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Invariant dependence structures and Archimedean copulas

Fabrizio Durante [a], Piotr Jaworski [b,*], Radko Mesiar [c,d]

[a] *School of Economics and Management, Free University of Bozen-Bolzano, Bolzano, Italy*

[b] *Institute of Mathematics, University of Warsaw, Warszawa, Poland*

[c] *Department of Mathematics and Descriptive Geometry, Slovak University of Technology, Bratislava, Slovakia*

[d] *Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic*

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ABSTRACT

We consider a family of copulas that are invariant under univariate truncation. Such a family has some distinguishing properties: it is generated by means of a univariate function; it can capture non-exchangeable dependence structures; it can be easily simulated. Moreover, such a class presents strong probabilistic similarities with the class of Archimedean copulas from a theoretical and practical point of view.

1. Introduction

The process of estimating the dependence structure of a multivariate extreme phenomenon (either minimum or maximum) is usually difficult. Generally, one may assume that a parametric family of copulas describes the dependence and may try to fit the parameters. However, even if several goodness-of-fit tests are now available, the model obtained may suffer from a degree of arbitrariness.

As underlined by Juri and Wüthrich (2002), a remedy to this situation is to find dependence models for conditional joint extremes, along the lines of the Pickands–Balkema–De Haan theorem. The most relevant example of this type is provided by the *Clayton family of copulas* (Cook and Johnson, 1981; Genest and MacKay, 1986; Oakes, 1982), whose members are given, for every $\theta \geq -1, \theta \neq 0$ and $uv \neq 0$, by

$$C_\theta^{Cl}(u,v) = \left(\max\left(0, u^{-\theta} + v^{-\theta} - 1\right)\right)^{-1/\theta}.$$

In fact, it is known (see, for instance, Charpentier and Juri, 2006, Juri and Wüthrich, 2002) that Clayton copulas can approximate the dependence behavior of a random pair (X, Y) when X and Y are less than their α-quantile and β-quantile, respectively, for sufficiently small $\alpha, \beta \in (0, 1)$. This is a consequence of the fact that, besides the comonotone copula M and the independence copula Π, Clayton copulas are the only copulas that are invariant under bivariate truncation; i.e. if a

* Corresponding author. Tel.: +48 22 5544523; fax: +48 22 554430.
E-mail addresses: fabrizio.durante@unibz.it (F. Durante), p.jaworski@mimuw.edu.pl, jwptxa@mimuw.edu.pl (P. Jaworski), mesiar@math.sk (R. Mesiar).

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Clayton copula C is associated with a random pair (X, Y), and x and y are given thresholds, then C is also the copula of (X, Y) supposing that $X \le x$ and $Y \le y$ provided that $\mathbb{P}(X \le x, Y \le y) > 0$ (Ahmadi-Javid, 2009; Charpentier and Juri, 2006; Durante and Jaworski, in press; Oakes, 2005).

With motivation from these investigations, copulas that are invariant under univariate truncation have been considered by Jágr et al. (2010), Mesiar et al. (2008) and characterized by Durante and Jaworski (in press). Analogously to the Clayton ones, such invariant copulas describe the limit behavior of (X, Y) when one component is taking on very small values. Along the lines of these investigations, the following class of bivariate copulas that are generated by a univariate function f has been introduced by Durante and Jaworski (in press).

Theorem 1.1. *Let $\mathbb{I} := [0, 1]$. Let $C_f: \mathbb{I}^2 \to \mathbb{I}$ be the function given by*

$$C_f(u, v) = \begin{cases} uf\left(\dfrac{f^{[-1]}(v)}{u}\right), & u \neq 0, \\ 0, & \textit{otherwise}, \end{cases} \tag{1}$$

where $f: [0, \infty] \to \mathbb{I}$ is surjective and monotone, and $f^{[-1]}: \mathbb{I} \to [0, \infty]$ denotes the right-inverse of f given by $f^{[-1]}(s) = \inf\{t \in [0, +\infty] \mid f(t) = s\}$.

If f is concave and nondecreasing (or convex and nonincreasing), then C_f is a copula.

