This uncertainty is expressed by a pair of smooth functions $p \mapsto B(y; p)$, $p \mapsto S(y; p)$ giving subjective probabilities of execution of a limit sell, respectively, buy at price p. These functions are strictly monotone in price: $B_p < 0, S_p > 0$ (subscripts stand for partial derivatives). Reducing order competitiveness makes the execution probability vanish: $\lim_{p \to +\infty} B(y; p) = 0$, $\lim_{p \to -\infty} S(y; p) = 0$, for every y. Although B and S are pinned down by the subjective market state density $f(y; .)$ (see the end of this subsection), of which the LO user is aware, B and S functions are all that matters for him, and his decision problem can be formulated without referring to f.

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A trader with private asset log-value x and information y will choose (log-)price p at which to place a LS so as to maximize the expected surplus $L^B(y, p) = B(y; p)(e^p - e^x)$. The expected surplus from a LB at price p is $L^S(y, p) = S(y; p)(e^x - e^p)$.

To characterize optimal limit prices, we introduce the following notation:

$$g^B(y, p) = p + \log\left(1 + \frac{B(y; p)}{B_p(y; p)}\right), g^S(y, p) = p + \log\left(1 + \frac{S(y; p)}{S_p(y; p)}\right)$$

Functions $p \mapsto g^{B,S}(y, p)$ will be assumed strictly increasing for every information parameter y. This is a generic property of the market state densities that generate B and S. In Section 3, we construct a class of subjective execution probabilities that satisfy it.

Proposition 1: *If the problem of choosing a price p optimizing the surplus from limit sell (buy) submission by a trader with private value x and information y has an interior solution, then* $x = g^B(y, p) < p$ ($p < x = g^S(y, p)$). *Traders who submit limit sells (buys) at prices not exceeding (not falling below) p are exactly those whose private values do not exceed* $g^B(y, p)$ *(do not fall below* $g^S(y, p)$*).*

Proof: See A.1 in Appendix.

Remarks:

1. Quantities $p - g^B(y, p)$ and $g^S(y, p) - p$ approximate the net welfare gains resulting to the trader with beliefs y from the limit order execution at price p.
2. There is a generic asymmetry of the buy and sell sides in this model. $g^B(y, p)$ may have an upper bound $\bar{g}(y)$. For traders with private values $x > \bar{g}(y)$, $L^B_p > 0$ for all p ($L^B_p > 0$ also holds when $B/B_p > -1$, i.e., when the log in the definition of g^B is not well-defined; this can happen for small values of p, meaning that the optimal ask

price is above p anyway). Such traders will always have an incentive to raise their LS price a little further, because the corresponding fall in the subjective execution probability will be compensated by a higher increase in the unconditional surplus $e^p - e^x$. This is tantamount to saying that these traders do not submit any (finite) LS price. The described phenomenon is absent on the market sell/limit buy side. For any information value y, $L_p^S(y, p)$ always changes sign once as p increases and $g^S(y, p)$ is a strictly increasing function of p mapping the real line on itself. Thus, every trader regardless of beliefs and private value optimally submits an LB with a finite price. This convenient property of limit buys can be explained as follows. For every information and private value, there always exists a sufficiently low price generating a positive expected surplus (i.e., the trader pays 'almost nothing' but receives a positive private value from the purchased asset unit). The trader would then raise that limit price until the loss in the unconditional surplus $e^x - e^p$ becomes equal to the gain in the execution probability. On the LS side, if the subjective execution probability does not fall to zero quickly enough with the rising price, traders with high private values can try out increasingly high prices and still increase surplus in expectation. Accordingly, depending on the distribution of beliefs, we may face markets in which all participants submit LB but only a fraction submits LS.

Proposition 1 implies that the order book satisfies the conditions:

$$Q^B(c,p) = \int dy \int_{-\infty}^{g^B(y,p)} z(x,y;c)dx,\ Q^S(c,p) = \int dy \int_{g^S(y,p)}^{\infty} z(x,y;c)dx \qquad \ldots(1)$$

2.1.2. Limit Order Crossing

The filled order book includes the best, i.e., lowest ask p^{ia} and the best, i.e., highest bid p^{ib}. One can obtain either $p^{ib} \leq p^{ia}$ or $p^{ib} > p^{ia}$. In the former case, (p^{ib}, p^{ia}) is the pair of well-defined inside quotes. For the latter case, we define an LO-crossing procedure based on the widespread principle of brokerage: a limit buy/sell order with a price above/below the best ask/bid is treated like a market order.

Formally, if $p^{ia} < p^{ib}$, there always exists such $p^m \in (p^{ia}, p^{ib})$ that the mass of limit sells with prices between p^{ia} and p^m equals the mass of limit buys with prices between p^m and p^{ib}:

$$Q^B\left(c,\ p^m(c)\right) = \int dy \int_{g^B\left(y,\ p^{ia}\right)}^{g^B\left(y,\ p^m\right)} z\left(x,\ y;\ c\right) dx = \int dy \int_{g^s\left(y,\ p^m\right)}^{g^s\left(y,\ p^{ib}\right)} z\left(x,\ y;\ c\right) dx = Q^S\left(c,\ p^m(c)\right) \qquad \ldots(2)$$

Given these two subsets of LO on both sides of the market with the same mass, the broker is assumed to use any matching procedure he prefers, to connect each of the so selected LB with a single one of the selected LS. LS at prices between p^{ia} and p^m and LB at prices between p^m and p^{ib} are executed with certainty. After having matched the 'stray' LO in the described fashion, the broker eliminates them from the book. The inside quotes are now both equal to

p^m and the inside spread is zero. MO are then executed against this updated book.

If $p^{ib} \le p^{ia}$, then any price p^m between p^{ib} and p^{ia} solves Equation (2), since $Q^B(c, p^m) = Q^S(c, p^m) = 0$. For definiteness, one can take p^m to be the midpoint between p^{ib} and p^{ia}. In the sequel, the cases $p^{ib} < p^{ia}$ and $p^{ib} \ge p^{ia}$ will be referred to as the 'wide' and the 'narrow inside spread' cases, respectively.

2.1.3. Market Order Execution

The market state uniquely determines the traders' decisions about MO placement, so that the number of market buys and sells, M^B and M^S, are functions of c. Here, we recall the distinction between the HL and HP cases defined in the **Introduction**. Both cases will be treated in parallel.

After the automatic LO-crossing described above has been carried out, when necessary, market orders are executed against the book $p \mapsto Q^B(c, p) - Q^B(c, p^m)$, $p > p^m$, on the buy side and $p \mapsto Q^S(c, p) - Q^S(c, p^m)$, $p < p^m$, on the sell side. In the present model, all MO must be executed in equilibrium. If it were not the case, e.g., $Q^B(c, \infty) - Q^B(c, p^m) < M^B(c)$, then, for a set of signals y of positive Lebesgue measure, there would exist a finite LS price p such that there would be almost no LS prices above p, even though $B(y, p) = 1$ (due to a positive mass of unsatisfied market buys). However, this cannot be a part of an equilibrium trade pattern, since traders with signals y from the said subset and private asset values below p would be better off placing LS at prices above p—a contradiction. A similar argument shows that one cannot have $Q^S(c, \infty) - Q^S(c, p^m) < M^S(c)$.

