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Berwald type inequality for Sugeno integral

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ABSTRACT

Nonadditive measure is a generalization of additive probability measure. Sugeno integral is a useful tool in several theoretical and applied statistics which has been built on non-additive measure. Integral inequalities play important roles in classical probability and measure theory. The classical Berwald integral inequality is one of the famous inequalities. This inequality turns out to have interesting applications in information theory. In this paper, Berwald type inequality for the Sugeno integral based on a concave function is studied. Several examples are given to illustrate the validity of this inequality. Finally, a conclusion is drawn and a problem for further investigations is given.

1. Introduction

The theory of nonadditive measures and integrals was introduced by Sugeno [20] as a tool for modeling nondeterministic problems. Sugeno integral is a useful tool in several theoretical and applied statistics. For instance, in decision theory, the Sugeno integral is a median, which is indeed a qualitative counterpart to the averaging operation underlying expected utility. The use of the Sugeno integral can be envisaged from two points of view: decision under uncertainty and multi-criteria decision-making [4]. Sugeno integral is analogous to Lebesgue integral which has been studied by many authors, including Pap [12], Ralescu and Adams [14] and, Wang and Klir [21], among others. Román-Flores et al. [5,15–18], started the studies of inequalities for Sugeno integral, and then followed by the authors [1,2,8–11].

Integral inequalities play important roles in classical probability and measure theory. The classical Berwald integral inequality is one of the famous inequalities. This inequality turns out to have interesting applications in information theory. In general, any integral inequality can be a very strong tool for applications. In particular, when we think of an integral operator as a predictive tool then an integral inequality can be very important in measuring and dimensioning such process. In this paper we prove Berwald type inequality for the Sugeno integral based on a concave function, which will be helpful for those areas in which the classical Berwald inequality plays a role whenever the environment is non-deterministic.

The paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Sections 3 and 4. Finally, a conclusion is drawn and a problem for further investigations is given in Section 5.

2. Preliminaries

In this section we recall some basic definitions and previous results which will be used in the sequel.

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As usual we denote by R the set of real numbers. Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Let $\mathbf{N}$ denote the set of all positive integers and $\overline{R_+}$ denote $[0,+\infty]$. Throughout this paper, we fix the measurable space $(X,\mathcal{F})$, and all considered subsets are supposed to belong to $\mathcal{F}$.

Definition 2.1 (*Ralescu and Adams [14]*). A set function $\mu:\mathcal{F}\to\overline{R_+}$ is called a nonadditive measure if the following properties are satisfied:

(FM1) $\mu(\emptyset)=0$;
(FM2) $A\subset B$ implies $\mu(A)\leqslant\mu(B)$;
(FM3) $A_1\subset A_2\subset\cdots$ implies $\mu(\bigcup_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$; and
(FM4) $A_1\supset A_2\supset\cdots$ and $\mu(A_1)<+\infty$ imply $\mu(\bigcap_{n=1}^{\infty}A_n)=\lim_{n\to\infty}\mu(A_n)$.

When μ is a nonadditive measure, the triple $(X,\mathcal{F},\mu)$ then is called a nonadditive measure space.

Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space, by $\mathcal{F}_+(X)$ we denote the set of all nonnegative measurable functions $f:X\to[0,\infty)$ with respect to $\mathcal{F}$. In what follows, all considered functions belong to $\mathcal{F}_+(X)$. Let f be a nonnegative real-valued function defined on X, we will denote the set $\{x\in X|f(x)\geqslant\alpha\}$ by $F\alpha$ for $\alpha\geqslant 0$. Clearly, F_α is nonincreasing with respect to α, i.e., $\alpha\leqslant\beta$ implies $F_\alpha\supseteq F_\beta$.