In this paper, we aim at investigating the class of copulas of type (1), by showing some strong similarities and connections with the class of Archimedean copulas. Specifically, Section 2 presents some facts about copulas of type (1). Section 3 discusses some similarities of the family introduced, with Archimedean copulas, while practical consequences are derived in Section 4. Section 5 concludes.

Notice that, while extreme value theory concerns with maxima of random variables, we are mainly interested in minima of random variables. However, it should be mentioned that similar results can also be obtained for maxima by using, for instance, the survival copula associated with a random pair.

2. Definitions and basic properties

For basic definitions and properties of copulas, we refer the reader to Durante and Sempi (2010), Joe (1997) and Nelsen (2006). We denote by $\mathscr{C}$ the class of bivariate copulas and by $\mathscr{A}$ the class of Archimedean copulas, whose members can be represented in the form

$$A_\psi(u, v) = \psi(\psi^{[-1]}(u) + \psi^{[-1]}(v)),$$

where $\psi: [0, +\infty] \to \mathbb{I}$ is a convex, surjective and nonincreasing function called the *generator*. The class of all generators of Archimedean copulas will be indicated by Ψ.

We denote by $\mathscr{C}_{\mathrm{LT}}$ the class of bivariate copulas that are invariant under univariate truncation, namely if C is the copula of a continuous random pair (X, Y), then C is also the copula of (X, Y) conditional on the fact that X is less than its α-quantile for all $\alpha \in (0, 1)$. The characterization of $\mathscr{C}_{\mathrm{LT}}$ has been given by Durante and Jaworski (in press). Namely, the copulas belonging to $\mathscr{C}_{\mathrm{LT}}$ can be represented as a g-ordinal sum of the independence copula Π and copulas of type (1).

The class of copulas of type (1) will be indicated by the symbol $\mathscr{C}_{\mathrm{LTI}}$. As known from Durante and Jaworski (in press), $\mathscr{C}_{\mathrm{LTI}} \subset \mathscr{C}_{\mathrm{LT}}$. Moreover, among all elements of $\mathscr{C}_{\mathrm{LT}}$ any copula $C \in \mathscr{C}_{\mathrm{LTI}}$ satisfies the additional property that no $\alpha \in (0, 1)$ exists such that $C(\alpha, v) = \alpha v$. We will use the term *irreducible* to denote such copulas.

Elements of $\mathscr{C}_{\mathrm{LTI}}$ are the Fréchet–Hoeffding upper and lower bounds given, respectively, by $M(u, v) = \min(u, v)$, obtained when $f(t) = \min(t, 1)$, and $W(u, v) = \max(u+v-1, 0)$, obtained when $f(t) = \max(0, 1-t)$. Moreover, the independence copula $\Pi(u, v) = uv$ belongs to $\mathscr{C}_{\mathrm{LT}}$ but not to $\mathscr{C}_{\mathrm{LTI}}$.

As regards some known dependence properties, irreducible copulas can be characterized by means of their generator f.

Proposition 2.1. *Let $C_f \in \mathscr{C}_{\mathrm{LTI}}$.*

(a) *If f is concave and increasing, then $C_f(u, v) > uv$ on $(0, 1)^2$, that is C_f is positively quadrant dependent (for short, PQD).*
(b) *If f is convex and decreasing, then $C_f(u, v) < uv$ on $(0, 1)^2$, that is C_f is negatively quadrant dependent (for short, NQD).*

As a consequence, if $f \in \Psi$, then f generates an NQD copula in $\mathscr{C}_{\mathrm{LTI}}$. Moreover, it can be easily proved that, if $\overline{f} = 1 - f$ for some $f \in \Psi$, then $C_{\overline{f}} \in \mathscr{C}_{\mathrm{LTI}}$ is PQD. Notice that it holds that $C_{\overline{f}}(u, v) = u - C_f(u, 1 - v)$, i.e. $C_{\overline{f}}$ and C_f are connected via a flipping transformation (De Baets et al., 2009). In other words, if (U, V) is distributed according to C_f, then $(U, 1 - V)$ is distributed according to $C_{\overline{f}}$. Setting $\overline{\Psi} = \{\overline{f}: \overline{f} = 1 - f$ for some $f \in \Psi\}$, we have $\mathscr{C}_{\mathrm{LTI}} = \{C_f\}_{f \in \Psi} \cup \{C_f\}_{f \in \overline{\Psi}}$.