Accordingly, there must exist a finite last executable limit ask price $K^B(c)$—the highest ask price the market buyer may have to pay under c. The number of LS with prices between p^m and $K^B(c)$ exactly equals M^B. The last executable limit buy price $K^S(c)$ is defined analogously. Formally,

$$Q^B\left(c; K^B(c)\right) - Q^B\left(c, p^m(c)\right) = M^B(c),\ Q^S\left(c; K^S(c)\right) - Q^S\left(c, p^m(c)\right) = M^S(c) \qquad \ldots(3)$$

Denote by $q^B(c; p)$ $(q^S(c; p))$ the partial p-derivative of $Q^B(Q^S)$—the LS-density at price $p > p^{ia}$ (LB-density at price $p < p^{ib}$). The broker's rule of assigning limit prices to incoming MO is assumed to use the logarithmic scale. The infinitesimal probability of executing an MB at price $p \in \left(p^m, K^B(c)\right)$ under market state c is $\dfrac{q^B(c; p)\,dp}{M^B(c)}$ and the infinitesimal probability of executing an MS at $p \in \left(K^S(c), p^m\right)$ is $\dfrac{q^S(c; p)\,dp}{M^S(c)}$. Consider the HP case, when c is known to the MO-submitter, first. The uncertainty only involves the broker's assignment of execution price from the known interval. The average logs of the market buy and sell prices equal:

$$\hat{a} = \int_{p^m}^{K^B(c)} \rho \frac{q^B(c;\rho)}{M^B(c)}\, d\rho,\ \hat{b} = \int_{K^S(c)}^{p^m} \rho \frac{q^S(c;\rho)}{M^S(c)}\, d\rho \qquad \ldots(4)$$

In the HL case, the MO submitter is uncertain about the market state as well. Accordingly, logs of the subjectively expected execution prices by the market buy and sell submitter are equal to

$$a(y) = \int \hat{a}(c)\, f(y;\, c)\, dc,\ b(y) = \int \hat{b}(c)\, f(y;\, c)\, dc \qquad \ldots(5)$$

A trader in an HL-market with a private asset value e^x and information y places a MB if and only if $x > a(y)$ and a market sell if and only if $x < b(y)$. In a HP market, these conditions simplify to $x > \hat{a}$ and $x < \hat{b}$, respectively. Traders with x between $a(y)$ and $b(y)$ ($\hat{a}$ and $\hat{b}$ in the HP case) only place limit orders. The masses of market buys and sells in an HL market are given by:

$$M^B(c) = \int dy \int_{a(y)}^{\infty} z(x,\, y;\, c)\, dx,\ M^S(c) = \int dy \int_{0}^{b(y)} z(x,\, y;\, c)\, dx \qquad \ldots(6)$$

In an HP market, $a(y)$ and $b(y)$ in Equation (6) must be replaced by $\hat{a}$ and $\hat{b}$.

2.1.4. Closing the Model

For a given trader histogram z, Equations (1) to (6) define the consistency restrictions on the pattern of trade and therewith determine the dimension of the market state space. In the market so defined, under a mild regularity assumption, there exists a space of equilibria parameterized by a 3-dimensional market state vector. Accordingly, let us assume that c consists of three components, c^m, c^h and c^l taking values in subsets of real line (*cf.* Section 3). A.2 in the Appendix lists the conditions that eliminate the remaining indeterminacy of the general model. Although not offering a formal proof of existence or uniqueness of equilibrium in full generality, these conditions can be useful in doing so for important special cases. Their role is to impose restrictions on the execution probability maps B and S, whose parameters we have not yet fully tied up by consistency requirements. To formulate them, one needs the following technical assumption (in the HL case).

Assumption 1: The trader histogram z is such that for every non-singular functions a and b of the signals, the map $(c^m,\, c^h,\, c^l) = c \mapsto T(c) = (p^m(c),\, M^B(c),\, M^S(c))$ with M^B and M^S defined in Equaton (6), is one-to-one and non-singular.

In Section 3, we construct a parametric version of the model for which the above assumption is easily verified. Note that in the HP case, an analogue of Assumption 1 is trivially satisfied for any histogram z with a non-degenerate dependence on c (in the opposite case, it suffices to redefine the range of market states to establish a one-to-one property).

Remark: Dealers Competing with the Broker – The modeled broker is not an exclusive trading venue. One can include a dealer acting in parallel with the discussed LO book. If the traders are risk-neutral and liquidity-unconstrained, they may accept a transaction with a dealer regardless of what they do in the brokered market segment.[16] So, if a dealer offers a pair of log-price quotes $p^b<p^a$, traders with $x<p^b$ will sell to, and traders with $p^a<x$ buy from, the dealer. Therefore, in market state c, the dealer will receive the purchasing and the selling order flows equal to:

$$\omega^B(c)=\int_{p^a}^{\infty}\int z(x,\,y;\,c)\,dxdy\,,\qquad \omega^S(c)=\int_{-\infty}^{p^b}\int z(x,\,y;\,c)\,dxdy$$

The true market state c_0 can be derived by the dealer from the pair of received order flows and a third parameter coming from the realized LO book (e.g., the earlier defined mid-price p^m; it is then required that the map $c\mapsto\left(p^m(c),\,\omega(c)\right)$ be non-singular). With this knowledge, she can offer an adjusted pair of quotes in order to extract the maximum surplus from another round of trades. The dealer's advantage is the ability to attract a representative—hence informative—order flow, a thing that an LO submitter in the brokered market cannot do. In this paper, we discuss only the easiest case when the private value/information distribution has just three unknown scalar parameters. These can be reconstructed by the dealer exactly from the two observed order flows and the broker's mid-price. Generalizations involving higher parameter dimensions and their imperfect extraction by the dealer are possible.

Author: plz clarify

2.2 Market State Space Parameterization and Equilibrium

The general model of the previous subsection, characterized by the nonlinear equation system stated in A.2 in Appendix, does not have an explicit closed-form solution and is too complex to justify an analysis in full generality. The specializing assumptions introduced below allow one, under a fixed structure of beliefs, to make the equilibrium a well-defined function of the market state.

In order to examine the effect of information quality on a trader with any private value, we abstract from the possible interdependence between private value and information.

Assumption 2: The number of traders with a given private value is independent on the number of traders with a given private information: $z(x,\,y;\,c)=\xi(c,\,x)\;\psi(c,\,y)$. The number of traders with a given private value (information) is normalized to unity: $\int_{-\infty}^{\infty}\xi(c,\,x)\,dx=1$,

Author: plz clarify

[16] Provided only trades of small standard quantities both through the broker and the dealer are allowed, traders are supposed to remain risk-neutral within the range of these position shifts. The model can be adjusted to reflect the possible influence of effectuated trades on the new private value, without changing the substance of the results.

