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A class of aggregation functions encompassing two-dimensional OWA operators

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ABSTRACT

In this paper we prove that, under suitable conditions, Atanassov's K_α operators, which act on intervals, provide the same numerical results as OWA operators of dimension two. On one hand, this allows us to recover OWA operators from K_α operators. On the other hand, by analyzing the properties of Atanassov's operators, we can generalize them. In this way, we introduce a class of aggregation functions – the generalized Atanassov operators – that, in particular, include two-dimensional OWA operators. We investigate under which conditions these generalized Atanassov operators satisfy some properties usually required for aggregation functions, such as bisymmetry, strictness, monotonicity, etc. We also show that if we apply these aggregation functions to interval-valued fuzzy sets, we obtain an ordered family of fuzzy sets.

1. Introduction

In 1983 Atanassov introduced a new operator [2] allowing to associate a fuzzy set with each Atanassov intuitionistic fuzzy set or interval-valued fuzzy set (IVFS) [17,20]. In fact, this operator, which we denote by K_α, takes a value from the interval representing the membership to the IVFS and defines that value to be the membership degree to a fuzzy set [26,27]. In this way, it is possible, for instance, to recover all the usual fuzzy set theoretic results when dealing with IVFS. In 1988 Yager presented the definition of an OWA operator [22].

Comparison of the results of Atanassov and Yager reveals that in two dimensions the numerical results provided by Atanassov operators and OWA operators are the same. This numerical coincidence prompted us to introduce and define new operators by suitably modifying the domain for the definition of Atanassov's operators. Analysis of the properties required for Atanassov's operators has allowed us to consider a class of aggregation functions that are a generalization of Atanassov's operators [6–8]. In particular, it would be interesting to determine whether some of the properties that are usually required for aggregation functions, such as bisymmetry, strictness, monotonicity, etc., also hold for this class of generalized Atanassov operators.

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As already stated, the original aim of Atanassov was to build fuzzy sets from IVFS. We have readdressed this aim for our generalized Atanassov operators. This enables us to use these new operators in all the fields in which Atanassov operators have worked well. For instance, because there is quite a simple way of associating each image with an IVFS in such a way that the membership interval represents to some extent the properties of a piece of the image, we can use our generalized Atanassov operators and the results linking them to OWA operators for image processing.

The remainder of the paper is organized as follows. The concepts of K_α operators, aggregation functions, OWA operators and IVFS are described in Section 2. Section 3 presents the relation between OWA and K_α operators. In Section 4 we present a generalization of the K_α operator properties and two construction theorems. In the same section, we define a new family of operators acting on pairs of real numbers and investigate their main properties. In Section 5 we propose two methods to obtain fuzzy sets from IVFS by means of generalized K_α operators. Section 6 concludes the paper.

2. Preliminary definitions

In fuzzy set theory, a strictly decreasing and continuous function $N : [0,1] \to [0,1]$ such that $N(0) = 1, N(1) = 0$ is called a strict negation. If, in addition, N is involutive, then we say that it is a strong negation. We call automorphism of the unit interval every function $\varphi : [0,1] \to [0,1]$ that is continuous, strictly increasing and such that $\varphi(0) = 0$ and $\varphi(1) = 1$.

In 1979 Trillas [21] presented the following theorem of characterization of strong negations.

Theorem 1. *A function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists an automorphism φ of the unit interval such that $N(x) = \varphi^{-1}(1 - \varphi(x))$.*

A function $T : [0,1]^2 \to [0,1]$ is said to be a t-norm if it is commutative, associative, increasing and has neutral element 1. In the same way, a function $S : [0,1]^2 \to [0,1]$ is said to be a t-conorm if it is commutative, associative, increasing and has neutral element 0.

A variation of t-norms with modification only of the axiom concerning the neutral element was recently proposed [9] in the following way:

Definition 1. A nullnorm is a binary operation V on the unit interval [0,1], i.e., a function $V : [0,1]^2 \to [0,1]$, which is commutative, associative and increasing, and there exists $a \in]0,1[$ such that $V(x,0) = x$ for all $x \in [0,a]$ and $V(x,1) = x$ for all $x \in [a,1]$.

We consider a finite, non-empty referential set $U = \{u_1, \ldots, u_n\}$. A fuzzy set is defined as $A = \{(u, \mu_A(u)) | u \in U\}$, with $\mu_A : U \to [0,1]$ a membership function. $FS(U)$ is the set of all fuzzy sets defined on U. For a given strict negation N the expression $A_N = \{(u, N(\mu_A(u))) | u \in U\}$ is used to denote the complement of the fuzzy set A with respect to N.

We consider the following order relationship in $FS(U)$, introduced by Zadeh [24]. For $A, B \in FS(U)$, $A \leqslant B$ if and only if $\mu_A(u) \leqslant \mu_B(u)$ for all $u \in U$.

We denote by $L([0,1])$ the set of all closed subintervals of the unit interval [0,1], that is:

$$L([0,1]) = \left\{ \mathbf{x} = [\underline{x}, \bar{x}] | (\underline{x}, \bar{x}) \in [0,1]^2 \text{ and } \underline{x} \leqslant \bar{x} \right\}.$$

$L([0,1])$ is a partially ordered set with respect to the order relationship $\leqslant_L$ defined in the following way. Given $\mathbf{x}, \mathbf{y} \in L([0,1])$,

$$\mathbf{x} \leqslant_L \mathbf{y} \quad \text{if and only if } \underline{x} \leqslant \underline{y} \quad \text{and} \quad \bar{x} \leqslant \bar{y}.$$

With this order relationship, $(L([0,1]), \leqslant_L)$ is a complete lattice [6,8,11,13], where the smallest element is $0_L = [0,0]$ and the largest is $1_L = [1,1]$.

Given $\mathbf{x}, \mathbf{y} \in L([0,1])$, we define:

$$\mathbf{x} <_L \mathbf{y} \quad \text{if and only if } \begin{cases} \underline{x} < \underline{y} \quad \text{and} \quad \bar{x} \leqslant \bar{y} \\ \text{or} \\ \underline{x} \leqslant \underline{y} \quad \text{and} \quad \bar{x} < \bar{y}. \end{cases}$$

Definition 2 [25]. An IVFS A on the universe U is defined by a membership function $\mathbf{M_A} : U \to L([0,1])$.

We use bold font to denote mappings that are defined over $L([0,1])$. *IVFS*(U) is the set of all IVFS over the universe U. $\mathbf{M_A}(u) = [\underline{A}(u), \overline{A}(u)] \in L([0,1])$ is the membership degree of $u \in U$, with $\underline{A}(u), \overline{A}(u) \in [0,1]$ denoting the lower and upper bound, respectively, of the membership associated with u. Observe that a fuzzy set can be considered to be a particular type of IVFS with the membership interval reduced to a single point, i.e., $\underline{A}(u) = \overline{A}(u)$.

Given an interval $\mathbf{x} = [\underline{x}, \bar{x}] \in L([0,1])$, $Length(\mathbf{x}) = \bar{x} - \underline{x}$ is its length.

2.1. Aggregation functions

An n-ary aggregation function was formally defined by Calvo et al. [10] (see also [19]) as a function

$$M : [0,1]^n \to [0,1],$$

such that,

(i) $M(x_1,\dots,x_n) \leqslant M(y_1,\dots,y_n)$ whenever $x_i \leqslant y_i$ for all $i \in \{1,\dots,n\}$.
(ii) $M(0,\dots,0) = 0$ and $M(1,\dots,1) = 1$.