While Archimedean copulas are exchangeable, most of the copulas in $\mathscr{C}_{\mathrm{LTI}}$ are not, as the following result shows.

Proposition 2.2. *Let $C_f \in \mathscr{C}_{\mathrm{LTI}}$. Then C_f is exchangeable if, and only if, either $C_f = M$ or C_f is a Clayton copula.*

Proof. Suppose that $C_f \in \mathscr{C}_{\mathrm{LTI}}$ is an exchangeable copula. Then $C_f^t(u, v) = C_f(v, u)$ is a copula that is invariant under right truncation (compare with Durante and Jaworski (in press)). It follows that C_f is invariant under left and right truncation and, by Durante and Jaworski (in press, Theorem 4.1), either $C_f = M$ or C is a Clayton copula. □

Generally, it follows that, starting with a generator $f \in \Psi$, two kinds of models can be built: an exchangeable model, based on the Archimedean copula A_f; a non-exchangeable model, based on a copula C_f or $C_{\bar{f}}$ belonging to $\mathscr{C}_{\mathrm{LTI}}$.

Example 2.1. Consider $f_\theta(t) = \min(t^\theta, 1) \in \overline{\Psi}$ for $\theta \in (0, 1)$. Then $C_f \in \mathscr{C}_{\mathrm{LTI}}$ is given by

$$C_f(u, v) = \begin{cases} u, & v > u^\theta, \\ u^{1-\theta} v, & \text{otherwise} \end{cases}$$

which is a member of the Marshall–Olkin family of copulas (Marshall and Olkin, 1967).

3. Similarities between $\mathscr{C}_{\mathrm{LTI}}$ and Archimedean copulas

Besides the fact that they are generated by some univariate function $f \in \Psi$, the classes $\mathscr{A}$ and $\mathscr{C}_{\mathrm{LTI}}$ have a strong connection, as the following result shows.

Theorem 3.1. (a) *Let (U, V) be a pair of continuous random variables distributed according to $A_f \in \mathscr{A}$. Then $C_f \in \mathscr{C}_{\mathrm{LTI}}$ is the distribution function of the random pair (X, Y), where it holds almost surely that*

$$X = \frac{f^{[-1]}(V)}{f^{[-1]}(U) + f^{[-1]}(V)}, \qquad Y = V.$$

(b) *Let (X, Y) be a pair of continuous random variables distributed according to $C_f \in \mathscr{C}_{\mathrm{LTI}}$, where $f \in \Psi$. Then $A_f \in \mathscr{A}$ is the distribution function of the random pair (U, V), where it holds almost surely that*

$$U = f\left(\frac{f^{[-1]}(Y)}{X} - f^{[-1]}(Y)\right), \qquad V = Y.$$

Proof. First, consider the following mappings:

$$T_1 : (0, 1)^2 \longrightarrow (0, 1)^2, \quad T_1(u, v) = \left(\frac{f^{[-1]}(v)}{f^{[-1]}(u) + f^{[-1]}(v)}, v\right),$$

$$T_2 : (0, 1)^2 \longrightarrow \mathbb{I}^2, \quad T_2(x, y) = \left(f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right), y\right).$$

Then, for almost all (x, y) and (u, v) in $(0, 1)^2$,

$$\partial_2 A_f(T_2(x, y)) = \partial_2 C_f(x, y) \quad \text{and} \quad \partial_2 C_f(T_1(u, v)) = \partial_2 A_f(u, v).$$