$\int \psi(c, y)\, dy = 1$ for every c. Functions ξ and ψ will be called the value and the information histogram, respectively.

Recalling Assumption 1, observe that there are many possible parameterizations of the market-state space. The choice between them is just a choice of perspective. To facilitate economic interpretation of technical results, it is convenient to define value histograms in terms of a triplet of parameters, θ, that correspond to some natural summary statistics (*cf*. the beginning of Section 3). Those statistics are functions of market state: $\theta = \Theta(c)$. Equilibrium conditions, by imposing a correspondence between θ and c, pin down the map Θ.

Keeping in mind that $\theta = \Theta(c)$, define density $\tilde{\xi}$ implicitly by $\tilde{\xi}(\theta, x) = \xi(c, x)$ and let $\Xi(\tilde{\Xi})$ be the cumulative distribution function corresponding to density $\xi(\tilde{\xi})$. Then,

$$Q^B(c, p) = \int \Xi\left(c, g^B(y, p)\right) \psi(c, y)\, dy,\ Q^S(c, p) = \int \left[1 - \Xi\left(c, g^S(y, p)\right)\right] \psi(c, y)\, dy \qquad ...(7)$$

and we redefine the LO densities q^B and q^S accordingly. Fix the following parameterization of the market-state space: $c^m = p^m$, $c^h = K^B - p^m$, $c^l = p^m - K^S$. For notational convenience, from now on the triplet (c^m, c^h, c^l) will be replaced with (m, h, l). So, m is the (logarithmic) mid-price of the LO book, whereas $h > 0$ and $l > 0$ characterize the trade intensity/execution depth on both sides of the market: $h(l)$ is the distance from m to the last executable limit sell (buy) price.

We now associate the information vector $y = (y^m, y^h, y^l)$ with a triplet of signals about the true value of the corresponding market-state component. Based on these signals, traders perform a Bayesian update of a prior distribution which is assumed to be the same for everyone. The information structure can be simplified as follows.

Assumption 3: The information parameter y is a 3-dimensional vector (y^m, y^h, y^l). Each component carries separately the information on the corresponding component of the market-state vector c. Given the information, components of c are mutually independent with probability densities driven by the respective components of y. That is, $f(y; c) = \varphi\left(y^m; m\right) \nu\left(y^h; h\right) w\left(y^l; l\right)$ for some univariate density functions φ, ν and w.

Note that, when the trader population is large, the law of large numbers suggests an analogue of independence statement of Assumption 3 for the information histogram as well: $\psi(c, y) = \psi^m\left(m, y^m\right) \psi^h\left(h, y^h\right) \psi^l\left(l, y^l\right)$.

Although the independence assumption may seem strong, we argue that it is fairly natural. Indeed, the central price m is a location parameter, whereas the two depth parameters (h, l) should be in a one-to-one correspondence with the shape regardless of location. Further, as follows from Equation (7) and Equations (8) to (11) below, the buy side of the market can be

characterized without the reference to the sale side 'depth' (just the location), and vice versa. So, what really matters for the buy/sell side is the marginal h- (l-) density obtained from the joint (h, l) density. Both should be independent of m. Thus, although it would be more accurate to formulate subsequent results in terms of marginal densities, it is more convenient to utilize the notation of Assumption 3.[17]

The following representation of the expected MO execution prices replaces Equation (4) under the present parameterization choice:

$$\hat{a}(c; m, h) = \frac{\int_{m}^{m+h} pq^{B}(c, p)\, dp}{Q^{B}(c, m+h) - Q^{B}(c, m)},\ a(y, m, h) = \int \hat{a}(c'; m, m+h)\, f(y; c')\, dc' \qquad \ldots(8a)$$

$$\hat{b}(c; m, l) = \frac{\int_{m-l}^{m} pq^{S}(c, p)\, dp}{Q^{S}(c, m-l) - Q^{S}(c, m)},\ b(y, m, l) = \int \hat{b}(c'; m-l, m)\, f(y; c')\, dc' \qquad \ldots(8b)$$

Finally,

$$M^{B}(c) = 1 - \Xi(c, \hat{a}(c; m, h)),\ M^{S}(c) = \Xi(c, \hat{b}(c; m, l)) \qquad \ldots(9\text{HP})$$

in the HP case and,

$$M^{B}(c) = \int [1 - \Xi(c, a(y, m, h))]\, \psi(c, y)\, dy,\ M^{S}(c) = \int \Xi(c, b(y, m, l))\, \psi(c, y)\, dy \qquad \ldots(9\text{HL})$$

in the HL case. The market-clearing conditions (2) and (3) take the form

$$M^{B}(c) = Q^{B}(c, m+h) - Q^{B}(c, m),\ M^{S}(c) = Q^{S}(c, m-l) - Q^{S}(c, m) \qquad \ldots(10)$$

$$Q^{B}(c, m) = Q^{S}(c, m) \qquad \ldots(11)$$

Obviously, Equation (11) is moot in the wide inside spread case.

Definition: For a given trader histogram satisfying Assumptions 2 and 3, an equilibrium is a trade pattern $(Q^{B}, Q^{S}, M^{B}, M^{S})$ corresponding to a market state $c = (m, h, l)$ such that

- Traders place LO optimally as given by *Proposition 1*;
- MO are optimally submitted by those traders who expect a positive surplus (the expected execution price is above the private value for MS and below it for MB);

[17] To examine possible links between market depth and price level, one can always consider other copulas in place of f from Assumption 3. There will be one equilibrium for every copula. We do not do this here because our prime interest is in the trade pattern dependence on marginal density properties for each market state component separately.

- The market-clearing conditions (10) equate the executed LO quantities and the total MO quantities; and
- The number of limit buys with prices above m is equal to the number of limit sells with prices below m, i.e., Equation (11) holds.

If one formally considers the value histogram a function of θ instead of c (i.e., replaces ξ by $\widetilde{\xi}$ and Ξ by $\widetilde{\Xi}$ everywhere in Equations (7) to (11)), then Equations (10) and (11) define three conditions on θ in terms of c that characterize the map Θ. In regular cases, there must exist a solution for every c-value. The following *proposition* is proved in A.3 in Appendix.

Proposition 2: *For every trader histogram defined in terms of value histogram $\widetilde{\xi}$ and satisfying Assumptions 2 and 3, and every fixed value of θ, there exists a solution (m, h, l) to the Equation system (10), (11) in both the HP and HL cases.*

We want to restrict attention to those histograms for which the correspondence Θ established in *Proposition 2*, becomes a regular mapping with a well-defined inverse. In that case, for a given trader histogram, the equilibrium as per above definition is fully described by the market state $c \in R^3$. Indeed, under Assumptions 2 and 3, the subjective probability of execution of a limit sell with price p is the 'y-probability' of the random event $\{p \leq m + h\}$. Similarly, the corresponding limit buy execution probability is calculated on the random event $\{p \geq m - l\}$. Based on Equations (A5) and (A6) in A.4 in Appendix, one can derive expressions for critical private values $g^B(y, p)$ and $g^S(y, p)$ in terms of densities φ, ν and w. Then, Equations (7) to (9) complete the characterization of the trade pattern in terms of the trader histogram.