Definition 2.2 (*Pap [12], Sugeno [20], Wang and Klir [21]*). Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space and $A\in\mathcal{F}$, the Sugeno integral of f on A, with respect to the nonadditive measure μ, is defined as

$$(S)\int_A f d\mu=\bigvee_{\alpha\geqslant 0}(\alpha\wedge\mu(A\cap F_\alpha)),$$

where $\vee$, $\wedge$ denotes the operations sup and inf on $[0,\infty]$, respectively. When $A=X$, then

$$(S)\int_X f d\mu=(S)\int f d\mu=\bigvee_{\alpha\geqslant 0}(\alpha\wedge\mu(F_\alpha)).$$

It is well known that Sugeno integral is a type of nonlinear integral [7]. I.e., for general case,

$$(S)\int(af+bg)d\mu=a(S)\int f d\mu+b(S)\int g d\mu$$

does not hold. Some basic properties of Sugeno integral are summarized in [12,21], we cite some of them in the next theorem.

Theorem 2.3 (*Pap [12], Wang and Klir [21]*). *Let $(X,\mathcal{F},\mu)$ be a nonadditive measure space, then*

(i) $\mu(A\cap F_\alpha)\geqslant\alpha\to\int_A f d\mu\geqslant\alpha$;
(ii) $(S)\int_A k d\mu=k\wedge\mu(A)$ *for any constant* $k\in[0,\infty)$;
(iii) $(S)\int_A f d\mu<\alpha\leftrightarrow$ *there exists* $\gamma<\alpha$ *such that* $\mu(A\cap F_\gamma)<\alpha$;
(iv) $(S)\int_A f d\mu>\alpha\leftrightarrow$ *there exists* $\gamma>\alpha$ *such that* $\mu(A\cap F_\gamma)>\alpha$;
(v) *If* $\mu(A)<\infty$, *then* $\mu(A\cap F_\alpha)\geqslant\alpha\leftrightarrow(S)\int_A f d\mu\geqslant\alpha$;
(vi) *If* $f\leqslant g$, *then* $(S)\int f d\mu\leqslant(S)\int g d\mu$.

Now, our results can be stated as follows.

3. Berwald's inequality for Sugeno integral

The well-known Berwald inequality is a part of the classical mathematical analysis, see [13].

Theorem 3.1. *Let f be a nonnegative concave function on [a,b]. Then*

$$\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{\int_a^b f^s(x)dx}{b-a}\right)^{\frac{1}{s}}\leqslant\left(\frac{\int_a^b f^r(x)dx}{b-a}\right)^{\frac{1}{r}},\tag{3.1}$$

for $0<r<s<\infty$.

Unfortunately, as we will see in the following example, in general, Berwald's inequality (3.1) is not valid for Sugeno integral.

Example 3.2. Let f be a real valued function defined as $f(x)=\sqrt{x}$ where $x\in[0,1]$. f is a nonnegative concave function. Let $r=\frac{1}{3}$ and $s=\frac{1}{2}$ in (3.1) and m be the Lebesgue measure. A straightforward calculus shows that

(i) $(S)\int_0^1 f^{\frac{1}{3}}(x)d\mu=\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge m\left(\left\{(\sqrt{x})^{\frac{1}{3}}\geqslant\alpha\right\}\right)\right]=\bigvee_{\alpha\in[0,1]}[\alpha\wedge(1-\alpha^6)]=0.778,$

(ii) $(S)\int_0^1 f^{\frac{1}{2}}(x)dm=\bigvee_{\alpha\in[0,1]}\left[\alpha\wedge m\left(\left\{(\sqrt{x})^{\frac{1}{2}}\geqslant\alpha\right\}\right)\right]=\bigvee_{\alpha\in[0,1]}[\alpha\wedge(1-\alpha^4)]=0.724.$

Therefore:

$$0.497\,56=\frac{\left(\frac{1}{2}+1\right)^2}{\left(\frac{1}{3}+1\right)^3}\left((S)\int_0^1 f^{\frac{1}{2}}(x)dm\right)^2>\left((S)\int_0^1 f^{\frac{1}{3}}(x)dm\right)^3=0.470\,91.$$

Thus, it can be concluded that the inequality (3.1) is not valid for Sugeno integral.

This section provides a Berwald's inequality derived from (3.1) for the Sugeno integral.