Note that *idempotency*, which establishes that $M(x,\dots,x) = x$ for all $x \in [0,1]$, is not universally recognized as a property required for an aggregation function and some alternative properties are quite often assumed (see [16] for a discussion). In any case, all the operators we consider here satisfy this property.

Here, unless otherwise stated, we consider binary aggregation functions; that is, n-ary aggregation functions with $n = 2$. Moreover, it should be recalled that the following properties can be required for an aggregation function.

- An aggregation function is called *commutative* or *symmetric* if
$$M(x,y) = M(y,x), \quad \text{for all } x,y \in [0,1].$$
- An aggregation operation M is called *bisymmetric* if
$$M(M(x,y),M(z,t)) = M(M(x,z),M(y,t)), \quad \text{for all } x,y,z,t \in [0,1].$$
- An element $a \in [0,1]$ is called an *annihilator* of an aggregation operation M if
$$M(a,x) = M(x,a) = a, \quad \text{for all } x \in [0,1].$$
- An aggregation function M is said to be *strictly increasing* if for any $x_1,x_2,y_1,y_2 \in [0,1]$ such that $x_1 \leqslant y_1, x_2 \leqslant y_2$ with $(x_1,x_2) \neq (y_1,y_2)$, the inequality $M(x_1,x_2) < M(y_1,y_2)$ holds.

In 1948, Aczél presented the following result [1].

Theorem 2. *Let $M : [0,1]^2 \to [0,1]$ be a function. Then M is continuous, strictly increasing, idempotent and bisymmetric if and only if there exists a continuous strictly increasing function $f : [0,1] \to [0,1]$ and a real number $p \in]0,1[$ such that*

$$M(x,y) = f^{-1}(pf(x) + (1-p)f(y)).$$

Later, Fodor and Marichal considered the general form of continuous, commutative, increasing, idempotent and bisymmetric functions M [14]. In other words, they analyzed Aczél's theorem when M is required to be increasing (not necessarily strictly increasing).

Consider three real numbers $x,y,z \in \mathbb{R}$. Their *median* (denoted as *median*(x,y,z)) is defined as:

$$median(x,y,z) = \begin{cases} x & \text{if } \min(y,z) \leqslant x \leqslant \max(y,z), \\ y & \text{if } \min(x,z) \leqslant y \leqslant \max(x,z), \\ z & \text{if } \min(x,y) \leqslant z \leqslant \max(x,y). \end{cases}$$

With this notion of median, Fodor and Marichal proved the following theorem for any interval, but we restrict ourselves to the unit interval.

Theorem 3. *$M : [0,1]^2 \to [0,1]$ is a continuous, commutative, increasing, idempotent, bisymmetric function if and only if there exist two real numbers λ and ρ fulfilling $0 \leqslant \lambda \leqslant \rho \leqslant 1$ such that:*

(a) $M(x,y) = M_{0,\lambda,\lambda}(x,y)$, *if* $x,y \in [0,\lambda]$;
(b) $M(x,y) = M_{\rho,1,\rho}(x,y)$, *if* $x,y \in [\rho,1]$;
(c) $M(x,y) = f^{-1}\left(\frac{f(median(\lambda,x,\rho))+f(median(\lambda,y,\rho))}{2}\right)$ otherwise,

with $M_{0,\lambda,\lambda} : [0,\lambda]^2 \to [0,\lambda]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M(0,\lambda) = \lambda$; $M_{\rho,1,\rho} : [\rho,1]^2 \to [\rho,1]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M(\rho,1) = \rho$ and f any continuous, bounded, strictly increasing function on $[\lambda,\rho]$.

2.2. Ordered weighted averaging aggregation operators

As already stated in the introduction, Yager introduced a particular type of aggregation function [22], the so-called ordered weighted averaging aggregation (OWA) operator.

Definition 3. A function $F : [0,1]^n \to [0,1]$ is called an OWA operator of dimension n if there exists a weighting vector W, $W = (w_1,w_2,\dots,w_n) \in [0,1]^n$ with $\sum_i w_i = 1$, and such that

$$F(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \quad \text{with } b_j \text{ the } j\text{th largest of the } a_i,$$

for any $(a_1, \ldots, a_n) \in [0,1]^n$.

Any OWA operator is completely defined by its weighting vector. However, in his original definition, Yager considered functions F defined on the whole Euclidean space $\mathbb{R}^n$ and taking values in $\mathbb{R}$, but for our interest it is more appropriate to reduce this to $[0,1]^n$. Observe that with restriction $\sum_i w_i = 1$, if any of the components of the weighting vector W is equal to 1, the other components should be zero.

Evidently, each OWA operator is a commutative, continuous, idempotent aggregation function [10,12,15,23]. Besides, any OWA operator F is an averaging function, i.e., it verifies $\min \leqslant F \leqslant \max$ (recall that this is a property fulfilled by any idempotent aggregation function). For an OWA operator the so-called stability under positive linear transformations with the same unit and independent zeros (SPLU) is also fulfilled [15]. That is, if F is an OWA operator of dimension n, and if $r > 0$ and $t \in [0,1]$, then, for any $(a_1, \ldots, a_n) \in [0,1]^n$, such that $(ra_1 + t, ra_2 + t, \ldots, ra_n + t) \in [0,1]^n$, the following holds:

$$F(ra_1 + t, ra_2 + t, \ldots, ra_n + t) = rF(a_1, a_2, \ldots, a_n) + t.$$

Yager defined and investigated the following particular types of OWA operators, which coincide with well-known specific cases of aggregation functions [23].

1. The "or" operator F^*: The weighting vector, denoted by W^*, is defined as $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$. Observe that $F^*(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n)$.
2. The "and" operator F_*: The weighting vector, denoted by W_*, is defined as $w_n = 1$ and $w_j = 0$ for all $j \neq n$. Observe that $F_*(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$.
3. The averaging operator F_A: The weighting vector, denoted by W_A, is defined as $w_j = 1/n$ for all $j \in \{1, \ldots, n\}$. Observe that in fact F_A recovers the arithmetic mean of $x_1, \ldots, x_n$.

Moreover, since OWA operators are averaging functions, the OWA operators F_* and F^* can be considered the "smallest" and "largest" OWA operators in the following sense.

$$F_*(a_1, \ldots, a_n) \leqslant F(a_1, \ldots, a_n) \leqslant F^*(a_1, \ldots, a_n) \quad \text{for all } (a_1, \ldots, a_n) \in [0,1]^n.$$

Given an OWA operator of dimension n, another OWA operator of the same dimension can be built by duality in the following way [22].

Definition 4. Let F be an OWA operator of dimension n with weighting vector $W = (w_1, \ldots, w_j, \ldots w_n)$. The dual operator of F, denoted by $\widehat{F}$, is the OWA operator given by the dual weighting vector $\widehat{W} = (w_n, \ldots, w_{n-j+1}, \ldots, w_1)$.

To measure how far a given OWA operator is from F_* and F^*, the following measure was introduced by Yager [23].