The economic essence of the model is encoded in the properties of Θ. In other words, the market is exhaustively described by the manner in which a value histogram is transformed into a trade pattern, given the information histogram. In Section 4 we will look at the dependence of this transformation on the information histogram precision and bias.

Despite qualitatively similar properties, there are quantitative distinctions between the HL and HP case.[18] Due to excessive numeric complexity of the HL case with no satisfactory calculation schemes at the present stage, in the remaining part of the paper we discuss equilibria for the HP market only, calculated under a selected parameterization of marginal densities φ, ν and w.

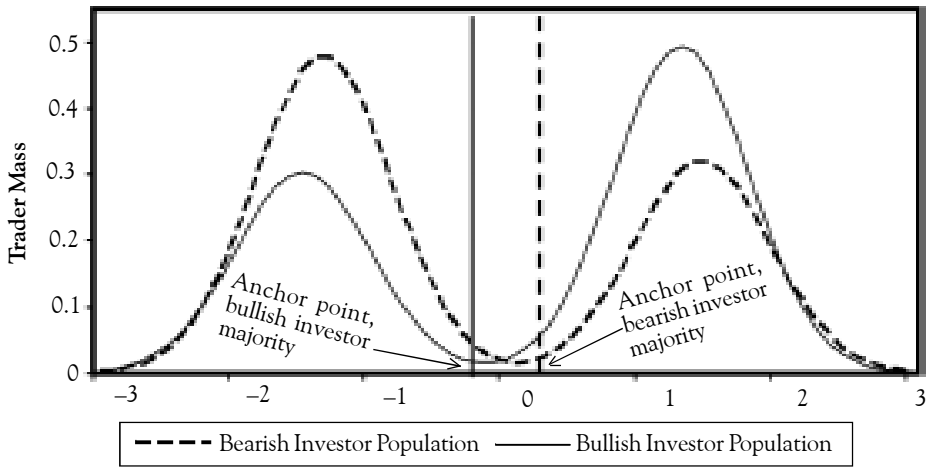
3. Private Value Histogram Parameters and Bayesian Belief Structure

For a numerical treatment of the model, we select the following functional form of the trader histogram. Let $\alpha \in [0, 1]$, $\beta > 0$ and $\gamma \in R$. Fixing a vector of parameters $\theta = (\alpha, \beta, \gamma)$, put[19]

[18] Although algorithms for equilibrium calculations in the HL case would be easy to design on paper, their feasible software implementation constitutes a separate problem.

[19] With a slight abuse of notation, ξ will now stand for a histogram depending on θ not c. One shall remember throughout that the focus of attention is on the properties of the map from (α, β, γ) to (m, h, l).

Figure 3: Log Mid-Price of Zero Generated By Two Imbalanced Private Value Distributions



Note: Anchor point means the mid-point between the high and the low mode of the private value histogram.

average generates non-trivial amounts of trade through market orders. Any genuinely Bactrian private value histogram, as well as any deviation from the case of equal bull and bear shares towards an asymmetric market state, produces even less automatic crossing relative to market orders.

Formally, the equilibrium trade pattern following from Equations (10) and (11) only depends on the difference $\Delta = \gamma - m$, the price shift relative to the private value histogram anchor point. This price shift is an increasing function of the bullish trader weight α. For a given distance between private value modes it is the bull-bear imbalance that shifts the equilibrium mid-price away from the private value histogram anchor point. Any given

Figure 4: Equilibrium Limit Order Book, Wide Inside Spread

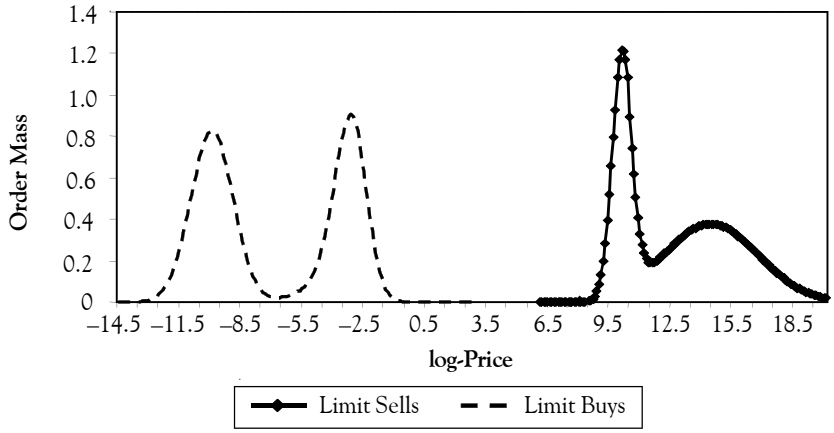
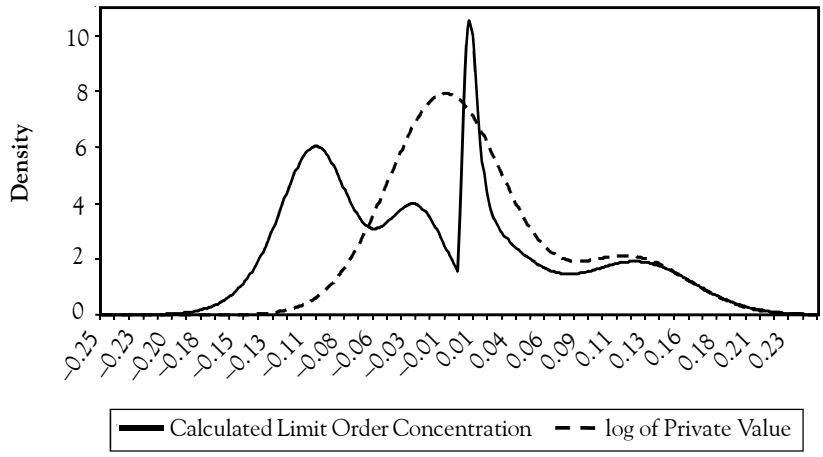


Figure 5: Equilibrium Limit Order Book, Narrow Inside Spread



mid-price m and the bull-bear difference parameter β can be generated as an equilibrium outcome by either shifting the anchor point γ to the right (i.e., increasing Δ) and at the same time increasing the bearish trader weight (i.e., reducing α below 0.5), or vice versa. Figure 3 depicts this situation for the (log) mid-price m equal to zero in a wide inside spread example.