Theorem 3.3. *Let $0<r<s<\infty$ and $f:[0,1]\to[0,\infty)$ be a concave function and m be the Lebesgue measure on R. Then*

(a) *if $f(0)<f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}}\geqslant\min\left\{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}},\left(1-\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}\right\}.$$

(b) *if $f(0)=f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}}\geqslant\min\{f(0),1\}.$$

(c) *if $f(0)>f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}}\geqslant\min\left\{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}},\left(\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}\right\}.$$

Proof. Let $0<r<s<\infty$ and $(S)\int_0^1 f^s(x)dm=t$. Since $f:[0,1]\to[0,\infty)$ is a concave function, for $x\in[0,1]$ we have

$$f(x)=f((1-x)\cdot 0+x\cdot 1)\geqslant(1-x)\cdot f(0)+x\cdot f(1)=h(x).$$

(a) If $f(0)<f(1)$, then

$$\left((S)\int_0^1 f^r(x)dm\right)^{\frac{1}{r}}\geqslant\left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}}=\left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{[(1-x)\cdot f(0)+x\cdot f(1)]\geqslant\alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$=\left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{x|x\geqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$\geqslant\left[\left(\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}\right)^r\wedge m\left([0,1]\cap\left\{x|x\geqslant\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right\}\right)\right]^{\frac{1}{r}}=\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}\wedge\left(1-\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}.$$

(b) If $f(0)=f(1)$, then

$$f(x)=f((1-x)\cdot 0+x\cdot 1)\geqslant f(0)=h(x).$$

Thus, by Theorem 2.3 (vi) and (ii) we have

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}}\geqslant\left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}}\geqslant\left((S)\int_0^1 f^r(0)dm\right)^{\frac{1}{r}}=[f^r(0)\wedge 1]^{\frac{1}{r}}=f(0)\wedge 1.$$

(c) If $f(0)>f(1)$, then

$$\left((S)\int_0^1 f^r(x)dm\right)^{\frac{1}{r}} \geqslant \left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}} = \left[\bigvee_{\alpha\in[0,1]}\left(\alpha \wedge m\left([0,1]\cap\left\{[(1-x)\cdot f(0)+x\cdot f(1)] \geqslant \alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$= \left[\bigvee_{\alpha\in[0,1]}\left(\alpha \wedge m\left([0,1]\cap\left\{x|x \leqslant \frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$\geqslant \left[\left(\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}\right)^r \wedge m\left([0,1]\cap\left\{x|x \leqslant \frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right\}\right)\right]^{\frac{1}{r}} = \frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}} \wedge \left(\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}},$$

and the proof is completed. □

Example 3.4. Let $f(x)=\sqrt{x}$ where $x\in[0,1]$ and m be the Lebesgue measure. Let $r = 2$ and $s = 4$. A straightforward calculus shows that

(i) $(S)\int_0^1 f^2(x)dm = \bigvee_{\alpha\in[0,1]}[\alpha\wedge m([0,1]\cap\{x \geqslant \alpha\})] = \bigvee_{\alpha\in[0,1]}[\alpha\wedge(1-\alpha)] = 0.5$,
(ii) $(S)\int_0^1 f^4(x)dm = \bigvee_{\alpha\in[0,1]}[\alpha\wedge m([0,1]\cap\{x^2 \geqslant \alpha\})] = \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\left(1-\alpha^{\frac{1}{2}}\right)\right] = 0.38197$,
(iii) $\frac{\frac{(1+4)^{\frac{1}{4}}}{(1+2)^{\frac{1}{2}}}\left((S)\int_0^1 f^4(x)dm\right)^{\frac{1}{4}}-f(0)}{f(1)-f(0)} = 0.67872$.

Therefore:

$$0.70711 = \left((S)\int_0^1 f^2 dm\right)^{\frac{1}{2}} \geqslant \frac{(1+4)^{\frac{1}{4}}}{(1+2)^{\frac{1}{2}}}\left(\int_0^1 f^4(x)dm\right)^{\frac{1}{4}} \wedge \sqrt{1-\frac{\frac{(1+4)^{\frac{1}{4}}}{(1+2)^{\frac{1}{2}}}\left((S)\int_0^1 f^4(x)dm\right)^{\frac{1}{4}}-f(0)}{f(1)-f(0)}} = 0.67872 \wedge \sqrt{0.32128}$$
$$= 0.56682.$$

Now, we provide a reverse inequality of (3.1) for Sugeno integral based on a convex function.