Definition 5. Let F be an OWA operator of dimension n and W its weighting vector. The orness measure of W is defined as

$$orness(W) = \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i.$$

From this definition it is easily shown that $orness(W^*) = 1, orness(W_*) = 0$ and $orness(W_A) = 0.5$. Yager also proved that the greater the orness of an OWA operator, the closer that operator is to the pure "or" operator F^* [23].

Yager introduced another measure to compare OWA operators that have the same orness.

Definition 6. Let F be an OWA operator of dimension n and W its weighting vector. Its dispersion measure is defined as

$$dispersion(W) = -\sum_{i=1}^{n} w_i \ln w_i.$$

Observe that dispersion can be understood as a measure of entropy, as it shows how "far" a given OWA operator is from the averaging operator F_A [22]. In particular:

(1) If $w_i = 1$ for some i, then $dispersion(W) = 0$, so the dispersion is minimal.
(2) If $w_i = 1/n$ for all i, then $dispersion(W) = \ln n$, so the dispersion is maximal.

One of the main advantages of OWA operators is the flexibility in the choice of the types of aggregation rules that can be modeled. However, a problem arises as to how to determine the weights to be used in a particular application.

3. OWA operators and K_α operators

As stated in the introduction, Atanassov proposed a family of operators to associate a fuzzy set to each IVFS [2,3].

Definition 7. The operator $K : [0,1] \times L([0,1]) \to [0,1]$ is given by $K = (K_\alpha)_{\alpha\in[0,1]}$, with each operator $K_\alpha : L([0,1]) \to [0,1]$ defined as a convex combination of its boundary arguments by

$$K_\alpha(\mathbf{x}) = \alpha \cdot \bar{x} + (1-\alpha) \cdot \underline{x},$$

where for any $\mathbf{x} \in L([0,1])$ we write $\mathbf{x} = [\underline{x}, \bar{x}]$.

Clearly the following properties hold.

(i) $K_0(\mathbf{x}) = \underline{x}$ for all $\mathbf{x} \in L([0,1])$.
(ii) $K_1(\mathbf{x}) = \bar{x}$ for all $\mathbf{x} \in L([0,1])$.
(iii) $K_\alpha(\mathbf{x}) = K_\alpha([K_0(\mathbf{x}), K_1(\mathbf{x})]) = K_0(\mathbf{x}) + \alpha(K_1(\mathbf{x}) - K_0(\mathbf{x})) = \underline{x} + \alpha(\bar{x} - \underline{x})$ for all $\mathbf{x} \in L([0,1])$.

Obviously $K_\alpha(\mathbf{x}) = \alpha \cdot \bar{x} + (1-\alpha) \cdot \underline{x}$, and because $\bar{x} \geqslant \underline{x}$, the family $(K_\alpha)_{\alpha\in[0,1]}$ is increasing.

Note that if we take the two-dimensional OWA operator F with weighting vector $W = (\alpha, 1-\alpha)$ and apply it to the bounds of the intervals, we obtain

$$F(\underline{x}, \bar{x}) = F(\bar{x}, \underline{x}) = K_\alpha(\mathbf{x})$$

for all $\mathbf{x} \in L([0,1])$. Nevertheless, although in these conditions the numerical value of both operators coincide, the two concepts are very different. K_α acts on elements of $L([0,1])$, whereas the OWA operator F acts over $[0,1] \times [0,1]$. In other words, the domains of both operators are different. In particular, K_α is defined on the set of pairs of points, extremes of the intervals, that are ordered. However, an OWA operator is defined on the unit square and requires an ordering operation.

This numerical coincidence prompted us to study possible relations between the two concepts, as in the following results.

Theorem 4 ($\mathbb{K}_\alpha$ *operators are OWA operators of dimension 2). Let* $\alpha \in [0,1]$ *and* $\mathbb{K}_\alpha = K_\alpha \circ i$, *where* K_α *is the operator given in Definition 7 and* $i : [0,1]^2 \to L([0,1])$ *given by*

$$i(x,y) = [\min(x,y), \max(x,y)].$$

Then, if $F(x,y)$ *is the OWA operator (of dimension 2) defined by the weighting vector* $W = (\alpha, 1-\alpha)$, *we have*

$$\mathbb{K}_\alpha(x,y) = F(x,y) \quad \textit{for all } x,y \in [0,1].$$

Proof. It is sufficient to take into account that any interval $\mathbf{x} \in L([0,1])$ is defined by a pair $(\underline{x}, \bar{x}) \in [0,1]^2$ with $\underline{x} \leqslant \bar{x}$, the shape of the operator $\mathbb{K}_\alpha$ and the definition of OWA operators. □

In particular, since the operator $\mathbb{K}_\alpha$ is an OWA operator, we have the following corollary.

Corollary 1. *Let* $\alpha \in [0,1]$. *Then the following hold:*

(a) $\mathbb{K}_\alpha$ is commutative and idempotent.
(b) $\mathbb{K}_0(x,y) = \min(x,y)$ and $\mathbb{K}_1(x,y) = \max(x,y)$.
(c) $\mathbb{K}_\alpha$ is increasing.
(d) *Let* $\beta \in [0,1]$. *If* $\alpha \leqslant \beta$, *then* $\mathbb{K}_\alpha(x,y) \leqslant \mathbb{K}_\beta(x,y)$ *for all* $(x,y) \in [0,1]^2$.
(e) $\mathbb{K}_\alpha(0,1) = \alpha$.

Proof. The proof directly follows from Theorem 4 and the well-known properties of OWA operators. □

Theorem 5 (OWA operators of dimension 2 are $\mathbb{K}_\alpha$ operators). *Let F be an OWA operator of dimension 2 with weighting vector* $W = (w_1, w_2)$. *Then for any* $(x,y) \in [0,1]^2$ *we have*

$$F(x,y) = \mathbb{K}_\alpha(x,y),$$

with $\alpha = w_1$.

Proof. The proof directly follows from the definition of $\mathbb{K}_\alpha$. □

Because K_α operators are written in terms of K_0 and K_1, we can also express the OWA operators in terms of these two OWA operators. In particular, observe that if F is an OWA operator of dimension 2 defined by the weighting vector (w_1, w_2), then, for any $x,y \in [0,1]$

$$F(x,y) = w_1\mathbb{K}_1(x,y) + w_2\mathbb{K}_0(x,y).$$

In particular,

(1) $F^*(x,y) = K_1([\min(x,y), \max(x,y)]) = \mathbb{K}_1(x,y)$;
(2) $F_*(x,y) = K_0([\min(x,y), \max(x,y)]) = \mathbb{K}_0(x,y)$; and

(3) $F_A(x,y) = (K_0([\min(x,y),\max(x,y)]) + K_1([\min(x,y),\max(x,y)]))/2 = \frac{1}{2}(x+y) = (\mathbb{K}_0(x,y) + \mathbb{K}_1(x,y))/2$

For the orness and dispersion we have the following results.

Proposition 1. *Let $\alpha \in [0,1]$. The following properties hold:*

(1) *$orness(\mathbb{K}_\alpha) = \alpha$.*
(2) *$dispersion(\mathbb{K}_\alpha) = \alpha \ln\left(\frac{1-\alpha}{\alpha}\right) - \ln(1-\alpha)$.*

Proof. The proof follows directly from the definitions of orness and dispersion. □

It is easy to see that dispersion is maximal when $\alpha = 1/2$ – which means that both weights are equal – and is minimal when $\alpha = 1$ or $\alpha = 0$, and, as in the general OWA case, could be used to measure the entropy of the transformation of an IVFS in a fuzzy set.