Next, we ask how the size of the bull-bear imbalance is reflected in executed market orders. At a first glance, the more imbalanced is the distribution of private asset values towards the bull side, the higher should be the net MB flow. Our results show that this is not necessarily so. Inversely, MO flows alone are insufficient to pin down the bull-bear split of the underlying private values. There are many shapes and positions of the value histogram that generate the same MO flows, depending on the parameters of the information distribution among traders. The role of informational dispersion in the failure of MO flows as a sufficient statistic for the price shift will be discussed in Section 4.

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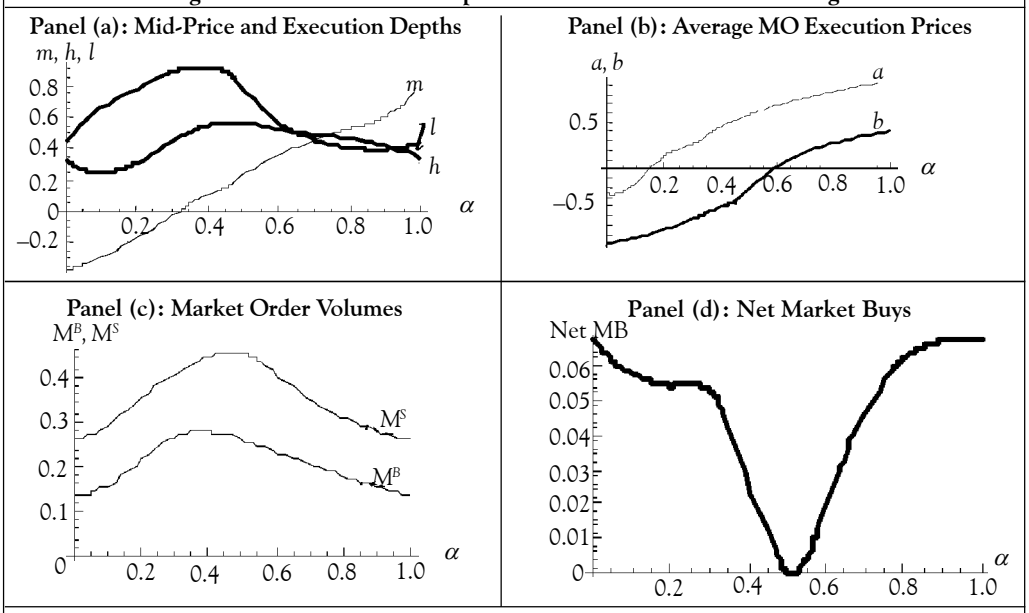
A typical equilibrium LO book in the wide inside spread case is shown in Figure 4, and in the narrow inside spread case is shown in Figure 5. Both have been obtained from the generic Bactrian-shaped histogram of private values. Figure 5 depicts the log-prices of limit orders around the log of mid-quote normalized to zero. The underlying average private value is higher than zero. However, the implied private value imbalance toward the bearish side causes the equilibrium price to fall below the midpoint between the two modes of the private value histogram.

4. Price Effect of the Private Value Distribution and the Information Bias

Structurally, our model draws a map from the trader histogram to both trades (i.e., market orders) and price statistics (mid-price and execution depths). In reduced form, one only sees links between trades and price. However, without a proper account of the informational dimension of the trader histogram, the reduced-form trade-price pattern can appear

Figure 6: Trade Pattern Dependence on the Bull Trader Weight



Note: The anchor point of the private asset value histogram is zero, β=0.6, $\sigma_H = \sigma_L$=0.3; the calculated trade patterns are for the case of zero bias in the prior beliefs; and in Panel (d), net market buy volume is shown as a difference from the base case of equal bull and bear weights ($\alpha = 0.5$).

counterintuitive. To see this, we first fix the information histogram with zero prior bias and look at the trade and price dependence on the bull-bear imbalance parameter of the value diagram.[21] Second, we fix a balanced bull-bear value histogram and analyze the role of prior bias changes. Outcomes of these two numeric experiments are then compared.

4.1 Bull-Bear Imbalance When the Traders are Unbiased on Average

Keeping the value histogram anchor point at zero and the distance between its modes fixed, we have calculated the market states satisfying the equilibrium conditions given by Equations (10) and (11) for a dense grid of bull-bear weights between 0 and 1. The results are shown in Figure 6. Panel (a) depicts the corresponding values of the mid-price, the buy and the sell-side LO execution depths. Panel (b) features average MB and MS execution prices, Panel (c) MB and MS volumes, and Panel (d) the net MB volume relative to the equal bull-bear partition benchmark, all as a function of the bull weight α.

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Whereas the three major price statistics (mid-price and the average MO execution prices) exhibit an intuitive increasing dependence on the bull weight, the considered statistics of the MO activity depend non-monotonically on it. Particularly, both MB and MS are highest for similar weights of bulls and bears and decline with growing bull-bear imbalance in either directions. This is understandable since most transactions are between bulls and bears rather than between traders with similar private values. Therefore, if private values close to one

[21] Clearly, α is the least trivial component of the parameter vector θ. Of the other two, γ simply determines location and β is the scale of the private value distribution.

mode dominate, they are less probable to find counterparties for trade. Conversely, net MB increase with the growing absolute bull-bear imbalance. That is, the liquidity-taking motive is dominant when there are more bears than bulls, whereas the value-trading is stronger when there are more bulls than bears. In any event, the mid-price has an involved non-monotonous reduced form relationship with the MB and MS quantities.

4.2 Common Prior Bias Under Equal Proportions of Bulls and Bears

The initial bias c_p enters the equilibrium trade pattern by influencing the mass of traders with a given information parameter (cf. Equation (12hist)). This quantity is present on both sides of the market-clearing conditions given by Equations (10) and (11). Of the three components of the bias, beliefs about the two execution depths (h and l) have a fairly predictable effect on price statistics. The farther the subjectively estimated last limit sell/buy is from the mid-price, the higher/lower is that price itself in equilibrium. At the same time, a high h-(l-) bias dampens (boosts) the MB orders and boosts (dampens) the MS orders, revealing a high elasticity of liquidity demand. However, quantitatively, those biases have a much weaker effect on MO flows than the mid-price bias, so we do not report the corresponding numbers here.

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We shall now illustrate the role of prior beliefs about the value of m in generating the same price as a given change in the bull-bear partition of the traders with unbiased information. More specifically, we calculate the changes in bull proportion that lead to the same price as four given (two positive and two negative) mid-price bias values. The calculation results are

Table 1: Relative Roles of Bull-Bear Balance Versus Prior Belief Bias in Price Formation and Market Order Flows (Deviations from the Base Case)

Common Parameters	Varying Parameters	m	M^B	M^S	M^B–M^S
c_p–c_0=0	α=0	–0.370930	–0.126402	–0.194218	0.067816
	α=0.445323	–0.07184	0.010832	0.000227	0.010605
	α=0.5 (Base Case)	0	0	0	0
	α=0.552321	0.072922	–0.012598	–0.017528	0.00493
	α=1	0.829070	–0.126402	–0.194218	0.067816
α=0.5 h_p=h_0 l_p=l_0	m_p–m_0= –0.880765	–0.370930	0.173985	–0.263275	0.437260
	m_p–m_0= –0.2	–0.07184	0.036833	–0.01568	0.052515
	m_p–m_0= 0.2	0.072922	–0.03936	0.012347	–0.051706
	m_p–m_0=0.795880	0.829070	–0.239466	0.044873	–0.284339

Note: All results are obtained from the solution of Equations (10) and (11) for the HP-market under γ=0, β=0.6, σ_H=σ_L=0.3. In all cases the obtained solution belongs to the narrow inside spread category. Market buy and sell quantities are given as deviations from the base case of zero bias and balanced bull-bear partition of the traders (α=0.5).

collected in Table 1.