In fact, at all of their respective points of differentiability,

$$\partial_2 C_f(x, y) = f'\left(\frac{f^{[-1]}(y)}{x}\right)(f^{[-1]})'(y)$$

and

$$\partial_2 A_f(u, v) = f'(f^{[-1]}(u) + f^{[-1]}(v))(f^{[-1]})'(v).$$

Thus, almost surely on $\mathbb{I}^2$,

$$\begin{aligned} \partial_2 C_f(T_1(u, v)) &= f'\left(\frac{f^{[-1]}(v)}{\frac{f^{[-1]}(v)}{f^{[-1]}(u)+f^{[-1]}(v)}}\right)(f^{[-1]})'(v) \\ &= f'(f^{[-1]}(u) + f^{[-1]}(v))(f^{[-1]})'(v) = \partial_2 A_f(u, v), \end{aligned}$$

and

$$\begin{aligned} \partial_2 A_f(T_2(x, y)) &= f'(f^{[-1]}(f(x^{-1} f^{[-1]}(y) - f^{[-1]}(y)) + f^{[-1]}(y)))(f^{[-1]})'(y) \\ &= f'\left(\frac{f^{[-1]}(y)}{x}\right)(f^{[-1]})'(y) = \partial_2 C_f(x, y). \end{aligned}$$

Now, let us prove part (a) (part (b) can be proved analogously). In view of Nelsen (2006, Theorem 4.3.7), X is uniformly distributed on $[0, 1]$. Therefore, we have only to show that the conditional distribution of X with respect to Y is a conditional distribution of the copula $C_f \in \mathscr{C}_{\mathrm{LTI}}$. In fact, for all $(x, y) \in \mathbb{I}$ we have

$$\begin{aligned}\mathbb{P}(X \le x \mid Y = y) &= \mathbb{P}\left(\frac{f^{[-1]}(V)}{f^{[-1]}(U) + f^{[-1]}(V)} \le x \mid V = y\right)\\ &= \mathbb{P}\left(U \le f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right) \mid V = y\right)\\ &= \partial_2 A_f\left(f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right), y\right) = \partial_2 C_f(x, y)\end{aligned}$$

and hence C_f is a copula of the pair (X, Y) (compare with Nelsen (2006, Eq. (2.9.1))).

Theorem 3.1 allows us to translate results from $\mathscr{A}$ to $\mathscr{C}_{\mathrm{LTI}}$ (and conversely). Some consequences of this relationship will be given below.

We recall that any copula C can be decomposed into the form $C = C_A + C_S$, where C_A and C_S are, respectively, the absolutely continuous (with respect to the Lebesgue measure) and the singular components of C.

Proposition 3.2. *Let $C_f \in \mathscr{C}_{\mathrm{LTI}}$. Then C_f is absolutely continuous if, and only if, the first derivative f' exists and is absolutely continuous on* $(0, \infty)$.

Proof. First, suppose that $f \in \Psi$. It holds that $C_f \in \mathscr{C}_{\mathrm{LTI}}$ is absolutely continuous if, and only if, $A_f \in \mathscr{A}$ is absolutely continuous. In fact, consider T_1^* and T_2^* given by

$$\begin{aligned}&T_1^* : (0, 1)^2 \to B, \quad T_1^*(u, v) = \left(\frac{f^{[-1]}(v)}{f^{[-1]}(u) + f^{[-1]}(v)}, v\right),\\ &T_2^* : B \to (0, 1)^2, \quad T_2^*(x, y) = \left(f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right), y\right),\end{aligned}$$

where $B = \{(x, y) \in (0, 1)^2 : f^{[-1]}(y) \cdot (1 - x) < f^{[-1]}(0) \cdot x\}$. Notice that these mappings are well-defined on their domains.

Moreover, T_2^* is an inverse of T_1^*. Indeed, for all $(x, y) \in B$ we get

$$\begin{aligned}T_1^* \circ T_2^*(x, y) &= T_1^*\left(f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right), y\right)\\ &= \left(\frac{f^{[-1]}(y)}{f^{[-1]}\left(f\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right)\right) + f^{[-1]}(y)}, y\right)\\ &= \left(\frac{f^{[-1]}(y)}{\left(\frac{f^{[-1]}(y)}{x} - f^{[-1]}(y)\right) + f^{[-1]}(y)}, y\right) = (x, y)\end{aligned}$$

and, similarly, for all $(u, v) \in (0, 1)^2$,

$$\begin{aligned}T_2^* \circ T_1^*(u, v) &= T_2^*\left(\frac{f^{[-1]}(v)}{f^{[-1]}(u) + f^{[-1]}(v)}, v\right)\\ &= \left(f\left(f^{[-1]}(v)\frac{f^{[-1]}(u) + f^{[-1]}(v)}{f^{[-1]}(v)} - f^{[-1]}(v)\right), v\right) = (u, v).\end{aligned}$$