Proposition 2. *Let $\alpha \in [0,1]$ and $\mathbb{K}_\alpha = K_\alpha \circ i$, where K_α is the operator given in Definition 7 and i is the function given in Theorem 4. The following properties hold.*

(1) *If $\alpha \in]0,1[$, then $\mathbb{K}_\alpha$ is strictly increasing;*
(2) *If $\alpha \in]0,1[$, then $\mathbb{K}_\alpha(x,y) = 0$ if and only if $x = y = 0$.*
(3) *If $\alpha \in]0,1[$, then $\mathbb{K}_\alpha(x,y) = 1$ if and only if $x = y = 1$.*
(4) $\mathbb{K}_\alpha$ has the SPLU property.
(5) *$\mathbb{K}_\alpha(x,y) + \widehat{\mathbb{K}}_\alpha(x,y) = x + y$, where $\widehat{\mathbb{K}}_\alpha$ denotes the dual mapping of $\mathbb{K}_\alpha$, as given in Definition 4.*
(6) $\mathbb{K}_\alpha = \widehat{\mathbb{K}}_{1-\alpha}$.

Proof. All these properties follow from the corresponding ones for two-dimensional OWA operators. □

Although some of these properties are already known for the K_α operators, they are now a consequence of the OWA perspective. The SPLU (stability under positive linear transformations with the same unit and independent zeros) property can be interpreted as a partial translation invariance and is useful in applications that require handling of general amplitudes.

Remark 1. Let $\alpha \in]0,1[$ and $\mathbb{K}_\alpha = K_\alpha \circ i$, where K_α is the operator given in Definition 7. Then:

(1) $\mathbb{K}_\alpha$ is not associative. Consider

$$\mathbb{K}_\alpha(\mathbb{K}_\alpha(0,1),1) = \mathbb{K}_\alpha(\alpha,1) = 2\alpha - \alpha^2,$$

whereas

$$\mathbb{K}_\alpha(0,\mathbb{K}_\alpha(1,1)) = \mathbb{K}_\alpha(0,1) = \alpha,$$

and $2\alpha - \alpha^2 \neq \alpha$ if $0 \neq \alpha \neq 1$.

(2) $\mathbb{K}_\alpha$ is not bisymmetric whenever $\alpha \neq \frac{1}{2}$. $\mathbb{K}_\alpha(\mathbb{K}_\alpha(0,1),\mathbb{K}_\alpha(\alpha,\alpha)) = \mathbb{K}_\alpha(\alpha,\alpha) = \alpha$. By contrast, $\mathbb{K}_\alpha(\mathbb{K}_\alpha(0,\alpha),\mathbb{K}_\alpha(1,\alpha)) = \mathbb{K}_\alpha(\alpha^2, 2\alpha-\alpha^2) = 3\alpha^2 - 2\alpha^3$. In this situation for $\alpha \notin \{0, 0.5, 1\}$, bisymmetry does not hold.

4. Generalized K_α operators

Observe that, if we denote by K the system of operators $(K_\alpha)_{\alpha\in[0,1]}$, then K can be regarded as an operator on $[0,1] \times L([0,1])$ with values in $[0,1]$. To generalize this operator, the following definition was proposed by Bustince et al. [6,8,7].

Definition 8. A GK operator is a mapping $GK : [0,1] \times L([0,1]) \to [0,1]$ such that, if we denote $GK_\alpha(\mathbf{x}) = GK(\alpha,\mathbf{x})$, the following properties hold:

(i) If $\underline{x} = \bar{x}$, then $GK_\alpha(\mathbf{x}) = \underline{x}$.
(ii) $GK_0(\mathbf{x}) = \underline{x}, GK_1(\mathbf{x}) = \bar{x}$ for all $\mathbf{x} \in L([0,1])$.
(iii) If $\mathbf{x} \leqslant_L \mathbf{y}$, with $\mathbf{x},\mathbf{y} \in L([0,1])$, then $GK_\alpha(\mathbf{x}) \leqslant GK_\alpha(\mathbf{y})$.
(iv) Let $\beta \in [0,1]$. If $\alpha \leqslant \beta$, then $GK_\alpha(\mathbf{x}) \leqslant GK_\beta(\mathbf{x})$ for all $\mathbf{x} \in L([0,1])$.
(v) $GK_\alpha([0,1]) = \alpha$.

From the theoretical point of view, condition (v) above might be very strong. From the applied point of view, it is quite important to ensure that $GK(\cdot,[0,1])$ provides a bijection from the unit interval onto itself. Condition (v) above is a very simple way of building this bijection, in such a way that, moreover, $[0,1]$ comes out to be a neutral element.

Example 1

$$GK_\alpha([\underline{x},\bar{x}]) = \begin{cases} \bar{x} & \text{if } \bar{x} \leqslant \alpha, \\ \underline{x} & \text{if } \underline{x} \geqslant \alpha, \\ \alpha & \text{otherwise}. \end{cases} \quad (1)$$

Note that the result obtained is nothing but the α-median of $\underline{x}$ and $\bar{x}$, or, equivalently, the result given by the idempotent nullnorm with annihilator α when applied to $\underline{x}$ and $\bar{x}$ (see, e.g. [10]).

Proposition 3. *Any system* $K = (K_\alpha)_{\alpha\in[0,1]}$ *is a GK operator.*

By contrast, the K_α operators can be considered the "simplest" GK_α operators, in the sense of the following proposition.

Proposition 4. *Let* $\mathscr{H} = (H_\alpha)_{\alpha\in[0,1]}$ *be a GK operator such that, for any* $\alpha \in [0,1], H_\alpha$ *is a linear mapping of the extremes of the interval, i.e.*

$$H_\alpha(\mathbf{x}) = a(\alpha)\underline{x} + b(\alpha)\bar{x},$$

for some mappings $a, b : [0,1] \to [0,1]$. *Then* $a(\alpha) = 1-\alpha$ *and* $b(\alpha) = \alpha$ *for any* $\alpha \in [0,1]$, *that is,* $H_\alpha = K_\alpha$.

Proof. Suppose that $H_\alpha(\mathbf{x}) = a(\alpha)\underline{x} + b(\alpha)\bar{x}$ as in the statement of the proposition. From (i) in the definition of GK_α operators we have that $H_\alpha([\underline{x},\underline{x}]) = a(\alpha)\underline{x} + b(\alpha)\underline{x} = \underline{x}$ for all $\underline{x} \in [0,1]$, so $a(\alpha) = 1 - b(\alpha)$. Since from (v) in the same definition $H_\alpha([0,1]) = (1-b(\alpha))\cdot 0 + b(\alpha)\cdot 1 = \alpha$, it follows that $b(\alpha) = \alpha$ and the result holds. $\square$

4.1. Construction of operators GK_α

Theorem 6. *Let* $\alpha \in [0,1]$ *and let* $f : [0,1] \to [0,1]$ *be a continuous and strictly increasing function. Then the operator* $GK_\alpha : L([0,1]) \to [0,1]$ *given by*

$$GK_\alpha(\mathbf{x}) = f^{-1}((1-p)f(\underline{x}) + pf(\bar{x})),$$

with $p = \frac{f(\alpha)-f(0)}{f(1)-f(0)}$, *is a continuous* GK_α *operator in the sense of Definition 8.*

Proof. We see that all the properties in Definition 8 hold. Continuity is clear. If $\alpha = 0$, then $p = 0$. In this case $GK_0([\underline{x},\bar{x}]) = f^{-1}(f(\underline{x})) = \underline{x}$. If $\alpha = 1$, then $p = 1$ and therefore $GK_1([\underline{x},\bar{x}]) = \bar{x}$.