Theorem 3.5. *Let $0 < r < s < \infty$ and $f:[0,1]\to[0,\infty)$ be a convex function and m be the Lebesgue measure on R. Then*

(a) *if $f(0) < f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}} \leqslant \max\left\{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}, \left(1-\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}\right\}.$$

(b) *if $f(0) = f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}} \leqslant \min\{f(0),1\}.$$

(c) *if $f(0) > f(1)$, then*

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}} \leqslant \max\left\{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}, \left(\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left((S)\int_0^1 f^s(x)dm\right)^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}\right\}.$$

Proof. Let $0 < r < s < \infty$ and $(S)\int_0^1 f^s(x)dm = t$. Since $f:[0,1]\to[0,\infty)$ is a convex function, for $x\in[0,1]$ we have

$$f(x) = f((1-x)\cdot 0 + x\cdot 1) \leqslant (1-x)\cdot f(0) + x\cdot f(1) = h(x).$$

(a) If $f(0) < f(1)$, then

$$\left((S)\int_0^1 f^r(x)dm\right)^{\frac{1}{r}} \leqslant \left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}} = \left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{[(1-x)\cdot f(0)+x\cdot f(1)]\geqslant\alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$= \left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{x|x\geqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$\leqslant \left[\bigwedge_{\alpha\in[0,1]}\left(\alpha\vee m\left([0,1]\cap\left\{x|x\geqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$\leqslant \left[\left(\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}\right)^r \vee m\left([0,1]\cap\left\{x|x\geqslant\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right\}\right)\right]^{\frac{1}{r}} = \frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}} \vee \left(1-\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}.$$

(b) If $f(0) = f(1)$, then

$$f(x) = f((1-x)\cdot 0 + x\cdot 1) \leqslant f(0) = h(x).$$

Thus, by Theorem 2.3 (vi) and (ii) we have

$$\left((S)\int_0^1 f^r dm\right)^{\frac{1}{r}} \leqslant \left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}} \leqslant \left((S)\int_0^1 f^r(0)dm\right)^{\frac{1}{r}} = [f^r(0)\wedge 1]^{\frac{1}{r}} = f(0)\wedge 1.$$

(c) If $f(0) > f(1)$, then

$$\left((S)\int_0^1 f^r(x)dm\right)^{\frac{1}{r}} \leqslant \left((S)\int_0^1 h^r(x)dm\right)^{\frac{1}{r}} = \left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{[(1-x)\cdot f(0)+x\cdot f(1)]\geqslant\alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$= \left[\bigvee_{\alpha\in[0,1]}\left(\alpha\wedge m\left([0,1]\cap\left\{x|x\leqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$\leqslant \left[\bigwedge_{\alpha\in[0,1]}\left(\alpha\vee m\left([0,1]\cap\left\{x|x\leqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\right)\right)\right]^{\frac{1}{r}}$$
$$\leqslant \left[\left(\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}\right)^r \vee m\left([0,1]\cap\left\{x|x\leqslant\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right\}\right)\right]^{\frac{1}{r}} = \frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}} \vee \left(\frac{\frac{(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}t^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right)^{\frac{1}{r}}$$

and the proof is completed. □

Example 3.6. Let $f(x) = x^2$ where $x\in[0,1]$ and m be the Lebesgue measure. Let $r=\frac{1}{2}$ and $s = 1$. A straightforward calculus shows that

(i) $(S)\int_0^1 f^{\frac{1}{2}}(x)dm = \bigvee_{\alpha\in[0,1]}[\alpha\wedge m([0,1]\cap\{x\geqslant\alpha\})] = \bigvee_{\alpha\in[0,1]}[\alpha\wedge(1-\alpha)] = 0.5$,

(ii) $(S)\int_0^1 f(x)dm = \bigvee_{\alpha\in[0,1]}[\alpha\wedge m([0,1]\cap\{x^2\geqslant\alpha\})] = \bigvee_{\alpha\in[0,1]}\left[\alpha\wedge\left(1-\alpha^{\frac{1}{2}}\right)\right] = 0.38197$,

(iii) $\frac{\frac{(1+1)}{\left(1+\frac{1}{2}\right)^2}\left((S)\int_0^1 f(x)dm\right)-f(0)}{f(1)-f(0)} = 0.33953$.