Thus, T_1^* and T_2^* are injective and locally Lipschitz on their respective domains, because the derivatives of f and $f^{[-1]}$ are locally bounded. In addition, C_f is vanishing on $(0, 1)^2 \setminus B$. In fact, for all $x \neq 0$, if $f^{[-1]}(y) \cdot (1 - x) \ge f^{[-1]}(0) \cdot x$, then $x^{-1} f^{[-1]}(y) \ge f^{[-1]}(0)$, and $C_f(x, y) = x f(f^{[-1]}(0)) = 0$. Therefore, it follows from the change of variable formula (Villani, 2009, chapter 1) that, if C_f is absolutely continuous, then so is A_f (and conversely). Now, the assertion follows by the characterization of absolutely continuous Archimedean copulas given by McNeil and Nešlehová (2009, Proposition 4.2).

Finally, if $f \in \overline{\Psi}$, then the assertion follows by considering that C_f is absolutely continuous if, and only if, $A_{\overline{f}} \in \mathscr{A}$ is absolutely continuous. □

Let f be a generator of a copula in $\mathscr{C}_{\mathrm{LTI}}$. Since f' is monotonic, f'' is defined almost everywhere on $(0, \infty)$. The density of the absolutely continuous component of the copula C_f is equal to its second mixed derivative, which is defined almost everywhere on $\mathbb{I}^2$ by

$$c_f(u, v) = -f''\left(\frac{f^{[-1]}(v)}{u}\right)\frac{f^{[-1]}(v)}{f'(f^{[-1]}(v))u^2}. \tag{2}$$

Therefore, if $C_f \in \mathscr{C}_{\mathrm{LTI}}$ is absolutely continuous, then c_f given by (2) is the density of C_f. Moreover, if $c_f = 0$ almost everywhere on $\mathbb{I}^2$, then C_f is singular. As a consequence, the following result follows.

Corollary 3.3. *Let $C_f \in \mathscr{C}_{\mathrm{LTI}}$. Then C_f is singular if, and only if, $f''(t) = 0$ almost everywhere on $(0, \infty)$.*

Three examples of singular copulas in $\mathscr{C}_{\mathrm{LTI}}$ are given below.

Example 3.1. Consider the piecewise linear function $f: [0, +\infty] \to \mathbb{I}$, given by $f(x) = \max(0, 1 - 2x, (1 - x)/2)$. Then $f \in \Psi$ and, since $f'' = 0$ except for finitely many points, it generates the singular copula C_f given by

$$C_f(x, y) = \begin{cases} 0, & y \le f(x), \\ W(x, y), & y \ge 1 - 2x/3, \\ y + \dfrac{x-1}{2}, & \dfrac{3-x}{6} \le y \le \dfrac{1}{3}, \\ \dfrac{x}{2} + \dfrac{y-1}{4}, & \text{elsewhere.} \end{cases}$$

Note that the support of C_f consists of the three segments connecting, respectively, $(0, 1)$ with $(1/3, 1/3)$, $(1/3, 1/3)$ with $(1, 0)$, and $(1, 1)$ with $(1/3, 1/3)$.

Example 3.2. Let π be the bijection defined by

$$\pi: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}, \quad \pi(p, q) = \frac{1}{2}(p+q)(p+q+1) + q,$$

called the *Cantor pairing*. Consider $f: [0, +\infty] \to \mathbb{I}$ given by

$$f(x) = \sum_{p,q=1}^{+\infty} 2^{-\pi(p,q)}\left(1 - \frac{q}{p}x\right)^+,$$

where $x^+ := \max(x, 0)$ for all $x \in \mathbb{R}$. Then $f \in \Psi$ and, since $f'' = 0$ except for a countable set of points, it generates a singular copula $C_f \in \mathscr{C}_{\mathrm{LTI}}$.

Note that the probability mass of C_f is located on a dense subset of $\mathbb{I}^2$, which consists of a countable set of lines (compare with Proposition 4.3).