If $[x,y] <_L [z,y]$, because f is continuous and strictly increasing, we have, for p as in the statement of the theorem and fixed, $GK_\alpha([x,y]) = f^{-1}((1-p)f(x) + pf(y)) < f^{-1}((1-p)f(z) + pf(y)) = GK_\alpha([z,y])$.

Take $\alpha, \beta \in [0,1]$. If $\alpha \leqslant \beta$, then $f(\alpha) \leqslant f(\beta)$. Since p is increasing in α and f is a strictly increasing function, it follows that $GK_\alpha([\underline{x},\bar{x}]) \leqslant GK_\beta([\underline{x},\bar{x}])$.

$GK_\alpha([0,1]) = f^{-1}((1-p)f(0) + pf(1)) = f^{-1}(f(0) + p(f(1) - f(0))) = f^{-1}\left(f(0) + \frac{f(\alpha)-f(0)}{f(1)-f(0)}(f(1) - f(0))\right) = f^{-1}(f(\alpha)) = \alpha.$ $\square$

Remark 2. Continuity of the function f is necessary in Theorem 6, as the following example shows. Define

$$f(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}x + \frac{1}{2} & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

Then it is clear that f is not surjective, because, for instance, there is no inverse image for the point $x = \frac{5}{8}$, so the function f^{-1} is not defined over the whole $[0,1]$. Since a strictly increasing, surjective function under the conditions of Theorem 6 should be continuous – it cannot have any point of discontinuity, because it cannot have jumps – the hypothesis of continuity is natural.

Corollary 2. *Let* φ *be an automorphism of the unit interval and a GK operator* $GK = (GK_\alpha)_{\alpha\in[0,1]}$. *Then the system* $GK^{(\varphi)} = \left(GK_\alpha^{(\varphi)}\right)_{\alpha\in[0,1]}$ *where, for any* $\alpha \in [0,1], GK_\alpha^{(\varphi)} : L([0,1]) \to [0,1]$ *is defined by*

$$GK_\alpha^{(\varphi)}(\mathbf{x}) = \varphi^{-1}(\varphi(\alpha)\varphi(\bar{x}) + (1-\varphi(\alpha))\varphi(\underline{x})),$$

is a continuous GK operator in the sense of Definition 8.

Proof. The proof is immediate. $\square$

Example 2. Just by taking $\varphi(x) = x$ we recover Atanassov's operator.

Example 3. For each $q > 0$ the function $\varphi(x) = x^q$ defines the GK operator given by

$$GK_\alpha^{(\varphi)}(\mathbf{x}) = \left(\alpha^q(\bar{x})^q + (1-\alpha^q)(\underline{x})^q\right)^{\frac{1}{q}},$$

for $\alpha \in [0,1]$, that is, a weighted root-power mean (see, e.g. [10]) of $\underline{x}$ and $\bar{x}$.

Corollary 3. *Let N be the strong negation generated by Theorem 1 using the automorphism in Corollary 2 . Then we have:*

$$GK_\alpha^{(\varphi)}(\mathbf{x}) = \varphi^{-1}(\varphi(\alpha)\varphi(\bar{x}) + \varphi(N(\alpha))\varphi(\underline{x})).$$

Proof. The proof is direct. $\square$

4.2. $G\mathbb{K}_\alpha$ operators

Operators GK_α act on elements of $L([0,1])$. Now we generalize them in such a way that they are defined over $[0,1]^2$, at the same time that they retain most of the properties of GK_α. This leads to the following theorem.

Theorem 7. *Consider $i : [0,1]^2 \to L([0,1])$ given by*

$$i(x,y) = [\min(x,y), \max(x,y)].$$

Let $\alpha \in [0,1]$ and consider $G\mathbb{K}_\alpha : [0,1]^2 \to [0,1]$ given by

$$G\mathbb{K}_\alpha(x,y) = (GK_\alpha \circ i)(x,y),$$

where GK_α is the operator given in Definition 8 . Then the following observations hold.

(a) $G\mathbb{K}_\alpha$ is commutative and idempotent.
(b) $G\mathbb{K}_0(x,y) = \min(x,y)$ and $G\mathbb{K}_1(x,y) = \max(x,y)$.
(c) $G\mathbb{K}_\alpha$ is increasing.
(d) *Let $\beta \in [0,1]$. If $\alpha \leqslant \beta$, then $G\mathbb{K}_\alpha(x,y) \leqslant G\mathbb{K}_\beta(x,y)$ for all $(x,y) \in [0,1]^2$.*
(e) $G\mathbb{K}_\alpha(0,1) = \alpha$.

Proof. The proof is direct. □

Proposition 5. *Let $\alpha \in [0,1]$ and $G\mathbb{K}_\alpha = GK_\alpha \circ i$, where GK_α is the operator given in Definition 8 and i is the function given in Theorem 7 . The following properties hold.*

(1) *If $\alpha = G\mathbb{K}_\alpha(\alpha,1)$ with $\alpha \in]0,1]$, then $G\mathbb{K}_\alpha$ is not strictly increasing.*
(2) *If $\alpha = G\mathbb{K}_\alpha(\alpha,0)$ with $\alpha \in [0,1[$, then $G\mathbb{K}_\alpha$ is not strictly increasing.*
(3) *If GK_α is strictly increasing, then $G\mathbb{K}_\alpha$ is strictly increasing.*
(4) $G\mathbb{K}_\alpha(\min(x,y), \max(x,y)) = G\mathbb{K}_\alpha(x,y)$ for all $(x,y) \in [0,1]^2$.

Proof. (1) Suppose that $G\mathbb{K}_\alpha$ is strictly increasing. By taking $\alpha \in]0,1]$ we have $\alpha = G\mathbb{K}_\alpha(\alpha,1) > G\mathbb{K}_\alpha(0,1) = \alpha$, which is contradictory. (2) As for item (1). (3)–(4) The proof is direct. □

Example 4. If in the construction of $G\mathbb{K}_\alpha = GK_\alpha \circ i$ we take expression (1) for GK_α, the operator we obtain is an α-median (an idempotent nullnorm with annihilator α). Among other properties, it satisfies $\alpha = G\mathbb{K}_\alpha(\alpha,0)$ and $\alpha = G\mathbb{K}_\alpha(\alpha,1)$ (in fact it satisfies $\alpha = G\mathbb{K}_\alpha(\alpha,x)$ for any $x \in [0,1]$) and it is not strictly increasing.

4.3. Bisymmetric $G\mathbb{K}_\alpha$ operators

In Remark 1 we have seen that the operator $\mathbb{K}_\alpha = K_\alpha \circ i$ constructed from the K_α operator is not bisymmetric. Hence, if we take $GK_\alpha = K_\alpha$, the operator

$$\begin{aligned} G\mathbb{K}_\alpha(x,y) &= (GK_\alpha \circ i)(x,y) = GK_\alpha([\min(x,y), \max(x,y)]) = K_\alpha([\min(x,y), \max(x,y)]) \\ &= \min(x,y) + \alpha(\max(x,y) - \min(x,y)), \end{aligned}$$

is not bisymmetric. Nevertheless, if we take expression (1) for GK_α, then $G\mathbb{K}_\alpha = GK_\alpha \circ i$ is bisymmetric. All these considerations prompted us to study the bisymmetry of $G\mathbb{K}_\alpha$.