Table 1 reveals that, at least in relatively balanced trader populations, net MB are a decreasing function of *m*-bias. This means that unjustified pessimism/optimism increases/decreases buy pressure, speaking for the dominant role of the liquidity-taking motive. Looking at the reduced form link between net market buys and mid-price, we come at a negative, instead of a positive, relationship. Unless one takes into account the underlying common bias factor, one arrives at a violation of the conventional wisdom of microstructure finance.

Quantitatively, for a given baseline market price, changes in prior beliefs have a stronger effect on the equilibrium MO flows than the actual change in the bull-bear balance.[22] There are at least two pieces of evidence corroborating this claim.

First, for a given price shift, biased information induces more trading activity in gross terms than a bull-bear imbalance under unbiased information. Second, under fixed prior mid-price bias, there is an upper bound on the shift from the average private value to the mid-price. (The 1st and the 5th lines of Table 1 show this for the zero bias case: the value of *m* for $\alpha=0$ gives its lower, and for $\alpha=1$ – its upper bound.) On the contrary, the same shifts due to a prior bias are, in theory, unlimited. Any bias below the value in the 6th line or above the one in the 9th line would take the mid-price outside the range attainable by bull weight shifts.[23]

4.3 Implications for Asset Price Modeling

As is usual in microstructure finance, the market price in our model is a result of liquidity-suppliers' (LO providers here) quotes reaction to the liquidity demanders' (MO submitters) order flow. Nonetheless, the model demonstrates the perils of the widespread logical shortcut, which identifies the price-generating order flow with the actions of 'liquidity-consumers'. In our model, liquidity-consumers (MO-submitters) go in the direction of favorably offered liquidity rather than in the direction of the average asset value. The latter is much better reflected in the 'liquidity suppliers' actions. Accordingly, to extract the bull-bear composition of traders, the MO flows alone are insufficient. Only the full geometry of the order book would reveal both the market state and the bias. Equally insufficient are MO flows as statistics of the market price. They do not reflect enough of the true market state, but, at the same time, reflect too much of the bias in private information. With this qualifier, the main thesis of Evans and Lyons (2002), that order flow contains price-relevant information, is sustained by our model of brokered trading. Nevertheless, in our context, 'information' has two components: the true value and the systematic prior prejudice.

The relative importance of prior information about, as opposed to the actual momentary state of, private value profile for a persistent price change is yet to be explored. We have defined heterogeneous trader information as an outcome of Bayesian updating of a common biased prior belief by heterogeneous unbiased signals. This definition might evoke an impression of the prior beliefs representing a deviation from the true market state responsible

[22] The above statement refers to the cases when MO flows at least change in the same direction as in the unbiased case, but even this regularity does not hold. So, a positive bias increases the MS-volume whereas, for values of $\alpha>0.5$ the increase in the number of bullish traders with the same price impact decreases it.

[23] In practice, of course, extreme bias values are very improbable.

for the fundamental value. However, an inverse interpretation is equally possible. Hypothetically, there may be more fundamental substance to the common prior knowledge c_p of the market state than to the specific realization c of this state at a point in time. Seen from this perspective, the only fundamental input into the price is fed through the initial belief c_p, whereas all other determinants of equilibrium trades and quotes, summarized by $c–c_p$, are transient. As usual, the reality most likely lies between the extremes, so that we can never be sure as to whether the observed order flow takes the price towards or away from, the unobserved fundamental value.

4.4 Tentative Empirical Implications

As mentioned earlier, the aim of our numeric experiments was to evaluate the relative importance of market-wide changes in private values on the one hand, and sentiment on the other, for price setting and transaction intensity in a fast market. As the numbers reproduced in Table 1 testify, the model generates price adjustment in the direction of either value or sentiment shift, as expected. Less obvious is the finding that the same observed price shift, if it occurs as a result of biased sentiment, is accompanied by a higher gross and net volume of market orders than would be the case in which a value shift takes place under unbiased sentiment. Accordingly, if our model reflects the said aspect of fast markets adequately, one should see, after controlling for average private value movements and imbalance (bull-bear division), a higher transaction volume around times of abrupt market sentiment revisions.

Naturally, especially for very frequently traded assets with a fuzzy consensus value and no obvious anchor for sentiment movements (FX, gilt bonds), neither the positioning, let alone bull-bear symmetry, of private values nor informational bias is observable. Still, an empirical analysis based on our results could try to associate the timing of strongly biased sentiment with the moments of reversal in the preceding non-zero trend. Given that our model deals with order-driven market snapshots at one's moments of choice, one is free to censor the continuous real-time trading data by selecting either representative trend-following or trade-reversing moments, and compare them. As a first approximation, the actual bull-bear spread in the private value distribution may be proxied with the help of transaction-frequency data on aggregate forward positions (through which the bulk of actual inventory management,

Table 2: Market Order Response to Actual and Presumed Bull-Bear Imbalance

	Growing Bull Weight		Growing Bullish Sentiment
	$\alpha<1/2$	$\alpha>1/2$	
MB	+	–	–
MS	+	–	+
Net MB	–	+	–

Note: + Stands for increase; – for decrease; Net MB is the difference between market buy and market sell volumes.

hedging and speculation takes place). One might then look at the evolution of executed MO volumes, e.g., in the brokered FX trading with less prominent currency pairs. For many 'lesser' currencies whose spot markets still qualify as fast, there exist only a few market-making banks, particularly in the forward/swap segment. Therefore, in principle, aggregate open forward positions are observable. And, indeed, as some informal reports from dealer banks suggest, when such currencies undergo a sustained change in the direction of price movement, one sees much more spot trading activity, including MO executions, than would be suggested by the observed intraday evolution of forward positions.

A more detailed qualitative summary of MO-reaction to both actual (higher bull weight) and perceived (higher bullish prior bias) preference shifts, providing a possible empirical guidance, is given in Table 2. According to our evidence, the results there should be fairly robust to changes in parametric specifications and therefore, provide an opportunity of real market data interpretation. For example, Table 2 predicts that an observed simultaneous growth of MB-volume and decrease of MS-volume is consistent with growing bearish moods without an actual growth in the percentage of bears. On the contrary, the true bear weight changes would be consistent with MB- and MS-volume movements in the same direction. Adding the information about the direction of change in net MB-pressure and the central price change, one can further identify the case of either the bull or bear majority dwindling (i.e., α reverting to the 50% mark) or expanding (α diverging to either 0 or 1).