Example 3.3. Consider $f: [0, +\infty] \to \mathbb{I}$ given by

$$f(x) = \begin{cases} 2\displaystyle\int_0^{1-x} \Gamma(\xi) d\xi & \text{for } x \in [0, 1], \\ 0 & \text{for } x > 1, \end{cases}$$

where $\Gamma: \mathbb{I} \to \mathbb{I}$ is the Cantor function.[1] Note that, since $\Gamma(1 - \xi) = 1 - \Gamma(\xi)$, we have that $f(0) = 1$ and $\int_0^1 \Gamma(\xi)\mathrm{d}\xi = \frac{1}{2}$. Then $f \in \Psi$ and, since $f''(t) = 0$ except for the Cantor set, it generates a singular copula $C_f \in \mathscr{C}_{\mathrm{LTI}}$. See also McNeil and Nešlehová (2009, Example 4.3).

Remark 3.1. Associative copulas have been characterized in terms of ordinal sums of Archimedean copulas (see Ling, 1964; Klement et al., 2000). We may note the analytical correspondence between this result and the representation of copulas in $\mathscr{C}_{\mathrm{LT}}$ as given in Durante and Jaworski (in press, Theorem 3.1).

The similarities between $\mathscr{A}$ and $\mathscr{C}_{\mathrm{LTI}}$ also extend to constructions applied to the Archimedean class. For instance, in order to extend $\mathscr{A}$, Archimax copulas have been considered by Capéraà et al. (2000). An analogous extension of the $\mathscr{C}_{\mathrm{LTI}}$ is given by DUCS copulas, introduced by Mesiar and Pekárová (2010).

[1] Its graph is called figuratively the "Devil's staircase".

4. Practical implications

4.1. Modelling tail dependence

As already said, copulas of type (1) can capture non-exchangeable dependence structures. Moreover, another interesting feature of this family is its ability to model non-trivial tail dependence.

We recall that, given a copula C, we can describe the tail behavior of C on the four corners of $\mathbb{I}^2$ by means of the *tail dependence functions* L_{ij} $(i, j \in \{0, 1\})$. Following Joe et al. (2010) and Klüppelberg et al. (2008), these functions are given for every $x, y \in [0, \infty)$ by the following formulas:

$$L_{0,0}(x, y) = \lim_{t\to0^+} t^{-1}\mathbb{P}(U \le tx, V \le ty) = \lim_{t\to0^+} \frac{C(tx, ty)}{t},$$
$$L_{0,1}(x, y) = \lim_{t\to0^+} t^{-1}\mathbb{P}(U \le tx, V \ge 1 - ty) = x - \lim_{t\to0^+} \frac{C(tx, 1 - ty)}{t},$$
$$L_{1,0}(x, y) = \lim_{t\to0^+} t^{-1}\mathbb{P}(U \ge 1 - tx, V \le ty) = y - \lim_{t\to0^+} \frac{C(1 - tx, ty)}{t},$$
$$L_{1,1}(x, y) = \lim_{t\to0^+} t^{-1}\mathbb{P}(U \ge 1 - tx, V \ge 1 - ty) = x + y + \lim_{t\to0^+} \frac{C(1 - tx, 1 - ty) - 1}{t},$$

provided that the limits exist and (U, V) is a pair of uniform random variables distributed according to C. Actually, such L_{ij} are the leading parts of the tail expansion (of degree 1) of the copula C near the corners of the unit square (Jaworski, 2004, 2010). Notice that the upper and lower tail dependence coefficients are equal to $\lambda_U = L_{1,1}(1, 1)$ and $\lambda_L = L_{0,0}(1, 1)$, respectively.

Here we calculate the tail dependence functions associated with C_f. Notice that, for a function g, $g'(x_0^+)$ and $g'(x_0^-)$ denote, respectively, the right and left first derivatives of g.