Theorem 8. *Let $G\mathbb{K}_\alpha = GK_\alpha \circ i$ be a bisymmetric operator and GK_α the operator given in Definition 8 . Then:*

$$G\mathbb{K}_\alpha(\alpha,1) = \alpha \quad \textit{if and only if} \quad G\mathbb{K}_\alpha(\alpha,0) = \alpha.$$

Proof. By Theorem 7 we know that $G\mathbb{K}_\alpha$ is idempotent and $G\mathbb{K}_\alpha(0,1) = \alpha$.

(**Necessity**) By hypothesis, $G\mathbb{K}_\alpha(\alpha,1) = \alpha$. Therefore

$$G\mathbb{K}_\alpha(\alpha,0) = G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,1), G\mathbb{K}_\alpha(0,0)) = G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,0), G\mathbb{K}_\alpha(1,0)) = G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,0), \alpha).$$

On the other hand, we have that

$$G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,0), \alpha) = G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,0), G\mathbb{K}_\alpha(\alpha,1)) = G\mathbb{K}_\alpha(G\mathbb{K}_\alpha(\alpha,\alpha), G\mathbb{K}_\alpha(0,1)) = G\mathbb{K}_\alpha(\alpha,\alpha) = \alpha.$$

(**Sufficiency**) The proof is similar. □

4.3.1. Increasing and bisymmetric $G\mathbb{K}_\alpha$ operators

In the next theorem we analyze when the increasing bisymmetric operators considered by Fodor and Marichal [14] satisfy $M_\alpha(0,1) = \alpha$ (which is an adaptation of $G\mathbb{K}_\alpha(0,1) = \alpha$).

Theorem 9. *$M_\alpha : [0,1]^2 \to [0,1]$ with $\alpha \in [0,1]$ and $M_\alpha(0,1) = \alpha$ is a continuous, commutative, increasing, idempotent, bisymmetric function if and only if there exist two real numbers λ and ρ fulfilling $0 \leqslant \lambda \leqslant \rho \leqslant 1$ and a bounded function f that is continuous and strictly increasing on $[\lambda,\rho]$ fulfilling $2f(\alpha) = f(\lambda) + f(\rho)$ such that*

$$M_\alpha(x,y) = \left\{ \begin{array}{ll} M_{0,\lambda,\lambda}(x,y) & \text{if } x,y \in [0,\lambda] \\ M_{\rho,1,\rho}(x,y), & \text{if } x,y \in [\rho,1] \\ f^{-1}\left(\frac{f(median(\lambda,x,\rho)) + f(median(\lambda,y,\rho))}{2}\right) & \text{otherwise} \end{array} \right\}$$

with $M_{0,\lambda,\lambda} : [0,\lambda]^2 \to [0,\lambda]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M_\alpha(0,\lambda) = \lambda; M_{\rho,1,\rho} : [\rho,1]^2 \to [\rho,1]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M_\alpha(\rho,1) = \rho$.

Proof. (**Necessity**) By Theorem 3 we have that there exist two real numbers λ and ρ fulfilling $0 \leqslant \lambda \leqslant \rho \leqslant 1$ such that:

(a) $M_\alpha(x,y) = M_{0,\lambda,\lambda}(x,y)$, if $x,y \in [0,\lambda]$.
(b) $M_\alpha(x,y) = M_{\rho,1,\rho}(x,y)$, if $x,y \in [\rho,1]$.
(c) $M_\alpha(x,y) = f^{-1}\left(\frac{f(median(\lambda,x,\rho)) + f(median(\lambda,y,\rho))}{2}\right)$ otherwise,

with $M_{0,\lambda,\lambda} : [0,\lambda]^2 \to [0,\lambda]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M_\alpha(0,\lambda) = \lambda; M_{\rho,1,\rho} : [\rho,1]^2 \to [\rho,1]$ a continuous, commutative, increasing, idempotent and bisymmetric function such that $M_\alpha(\rho,1) = \rho$ and f any continuous, bounded, strictly increasing function on $[\lambda,\rho]$.

Moreover, by hypothesis, $M_\alpha(0,1) = \alpha$. Therefore,

$$M_\alpha(0,1) = \alpha = f^{-1}\left(\frac{f(\lambda) + f(\rho)}{2}\right),$$

and then, since by our hypothesis, f can be inverted, $2f(\alpha) = f(\lambda) + f(\rho)$.

(**Sufficiency**) By hypothesis, $0 \leqslant \lambda \leqslant \rho \leqslant 1$ and $2f(\alpha) = f(\lambda) + f(\rho)$. Besides, $median(\lambda,0,\rho) = \lambda$ and $median(\lambda,1,\rho) = \rho$, and therefore $M_\alpha(0,1) = f^{-1}\left(\frac{f(\lambda)+f(\rho)}{2}\right) = f^{-1}(f(\alpha)) = \alpha$. □

Under the conditions of Theorem 9 we have that the values of λ and ρ are related to the value of α we are considering. For this reason we denote them by $\lambda(\alpha)$ and $\rho(\alpha)$, respectively.

Corollary 4. *Under the conditions of Theorem 9, if the family M_α is increasing in α, that is, if for any $\alpha,\beta \in]0,1[$, if $\alpha \geqslant \beta$, then $M_\alpha(x,y) \geqslant M_\beta(x,y)$ for all $x,y \in [0,1]$, and then $\lambda(\alpha)$ and $\rho(\alpha)$ are increasing, i.e. if $1 > \alpha \geqslant \beta > 0$, then $\lambda(\alpha) \geqslant \lambda(\beta)$ and $\rho(\alpha) \geqslant \rho(\beta)$.*

Proof. Take $\alpha,\beta \in]0,1[$. By definition (see [14]),

$$\lambda(\alpha) = \sup\{x \in [0,1] | M_\alpha(0,x) = x\}.$$

Take $x \in]0,1[$. If $x > \lambda(\alpha)$, by continuity, $M_\alpha(0,x) < x$. Since, if $M_\alpha(0,x) > x$, as $M_\alpha(0,1) = \alpha < 1$, by the mean value theorem there exists $x_0 \in]0,1[$ such that $M_\alpha(0,x_0) = x_0$, which is a contradiction.

Take $0 < \beta < \alpha$. If $\lambda(\alpha) = 1, \lambda(\beta) \leqslant \lambda(\alpha)$ trivially. If $\lambda(\alpha) < 1$, then for all $x \in]\lambda(\alpha),1], M_\alpha(0,x) < x$ and $M_\beta(0,x) \leqslant M_\alpha(0,x) < x$, and therefore $\lambda(\beta) \leqslant \lambda(\alpha)$. Analogously, it is evident that $\rho(\beta) \leqslant \rho(\alpha)$. □

In Theorem 9 we studied the operators defined by Fodor and Marichal satisfying $M_\alpha(0,1) = \alpha$. We also know that operators $G\mathbb{K}_\alpha$ have the following properties: $G\mathbb{K}_0(x,y) = \min(x,y)$ and $G\mathbb{K}_1(x,y) = \max(x,y)$. For this reason, in the next theorem we use the functions of Fodor and Marichal to define operators G_α such that $G_\alpha(0,1) = \alpha, G_0(x,y) = \min(x,y)$ and $G_1(x,y) = \max(x,y)$.