Conclusion

We model price formation in a fast order-driven market at a point in time. There are many participants who may submit both LO and MO for a small standard quantity of the asset. Terms of trade facing every potential order are partially unobservable. LO submitters cannot keep up with book changes, whereas MO submitters do not observe for sufficient time the book against which their orders will be matched. At least, due to a large number of market orders arriving nearly simultaneously, no trader, for uncertain priority reasons, can be sure about the exact execution price. We have derived the equations for equilibrium mid price and execution depths for limit buys and sells.

The model can be useful in addressing many issues of order driven market operation, such as execution depth, execution costs or the value of information. We show that the market buy and sell order quantities together with the value histogram location do not make up a vector of sufficient statistics for the asset price. Generically, informational parameters of the trader population (including ones that can be loosely identified with the popular notion of 'market sentiment') influence the price, making it indeterminate for any given MO flow value. Formally, we have associated market sentiment with the initial bias in the investor private information.

The above findings may be important for the ongoing debate between the 'flow-centrists' and 'fundamentalists' among asset price theorists. In our model, trades drive prices subject to a given attitude profile (private value + information) of the investor population. Possibly, the longer the horizon, the more important becomes the common element in beliefs. So, for

time spans exceeding just one period of intensive trade, price evolution may mirror the attitude evolution more than anything else, even though traded order flow is always instrumental in channeling the attitude into the price.

Our conclusion on the insufficiency of order flow statistics for price determination by a broker may cast a new light on the similar proposition for the dealer markets. Specifically, if a LO book looks differently under the same values but different common biases in beliefs, then also dealer quotes may react differently to the same incoming order flow under different opinions she might share with her customers.

As should be clear from the technical exposition of the model (Sections 2 and 3), cognitive requirements on agents needed to achieve an equilibrium are substantial. Nevertheless, our rationality-based formalism can be useful for comparison with minimal-intelligence order-driven market models in the econophysics literature. The difference of predictions coming out of the two model classes could give an idea of the role of rationality in the studied market. One way of comparison is offered by our results on the role of sentiment (common bias in beliefs). One can imagine minimally rational traders endowed with a sentiment, but having an insufficient ability to infer other participants' order distributions accurately from own information. Then, our preliminary findings about the quantitative dominance of belief bias over belief dispersion effects on price discovery indicate that, to an extent, relaxing the rationality assumption has a second-order effect on price compared to allowing for a common prior bias. Naturally, if the traders are too naïve in terms of their order book and market order expectations (e.g., misconceptions about order distribution shape are stronger than misconceptions about the location of its modes), the sentiment effect may be overshadowed by those other cognitive distortions. Exact quantitative analysis would crucially depend on the choice of bounded rationality specification, which is a pre-requisite of future research in this direction. ❖

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Appendix

A.1 Proof of *Proposition 1*

The partial derivatives of an LS and LB submitter's surplus w.r.t. p can be expressed as:

$$L_p^B(y,p) = -B_p(y;p)\left[e^x - e^p\left(1+\frac{B(y;p)}{B_p(y;p)}\right)\right],\ L_p^S(y,p) = S_p(y;p)\left[e^x - e^p\left(1+\frac{S(y;p)}{S_p(y;p)}\right)\right]$$

If there is an interior solution obeying the first order conditions, then the limit sell and buy prices p will be chosen by traders with private values x so that,

$$e^x = e^p\left(1+\frac{B(y;p)}{B_p(y;p)}\right) = e^{g^B(y,p)},\ e^x = e^p\left(1+\frac{S(y;p)}{S_p(y;p)}\right) = e^{g^S(y,p)}$$

in the LS and LB case, respectively. The result about cumulative LO quantities follows from the assumed strict monotonicity of functions g^B and g^S with respect to the price variable.

A.2 Consistency Restrictions on Parameters of Subjective Execution Probabilities

Equations (1) to (6) of Subsection 2.1 define necessary conditions for equilibrium in the fast order-driven market constructed there. These conditions tie together individual components of the trade pattern (Q^B, Q^S, M^B, M^S), given trader histogram z. The 'free parameters' in this system of functional equations are the market state c and the pair of critical private value functions $g = (g^B, g^S)$, in their turn derived from execution probability

Appendix (Cont.)

functions B and S. However, the latter are themselves fully determined by the trader histogram and the trade pattern.

Indeed, probability $B(y;p)$ ($S(y;p)$) can be equivalently characterized as the y-probability that the number of market buys (sells) exceeds $Q^B(c, p)$ ($Q^S(c, p)$). For a trader with information y, the MO flow pair of m^B market buys and m^S market sells, is a random variable with density $m=\left(m^B,m^S\right)\mapsto h(y;m)$. Let $h^B(y;.),h^S(y;.)$ stand for the corresponding marginal densities and $H^B(y;.),H^S(y;.)$—their cumulative distribution functions. Put $G^{B,S}=1–H^{B,S}$. Then

$$B(y;p)=\int G^B\left(y;Q^B(c,p)\right)f(y;c)dc, S(y;p)=\int G^S\left(y;Q^S(c,p)\right)f(y;c)dc \quad ...(A1)$$

and

$$B_p(y;p)=-\int h^B\left(y;Q^B(c,p)\right)q^B(c;p)f(y;c)dc \quad ...(A2a)$$

$$S_p(y;p)=\int h^S\left(y;Q^S(c,p)\right)q^S(c;p)f(y;c)dc \quad ...(A2b)$$

From Equation (1) it follows that:

$$q^B(c;p)=\int z\left(g^B(y,p),y;c\right)g_p^B(y,p)dy, q^S(c;p)=\int z\left(g^S(y,p),y;c\right)g_p^S(y,p)dy \quad ...(A3)$$

Using Equations (A1) to (A3), one can express $[g^B, g^S]$ as a functional of Q^B, Q^S, h and f.

Finally, Assumption 1 together with the standard change-of-variable argument show that the subjective MO flow density h is uniquely determined by the subjective market state density f and the mapping T ($|DT|$ denotes the determinant of the map T differential):

$$f(y;c)=h(y;T(c))|DT|(c) \quad ...(A4)$$

In view of Equation (A4), g is no longer a free parameter, and the equilibria of the model are indexed only by the 3-dimensional market-state vector c. The totality of consistency restrictions (1)-(6), (A1)-(A4) can be seen as a system of integro-differential equations for the private log-asset value functions $g^{B,S}$. It can be reduced to a c-indexed system of algebraic equations in the HP case. In the HL case, however, even after possible dimensionality reductions, the equilibrium remains a system of non-linear integral equations for the order book function $Q=[Q^B,Q^S]$. Finding a numerical solution method for this system represents a separate problem.