Proposition 4.1. *Let $C_f \in \mathscr{C}_{\mathrm{LTI}}$, where $f \in \overline{\Psi}$. Then the following expressions hold:*

$$L_{0,0}(x, y) = xf\left(\frac{y}{x}(f^{[-1]})'(0^+)\right),$$
$$L_{0,1}(x, y) = L_{1,0}(x, y) = 0,$$
$$L_{1,1}(x, y) = \begin{cases} \min\left(y, xf'(f^{[-1]}(1)^-)f^{[-1]}(1)\right), & f^{[-1]}(1) < \infty, \\ 0, & f^{[-1]}(1) = \infty. \end{cases}$$

Proof. Let $f \in \overline{\Psi}$ and $C_f \in \mathscr{C}_{\mathrm{LTI}}$. Since C_f is PQD, it follows that $L_{0,1} = L_{1,0} = 0$. Let $(x, y) \in (0, \infty)^2$. We have

$$\begin{aligned} L_{0,0}(x, y) &= \lim_{t\to0^+} \frac{C(tx, ty)}{t} = \lim_{t\to0^+} xf\left(\frac{f^{[-1]}(ty)}{tx}\right) \\ &= xf\left(\lim_{t\to0^+} \frac{f^{[-1]}(ty)}{tx}\right) = xf\left(\frac{y}{x}(f^{[-1]})'(0^+)\right). \end{aligned}$$

In order to calculate $L_{1,1}$, we have to distinguish two cases.

- Suppose that $f^{[-1]}(1) < \infty$. It follows that

$$\begin{aligned} L_{1,1}(x, y) &= x + y + \lim_{t\to0^+} \frac{-1 + (1 - tx)f\left(\frac{f^{[-1]}(1-ty)}{1-tx}\right)}{t} \\ &= x + y - xf(f^{[-1]}(1)) + \lim_{t\to0^+} \frac{-f(f^{[-1]}(1)) + f\left(\frac{f^{[-1]}(1-ty)}{1-tx}\right)}{t}. \end{aligned}$$

If $f^{[-1]}(1)x > (f^{[-1]})'(1^-)y$, then $f'(f^{[-1]}(1)^+) = 0$ implies that

$$L_{1,1}(x, y) = y + f'(f^{[-1]}(1)^+)(-y(f^{[-1]})'(1^-) + xf^{[-1]}(1)) = y.$$

If $f^{[-1]}(1)x < (f^{[-1]})'(1^-)y$, then $f'(f^{[-1]}(1)^+)(f^{[-1]})'(1^-) = 1$ implies that

$$L_{1,1}(x, y) = y + f'(f^{[-1]}(1)^-)(-y(f^{[-1]})'(1^-) + xf^{[-1]}(1)) = xf'(f^{[-1]}(1)^-)f^{[-1]}(1).$$

If $f^{[-1]}(1)x = (f^{[-1]})'(1^-)y$, then for sufficiently small $t > 0$ we get

$$\left| \frac{-f(f^{[-1]}(1)) + f\left(\frac{f^{[-1]}(1-ty)}{1-tx}\right)}{t} \right| \leq \left|f'(f^{[-1]}(1-ty)^-)\right| \left| \frac{f^{[-1]}(1-ty) - f^{[-1]}(1)}{t} + xf^{[-1]}(1) \right| \frac{1}{1-tx}$$

$$\xrightarrow{t\to 0^+} |f'(f^{[-1]}(1)^-) \parallel xf^{[-1]}(1) - y(f^{[-1]})'(1^-)| = 0.$$

Summarizing, we get

$$L_{1,1}(x, y) = \min\left(y, xf'(f^{[-1]}(1)^-)f^{[-1]}(1)\right).$$

- Suppose that $f^{[-1]}(1) = \infty$. Then we have

$$L_{1,1}(x, y) = \lim_{t\to 0^+} \frac{-f(f^{[-1]}(1-ty)) + f\left(\frac{f^{[-1]}(1-ty)}{1-tx}\right)}{t}$$
$$\leq \lim_{t\to 0^+} f'(f^{[-1]}(1-ty)^-)f^{[-1]}(1-tx)\frac{x}{1-tx} = x \lim_{\xi\to+\infty} f'(\xi)\xi = 0.$$

This concludes the proof. □

The tail behavior of C_f for $f \in \Psi$ can be obtained analogously.