Theorem 10. *Consider the function $G_\alpha : [0,1]^2 \to [0,1]$ with $\alpha \in [0,1]$ defined as:*

$$G_\alpha(x,y) = \begin{cases} \max(x,y) & \text{if } x,y \in [0,\lambda(\alpha)]; \\ \min(x,y) & \text{if } x,y \in [\rho(\alpha),1]; \\ f^{-1}\left(\frac{f(median(\lambda(\alpha),x,\rho(\alpha))) + f(median(\lambda(\alpha),y,\rho(\alpha)))}{2}\right) & \text{otherwise}, \end{cases}$$

with $\lambda(\alpha)$ and $\rho(\alpha)$ two real numbers such that $0 \leqslant \lambda(\alpha) \leqslant \rho(\alpha) \leqslant 1$, and f a function that is continuous and strictly increasing on $[\lambda(\alpha),\rho(\alpha)]$ fulfilling $2f(\alpha) = f(\lambda(\alpha)) + f(\rho(\alpha))$. Then G_α is continuous, commutative, increasing, idempotent and bisymmetric and satisfies $G_\alpha(0,1) = \alpha, G_0(x,y) = \min(x,y)$ and $G_1(x,y) = \max(x,y)$.

Proof. Just observe that G_α has the structure of the mapping M_α in the sufficiency condition in Theorem 9. $\square$

Corollary 5. *In the setting of Theorem 10 we have that:*

$$G_\alpha(\alpha, 1) = \alpha \quad \text{if and only if} \quad \alpha = \lambda(\alpha) = \rho(\alpha).$$

Proof. (**Necessity**) By Proposition 8 we know that $G_\alpha(\alpha, 1) = \alpha$ if and only if $G_\alpha(\alpha, 0) = \alpha$. Taking into account Theorem 10, we have that $0 \leqslant \lambda(\alpha) \leqslant \alpha \leqslant \rho(\alpha) \leqslant 1$. Therefore, if $G_\alpha(\alpha, 1) = \alpha = f^{-1}\left(\frac{f(\alpha)+f(\rho(\alpha))}{2}\right)$, then $\alpha = \rho(\alpha)$.

If $G_\alpha(\alpha, 0) = \alpha = f^{-1}\left(\frac{f(\alpha)+f(\lambda(\alpha))}{2}\right)$, then $\alpha = \lambda(\alpha)$.

(**Sufficiency**) The proof is direct. $\square$

Note that if $\alpha = \lambda(\alpha) = \rho(\alpha)$, then the function G_α of Theorem 10 becomes an α-median (i.e., the idempotent nullnorm with annihilator α).

From Theorem 10 and Corollary 5, if we impose the condition that α is an annihilator on the operators of Fodor and Marichal, we have the following result.

Theorem 11. *The idempotent nullnorms with annihilator $\alpha \in]0, 1[$ are $G\mathbb{K}_\alpha = GK_\alpha \circ i$ operators.*

Proof. In the construction $G\mathbb{K}_\alpha$, take GK_α given in (1). We have

$$G\mathbb{K}_\alpha(x, y) = GK_\alpha([\min(x, y), \max(x, y)]) = \begin{cases} \max(x, y) & \text{if } x, y < \alpha, \\ \min(x, y) & \text{if } x, y \geqslant \alpha, \\ \alpha & \text{otherwise}. \end{cases} \tag{2}$$

So the theorem is proved by recalling Theorem 10, Corollary 5 and the fact that for each $\alpha \in]0, 1[$ there exists a unique idempotent nullnorm V_α with annihilator α [18] which is given by (2). $\square$

Remark 3. Note that under the hypothesis of Theorem 11:

- $G\mathbb{K}_\alpha$ is bisymmetric because it is commutative and associative.
- $G\mathbb{K}_\alpha(0, 1) = \alpha = G\mathbb{K}_\alpha(\alpha, 1) = G\mathbb{K}_\alpha(\alpha, 0) = G\mathbb{K}_\alpha(\alpha, \alpha)$, and therefore by Proposition 5 we have that $G\mathbb{K}_\alpha$ is not strictly increasing.
- $G\mathbb{K}_\alpha = GK_\alpha \circ i$ with GK_α the operator given in (1); then we have that $G\mathbb{K}_0(x, y) = \min(x, y)$ and $G\mathbb{K}_1(x, y) = \max(x, y)$.

4.3.2. Construction from Aczél's theorem

In this subsection we consider a family of strictly increasing operators $\mathbb{H}_\alpha$ that satisfy properties (a)–(e) of Theorem 7 and that are obtained from Aczél's theorem.

Theorem 12. *Consider the function $j : [0, 1]^2 \to [0, 1]^2$ given by*

$$j(x, y) = (\min(x, y), \max(x, y)),$$

and consider $H_\alpha : [0, 1]^2 \to [0, 1]$ given by

$$H_\alpha(x, y) = f^{-1}(pf(x) + (1 - p)f(y)),$$

where $f : [0, 1] \to [0, 1]$ is continuous and strictly increasing, $p = \frac{f(1)-f(\alpha)}{f(1)-f(0)}$ and $\alpha \in [0, 1]$. In this setting, the function $\mathbb{H}_\alpha : [0, 1]^2 \to [0, 1]$ given by

$$\mathbb{H}_\alpha(x, y) = (H_\alpha \circ j)(x, y) = f^{-1}(pf(\min(x, y)) + (1 - p)f(\max(x, y))) = f^{-1}(f(\max(x, y)) - p|f(x) - f(y)|)$$

satisfies the following:

(a) *It is commutative and idempotent.*
(b) $\mathbb{H}_0(x, y) = \min(x, y)$ and $\mathbb{H}_1(x, y) = \max(x, y)$.
(c) *If $\alpha \in]0, 1[$, it is strictly increasing.*
(d) *Take $\alpha, \beta \in [0, 1]$. If $\alpha \leqslant \beta$, then $\mathbb{H}_\alpha(x, y) \leqslant \mathbb{H}_\beta(x, y)$ for all $(x, y) \in [0, 1]^2$.*
(e) $\mathbb{H}_\alpha(0, 1) = \alpha$.

Proof. The proof is direct from Theorem 2. $\square$

Note that the above operator is a generalized OWA (a "quasi-OWA"), i.e., the transformation (see e.g. [10] p. 30) of an OWA by means of a monotonic function (see e.g. [10] p. 56, where it is called OWQA). Note that these functions can also be seen as "symmetrization" (see e.g. [10] p. 15) of the weighted quasi-arithmetic mean, because they can be constructed from the latter (see [10 p. 56]).

Remark 4. Note that in the construction of $\mathbb{H}_\alpha$, we use Aczél's theorem (Theorem 2). Nevertheless, the operators given by Aczél are bisymmetric, whereas the function $\mathbb{H}_\alpha$, in general, is not.