Therefore:

$$0.25 = \left((S)\int_0^1 f^{\frac{1}{2}}dm\right)^2 \leqslant \frac{(1+1)}{\left(1+\frac{1}{2}\right)^2}\left(\int_0^1 f(x)dm\right) \vee \left(1-\frac{\frac{1+1}{\left(1+\frac{1}{2}\right)^2}\left((S)\int_0^1 f(x)dm\right)-f(0)}{f(1)-f(0)}\right)^2 = 0.33953\vee 0.43622$$
$$= 0.43622.$$

Remark 3.7. The last step in demonstration of Theorems 3.3 and 3.5 (part (a)) is valid whenever $[0,1]\cap\left\{x|x\geqslant\frac{\alpha^{\frac{1}{r}}-f(0)}{f(1)-f(0)}\right\}\neq\emptyset$. Analogously, the last step in demonstration of Theorems 3.3 and 3.5 (part (c)) is valid whenever $[0,1]\cap\left\{x\leqslant\frac{\alpha^{\frac{1}{s}}-f(0)}{f(1)-f(0)}\right\}\neq\emptyset$. Anyway, in any other case the inequality is trivially verified.

4. Further discussions

In this section, we provide the general cases of the Theorems 3.3 and 3.5.

Theorem 4.1. *Let $0<r<s<\infty$ and $f:[a,b]\to[0,\infty)$ be a concave function and m be the Lebesgue measure on R. Then*

(a) *if $f(a)<f(b)$, then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}}\geqslant\min\left\{\begin{array}{c}\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}},\\ \left(b-\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right)^{\frac{1}{r}}\end{array}\right\}.$$

(b) *if $f(a)=f(b)$, then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}}\geqslant\min\left\{f(a),(b-a)^{\frac{1}{r}}\right\}.$$

(c) *if $f(a)>f(b)$, then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}}\geqslant\min\left\{\begin{array}{c}\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}},\\ \left(\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}-a\right)^{\frac{1}{r}}\end{array}\right\}.$$

Proof. Let $0<r<s<\infty$ and $(S)\int_a^b f^s(x)dm=t$. Since $f:[a,b]\to[0,\infty)$ is a concave function, for $x\in[a,b]$ we have

$$f(x)=f\left(\left(1-\frac{x-a}{b-a}\right)a+\left(\frac{x-a}{b-a}\right)b\right)\geqslant\left(1-\frac{x-a}{b-a}\right)f(a)+\left(\frac{x-a}{b-a}\right)f(b)=h(x).$$

(a) If $f(a)<f(b)$, then

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}}\geqslant\left((S)\int_a^b h^r(x)dm\right)^{\frac{1}{r}}=\left[\bigvee_{\alpha\in[a,b]}\left(\alpha\wedge m\left([a,b]\cap\left\{\left[\left(1-\frac{x-a}{b-a}\right)f(a)+\left(\frac{x-a}{b-a}\right)f(b)\right]\geqslant\alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$=\left[\bigvee_{\alpha\in[a,b]}\left(\alpha\wedge m\left([a,b]\cap\left\{x|x\geqslant\frac{\alpha^{\frac{1}{r}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$\geqslant\left[\left(\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\right)^r\wedge m\left([a,b]\cap\left\{x|x\geqslant\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right]^{\frac{1}{r}}$$

$$=\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\wedge\left(b-\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right)^{\frac{1}{r}}.$$