A.3 Proof of Proposition 2

Let us drop the 0-subsrcipt by the components of c_0 for notational simplicity. Denote by $L^B(\theta,m,h)$ and $L^S(\theta,m,l)$ the quantities of limit orders hit by market orders, given by the right hand sides of Equation (10). Obviously, L^B is strictly increasing in h, whereas L^S is strictly increasing in l. We shall also prove that the market buy quantity M^B, which depends on h through the expected execution price, is a strictly decreasing function of h. Analogously, M^S is strictly decreasing in l. This is true for both HP- and HL-markets. To prove this, let us observe that in both HP- and HL-markets, the dependence of market order quantity $M^B(M^S)$ on the market state-conditional expected execution price $\hat{a}(\hat{b})$ is strictly monotonic. Therefore, it is sufficient to demonstrate for the auxiliary functions $\tilde{a}$ and $\tilde{b}$ introduced in Subsection 2.2 prior to Equation (8), that the first one of them is strictly increasing in p_2, whereas the second one is strictly decreasing in p_1.

We prove the statement for $\tilde{a}$, the $\tilde{b}$ case being fully analogous. Recall that:

$$\tilde{a}(\theta;p_1,p_2)=\frac{p_2Q^B(\theta,p_2)-p_1Q^B(\theta,p_1)-\int_{p_1}^{p_2}Q^B(\theta,p)dp}{Q^B(\theta,p_2)-Q^B(\theta,p_1)}$$

The p_2-partial derivative of the numerator of this expression is equal to $p_2q^B(\theta, p_2)$, whereas that of the denominator equals $q^B(\theta, p_2)$. Accordingly, the derivative of the whole fraction is a fraction with $L^B(\theta,p_1,p_2-p_1)^2$ in the denominator and $q^B(\theta,p_2)\left(\int_{p_1}^{p_2}Q^B(\theta,p)dp-(p_2-p_1)Q^B(\theta,p_1)\right)$ in the numerator. The expression in parentheses is strictly positive since the integral there is over a strictly increasing function with lower-end value $Q^B(\theta, p_1)$ on the interval with length p_2-p_1. Therefore, $\tilde{a}$ is indeed increasing in p_2.

Consequently, function $J^B=L^B-M^B$ (the difference between the left and the right hand side of the first equation in Equation (10)) is strictly increasing in h, converges to a negative quantity when $h\to0$ and has a positive limit below 1 when $h\to\infty$. Function $J^S=L^S-M^S$ (the difference between the left and the right hand side of the second equation in Equation (10)) is strictly increasing in l, has a negative limit when $l\to0$ and limit 1 when $l\to\infty$. Therefore, for every m, (h, l) can be found as the unique zero of the 2-dimensional map (J^B, J^S). In the wide inside spread case, this is sufficient for the solution.

Let us turn to the narrow inside spread case. We have just proved that the pair of Equations (10) has a unique solution (h, l) for every m. The solution to the whole system Equations (10) and (11) can be constructed by an iterative procedure, starting with any

initial value of m. Indeed, for the initial m, we select the initial (h, l) solving Equation (10). The next iteration for m is the one that solves Equation (11) for (h, l) fixed at the previously calculated value. Such a solution always exists as explained in Subsection 2.1. In this way, we generate the sequences of m-, h- and l-iterations. These sequences possess natural upper and lower limits. Indeed, m cannot explode since, when it grows to $+\infty$ $(-\infty)$, L^B (L^S) falls to zero, whereas M^B (M^S) stays positive, so that J^B (J^S) converges to a strictly negative value. Similarly, $\hat{a}$ has an upper limit for h approaching $+\infty$ (the quantity of limit sells quickly vanishes with the distance from the best ask and practically does not contribute to the average MB execution price). Therefore, J^B again has a strictly negative limit. The same is true for $l \to +\infty$. The iterations either end up in the wide inside spread region, for which the solution existence has been already proved, or converge to the limits forming the solution to Equations (10) and (11) in the narrow inside spread case.

A.4 Parameterization of the Trader Histogram and Subjective Limit Order Execution Probabilities

Let $\Phi(y^m, .)$, $V(y^h, .)$ and $W(y^l, .)$ be the cumulative distribution functions of m, h and l under information y. Then we can write:

$$B(y,p) = 1 - \Phi\left(y^m; p\right) + \int_{-\infty}^{p} \left[1 - V\left(y^h; p - m\right)\right] \varphi\left(y^m; m\right) dm$$

$$= 1 - \int_{-\infty}^{p} V\left(y^h; p - m\right) \varphi\left(y^m; m\right) dm \qquad \ldots(A5a)$$

$$S(y,p) = \Phi\left(y^m; p\right) + \int_{p}^{\infty} \left[1 - W\left(y^l, m - p\right)\right] \varphi\left(y^m; m\right) dm$$

$$= 1 - \int_{p}^{\infty} W\left(y^l; m - p\right) \varphi\left(y^m; m\right) dm \qquad \ldots(A5b)$$

and

$$B_p(y,p) = -\int_{-\infty}^{p} \nu\left(y^h; p - m\right) \varphi\left(y^m; m\right) dm, \quad S_p(y,p) = \int_{p}^{\infty} w\left(y^l; m - p\right) \varphi\left(y^m; m\right) dm \qquad \ldots(A6)$$

Under the information histogram parameterization defined in Section 3, integrals in Equation (A5) can be calculated exactly. The result is:

Appendix (Cont.)

$$B\left(y^m, y^h; p\right) = \frac{\varsigma}{\left[\varsigma^2 + \eta^2\right]^{1/2}} e^{-\frac{\left(p - y^m\right)^2}{2\left(\varsigma^2 + \eta^2\right)}} N\left(\frac{\varsigma\left(p - y^m\right)}{\eta\left[\varsigma^2 + \eta^2\right]^{1/2}}\right) + N\left(\frac{y^m - p}{\eta}\right)$$

with $\varsigma = Z\left(h_p, y^h\right) = \frac{h_p}{\left[1 + \left(h_p\right)^2 \left(y^h\right)^2\right]^{1/2}}$ and $\log h_p$ equal to the probability density mode of the prior trader belief about $\log h$. Analogously,

$$S\left(y^m, y^l; p\right) = \frac{\varsigma}{\left[\varsigma^2 + \eta^2\right]^{1/2}} e^{-\frac{\left(p - y^m\right)^2}{2\left(\varsigma^2 + \eta^2\right)}} N\left(\frac{\varsigma\left(y^m - p\right)}{\eta\left[\varsigma^2 + \eta^2\right]^{1/2}}\right) + N\left(\frac{p - y^m}{\eta}\right)$$

with $\varsigma = Z\left(l_p, y^l\right) = \frac{l_p}{\left[1 + \left(l_p\right)^2 \left(y^l\right)^2\right]^{1/2}}$ and $\log l_p$ equal to the probability density mode of the prior trader belief about $\log l$.

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