Corollary 6. *Under the conditions of Theorem 12 the following properties hold:*

(1) *If $p = \frac{1}{2}$, then $\mathbb{H}_\alpha$ is bisymmetric.*
(2) $\mathbb{H}_\alpha(x, T(y,z)) = T(\mathbb{H}_\alpha(x,y), \mathbb{H}_\alpha(x,z))$ if and only if $T = \min$.
(3) $\mathbb{H}_\alpha(x, S(y,z)) = S(\mathbb{H}_\alpha(x,y), \mathbb{H}_\alpha(x,z))$ if and only if $S = \max$.

Proof. 1. The proof is direct. Observe that with weight $p = \frac{1}{2}$ the operator becomes a quasi-arithmetic mean and hence fulfills bisymmetry. 2.

(**Sufficiency**) We know that $T(x,x) \leqslant x$ holds for any t-norm. Then $\mathbb{H}_\alpha(x, T(x,x)) = T(\mathbb{H}_\alpha(x,x), \mathbb{H}_\alpha(x,x)) = T(x,x)$; that is, $f^{-1}(pf(T(x,x) + (1-p)f(x)) = T(x,x)$. Therefore, $pf(T(x,x) + (1-p)f(x) = f(T(x,x))$ and then $(1-p)f(x) = (1-p)f(T(x,x))$. Taking into account that $p \in]0,1[$ and f is strictly increasing and continuous, $T(x,x) = x$ for all $x \in [0,1]$. Because the only idempotent t-norm is the minimum, we have that $T = \min$.

(**Necessity**) The proof is direct from the monotonicity of $\mathbb{H}_\alpha$. 3. Similar to item 1. $\square$

Corollary 7. *Under the conditions of Theorem 12 , if the function f is such that for any $x \in [0,1]$ and for any $\lambda \geqslant 0$ such that $\lambda x \in [0,1]$ the identity $f(\lambda x) = \lambda f(x)$ holds (first-order homogeneity), then*

$$\mathbb{H}_\alpha(\lambda x, \lambda y) = \lambda \mathbb{H}_\alpha(x,y).$$

Proof. Under the conditions of Theorem 12, if $f(\lambda x) = \lambda f(x)$, then $f^{-1}(\lambda x) = \lambda f^{-1}(x)$, as can easily be seen by applying f to each side of this last identity and taking into account the injectivity of f. $\square$

Remark 5. Observe that if f is first-order homogeneous, then $f(x) = f(1)x$ for all $x \in [0,1]$.

Corollary 8. *Under the conditions of Theorem 12 , if f is first-order homogeneous, then*

$$\mathbb{H}_\alpha(x,y) = (H_\alpha \circ j)(x,y) = \begin{cases} f^{-1}(f(1)\mathbb{K}_\alpha(x,y)) & \text{if } \alpha \in]0,1[, \\ \min(x,y) & \text{if } \alpha = 0, \\ \max(x,y) & \text{if } \alpha = 1. \end{cases}$$

Proof. Immediate. $\square$

Remark 6. Under the conditions of Theorem 12, if $f(x) = x$ for all $x \in [0,1]$ and $\alpha \in]0,1[$, then

$$\mathbb{H}_\alpha(x,y) = (H_\alpha \circ j)(x,y) = \mathbb{K}_\alpha(x,y) = (K_\alpha \circ i)(x,y)$$

for all $(x,y) \in [0,1]^2$.

5. Construction of a FS from a IVFS and an operator GK_α

In this section we present two methods to associate with each IVFS over the referential U a fuzzy set over the same referential. In both methods we use the operators GK_α.

5.1. Construction with fixed α

Let $A \in IVFS(U)$. Fix $\alpha \in [0,1]$. By means of the corresponding GK_α operator, we can associate a fuzzy set A_α with A in the following way [2,3,5]:

$$A_\alpha = \{(u, \mu_{A_\alpha}(u)) | u \in U\} \quad \text{with } \mu_{A_\alpha}(u) = GK_\alpha(\mathbf{M_A}(u)), \tag{3}$$

where $\mathbf{M_A}$ denotes the membership function of A.

Let A be an IVFS. We denote by $\{A_\alpha\}_{\alpha \in [0,1]}$ the family of all fuzzy sets associated with A by an operator GK_α when α varies in $[0,1]$.

From the definition of the GK_α operators, the following theorem follows [4].

Theorem 13. *Let $A \in IVFS(U)$, let $\beta \in [0,1]$ and consider an operator GK_α. Then $\{A_\alpha\}_{\alpha \in [0,1]}$ is a totally ordered family of fuzzy sets with respect to the order*

$$A_\alpha \leqslant A_\beta \quad \text{if and only if } \alpha \leqslant \beta.$$

5.2. Construction with variable α

The map between $IVFS(U)$ and $FS(U)$ given in (3) is such that, once α is fixed, this α is used to obtain the membership of every one of the elements u in U. This restriction is not necessary, and is not even advisable for some of the possible applications. In this sense, we show in the following definition a possible way of relating to each IVFS a fuzzy set in such a way that for each $u \in U$ we use a different value of α to obtain its membership.

Definition 9. Let $A \in IVFS(U)$. For each $u \in U$, take $\alpha(u) \in [0,1]$. Then $A_{\alpha(*)} = \left\{ \left(u, \mu_{A_{\alpha(*)}}(u) = GK_{\alpha(u)}(\mathbf{M_A}(u))\right) | u \in U \right\}$ is a fuzzy set.

From Definition 9 the following properties follow.

Proposition 6. *Let $A \in IVFS(U)$.*

(1) *If $\alpha(u) \leqslant \beta(u)$ for all $u \in U$, then $A_{\alpha(*)} \leqslant A_{\beta(*)}$.*
(2) *If for all $u \in U$ we have that $Length(\mathbf{M_A}(u)) = 0$, and then $A_{\alpha(*)} = A \in FS(U)$.*
(3) *Let $A, B \in IVFS(U)$. If $\mathbf{M_A}(u) \leqslant_L \mathbf{M_B}(u)$ for all $u \in U$, then $A_{\alpha(*)} \leqslant B_{\alpha(*)}$.*

Proof. The proof is immediate. □

Proposition 7. *Let $A \in IVFS(U)$ and take $GK_\alpha = K_\alpha$. Under the conditions of Definition 9 the following observations hold.*

(1) *If we take $N(x) = 1 - x$, then for any $u \in U$ and any choice of $\alpha(u) \in [0,1]$ we have $1 - \mu_{(A_N)_{\alpha(u)}} = \mu_{A_{1-\alpha(u)}}(u)$.*
(2) *If, for all $u \in U$, we take $\alpha(u) = \mu_{A_0}(u)/\left(1 + \mu_{A_0}(u) - \mu_{A_1}(u)\right)$ with $\mu_{A_1}(u) - \mu_{A_0}(u) \neq 1$, then*
$A_{\alpha(*)} = \{(u, \alpha(u)))|u \in U\}$.

Proof. The proof is direct. □

6. Conclusions and future research

We established a link between K_α operators and OWA operators of dimension 2. This relation led to the definition of a class of aggregation functions, the $\mathbb{K}_\alpha$ operators, in terms of K_α operators in such a way that the resulting class encompasses OWA operators of dimension 2.

This generalization retains most of the important features of Atanassov's operators. We presented two construction theorems for our functions and studied under which conditions they are bisymmetric.

Regarding future lines of research, a complete theoretical study of the general characterization of GK_α operators is required. In particular, because the resulting operators are instances of aggregation functions, we intend to study which aggregation functions give rise to generalized Atanassov's operators.

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