(b) If $f(a)=f(b)$, then $h(x)=f(a)$. Thus, by Theorem 2.3 (vi) and (ii) we have

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}} \geqslant \left((S)\int_a^b h^r(x)dm\right)^{\frac{1}{r}} \geqslant \left((S)\int_a^b f^r(a)dm\right)^{\frac{1}{r}} = [f^r(a)\wedge(b-a)]^{\frac{1}{r}} = f(a)\wedge(b-a)^{\frac{1}{r}}.$$

(c) if $f(a)>f(b)$, then

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}} \geqslant \left((S)\int_a^b h^r(x)dm\right)^{\frac{1}{r}} = \left[\bigvee_{\alpha\in[a,b]}\left(\alpha\wedge m\left([a,b]\cap\left\{\left[\left(1-\frac{x-a}{b-a}\right)f(a)+\left(\frac{x-a}{b-a}\right)f(b)\right]\geqslant\alpha^{\frac{1}{r}}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$= \left[\bigvee_{\alpha\in[a,b]}\left(\alpha\wedge m\left([a,b]\cap\left\{x|x\leqslant\frac{\alpha^{\frac{1}{r}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right)\right]^{\frac{1}{r}}$$

$$\geqslant \left[\left(\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\right)^r\wedge m\left([a,b]\cap\left\{x|x\leqslant\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)\right]^{\frac{1}{r}}$$

$$= \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}\wedge\left(\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{t}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}-a\right)^{\frac{1}{r}},$$

and the proof is completed. □

Example 4.2. Let $X\in[0,2]$, $f(x)=\ln(1+x)$, $s=2$, $r=1$ and m be the Lebesgue measure. A straightforward calculus shows that

(i) $(S)\int_0^2 f(x)dm = \bigvee_{\alpha\in[0,2]}[\alpha\wedge m([0,2]\cap\{\ln(1+x)\geqslant\alpha\})] = \bigvee_{\alpha\in[0,2]}[\alpha\wedge(3-e^{\alpha})] = 0.79206,$

(ii) $(S)\int_0^2 f^2(x)dm = \bigvee_{\alpha\in[0,2]}[\alpha\wedge m([0,2]\cap\{(\ln(1+x))^2\geqslant\alpha\})] = \bigvee_{\alpha\in[0,2]}\left[\alpha\wedge\left(3-e^{\sqrt{\alpha}}\right)\right] = 0.69637,$

(iii) $\dfrac{\frac{2^2(1+2)^{\frac{1}{2}}}{(1+1)}\left(\frac{(S)\int_0^2 f^2(x)dm}{2}\right)^{\frac{1}{2}}-2f(0)}{f(2)-f(0)} = \dfrac{\frac{2^2(1+2)^{\frac{1}{2}}}{(1+1)}\left(\frac{0.69637}{2}\right)^{\frac{1}{2}}}{\ln(3)} = 1.8606.$

Therefore:

$$0.79206 = (S)\int_0^2 f(x)dm \geqslant \min\left\{\begin{array}{c}\frac{2(1+2)^{\frac{1}{2}}}{(1+1)}\left(\frac{(S)\int_0^2 f^2(x)dm}{2}\right)^{\frac{1}{2}},\\ \left(2-\frac{\frac{2^2(1+2)^{\frac{1}{2}}}{(1+1)}\left(\frac{(S)\int_0^2 f^2(x)dm}{2}\right)^{\frac{1}{2}}-2f(0)}{f(2)-f(0)}\right)\end{array}\right\} = \min\{1.022, 0.1394\} = 0.1394.$$

In a similar way, we can prove the following theorem.

Theorem 4.3. *Let $0<r<s<\infty$ and $f:[a,b]\to[0,\infty)$ be a convex function and m be the Lebesgue measure on R. Then*

(a) *if $f(a)<f(b)$, then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}} \leqslant \max\left\{\begin{array}{c}\frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}},\\ \left(b-\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}}+af(b)-bf(a)}{f(b)-f(a)}\right)^{\frac{1}{r}}\end{array}\right\}.$$

(b) *if $f(a)=f(b)$, then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}} \leqslant \min\left\{f(a), (b-a)^{\frac{1}{r}}\right\}.$$

(c) *if f(a) > f(b), then*

$$\left((S)\int_a^b f^r dm\right)^{\frac{1}{r}} \leqslant \max\left\{\begin{array}{c} \frac{(b-a)^{\frac{1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}}, \\ \left(\frac{\frac{(b-a)^{\frac{r+1}{r}}(1+s)^{\frac{1}{s}}}{(1+r)^{\frac{1}{r}}}\left(\frac{(S)\int_a^b f^s(x)dm}{b-a}\right)^{\frac{1}{s}} + af(b) - bf(a)}{f(b)-f(a)} - a\right)^{\frac{1}{r}} \end{array}\right\}.$$

Example 4.4. Let $X \in [0,2],\ f(x) = 3x^2,\ s = \frac{1}{2},\ r = \frac{1}{4}$ and m be the Lebesgue measure. A straightforward calculus shows that

(i) $(S)\int_0^2 f^{\frac{1}{2}}(x)dm = \bigvee_{\alpha\in[0,2]}\left[\alpha \wedge m\left([0,2]\cap\left\{\sqrt{3}x \geqslant \alpha\right\}\right)\right] = \bigvee_{\alpha\in[0,2]}\left[\alpha\wedge\left(2-\frac{1}{3}\alpha\sqrt{3}\right)\right] = 1.2679,$

(ii) $(S)\int_0^2 f^{\frac{1}{4}}(x)dm = \bigvee_{\alpha\in[0,2]}\left[\alpha \wedge m\left([0,2]\cap\left\{\sqrt[4]{3x^2} \geqslant \alpha\right\}\right)\right] = \bigvee_{\alpha\in[0,2]}\left[\alpha\wedge\left(2-\frac{1}{3}\alpha^2\sqrt{3}\right)\right] = 1.1868,$

(iii) $\dfrac{\frac{2^5\left(1+\frac{1}{2}\right)^2}{\left(1+\frac{1}{4}\right)^4}\left(\frac{(S)\int_a^b f^{\frac{1}{2}}(x)dm}{2}\right)^2 - 2f(0)}{f(2)-f(0)} = \dfrac{\frac{2^5\left(1+\frac{1}{2}\right)^2}{\left(1+\frac{1}{4}\right)^4}\left(\frac{(S)\int_0^2 f^{\frac{1}{2}}(x)dx}{2}\right)^2}{12} = 0.98769.$

Therefore:

$$1.9839 = \left((S)\int_a^b f^{\frac{1}{4}}dm\right)^4 \leqslant \max\left\{\begin{array}{c} \frac{2^4\left(1+\frac{1}{2}\right)^2}{\left(1+\frac{1}{4}\right)^4}\left(\frac{(S)\int_0^2 f^{\frac{1}{2}}(x)dm}{2}\right)^2, \\ \left(2 - \frac{\frac{2^5\left(1+\frac{1}{2}\right)^2}{\left(1+\frac{1}{4}\right)^4}\left(\frac{(S)\int_a^b f^{\frac{1}{2}}(x)dm}{2}\right)^2 - 2f(0)}{f(2)-f(0)}\right)^4 \end{array}\right\} = \max\{5.9261, (2-0.98769)^4\}$$

$$= \max\{5.9261, 1.0502\} = 5.9261.$$

Remark 4.5. The last step in demonstration of Theorems 4.1 and 4.3 (part (a)) is valid whenever $[a,b]\cap\{x|x \geqslant \frac{\alpha^{\frac{1}{r}}(b-a)+af(b)-bf(a)}{f(b)-f(a)}\} \neq \emptyset$. An analogous restriction must be considered for obtaining part (c) of Theorems 4.1 and 4.3.

5. Conclusion

In this paper, we have investigated the classical Berwald inequality for Sugeno integral based on a concave function. For further investigations we propose to consider the Berwald inequality for the Choquet integral, and also for some other non-additive integrals, see [3,6,19]. In the future research, we will continue to explore other integral inequalities for nonadditive measures and integrals.

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