Proposition 4.2. *Let $C_f \in \mathscr{C}_{\text{LTI}}$, where $f \in \Psi$. Then the following expressions hold:*

$$L_{0,1}(x, y) = xf\left(\frac{y}{x}(f^{[-1]})'(1^-)\right),$$
$$L_{0,0}(x, y) = L_{1,1}(x, y) = 0,$$
$$L_{1,0}(x, y) = \begin{cases} \min\left(y, xf'(f^{[-1]}(0)^+)f^{[-1]}(0)\right), & f^{[-1]}(0) < \infty, \\ 0, & f^{[-1]}(0) = \infty. \end{cases}$$

4.2. Stochastic simulation

Several methods have been proposed for generating a random pair (U, V) with distribution $A_\psi \in \mathscr{A}$ (see, for instance, Hofert, 2010 and the references therein). The insights of Theorem 3.1 give a method for generating copulas in $\mathscr{C}_{\text{LTI}}$ based on the known algorithms for the Archimedean case. Specifically, we distinguish two cases.

Let $C_f \in \mathscr{C}_{\text{LTI}}$ be an NQD copula (thus, $f \in \Psi$). In order to generate a random pair (X, Y) from C_f the algorithm goes as follows:

1. Generate a random pair (U, V) from the Archimedean copula A_f.
2. Set $X = \frac{f^{[-1]}(V)}{f^{[-1]}(U)+f^{[-1]}(V)}$ and $Y = V$.
3. Return (X, Y).

Otherwise, let $C_f \in \mathscr{C}_{\text{LTI}}$ be a PQD copula. Then, in order to generate a random pair (X, Y) from C_f, one should use the following procedure:

1. Generate a random pair (U, V) from the Archimedean copula $A_{\bar{f}}$, where $\bar{f} = 1 - f$.
2. Set $X = \frac{\bar{f}^{[-1]}(V)}{\bar{f}^{[-1]}(U)+\bar{f}^{[-1]}(V)}$ and $Y = V$.
3. Return $(X, 1 - Y)$.

Examples of random samples generated from copulas in $\mathscr{C}_{\text{LTI}}$ are given in Fig. 1. Note that the two figures are identical up to a mirror symmetry $(x, y) \to (x, 1 - y)$, as a consequence of the sampling algorithms, which differ only in point 3. We may also notice the asymmetries of the samples with respect to the main diagonal of $\mathbb{I}^2$ and in the corners of $\mathbb{I}^2$.

4.3. Statistical inference

From Theorem 3.1, the following result can also be proved.

Proposition 4.3. *Let (X, Y) be a pair of continuous random variables distributed according to $C_f \in \mathscr{C}_{\text{LTI}}$, where $f \in \Psi$. Let $Z = C_f(X, Y)/X$. Then the following statements hold:*

(a) *The distribution function of Z is given, for all $t \in (0, 1)$, by*

$$F_Z(t) = t - f^{[-1]}(t)f'(f^{[-1]}(t)^-). \tag{3}$$

(b) *The random variables X and Z are independent.*

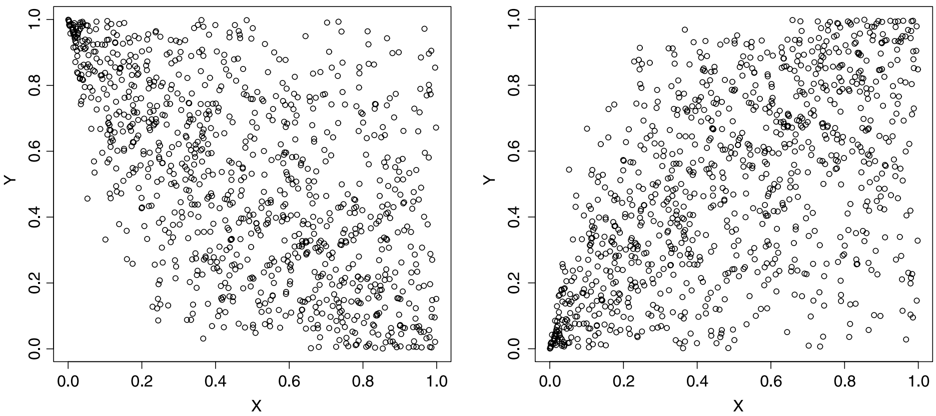

Fig. 1. Sampling 1000 points from an NQD copula $C \in \mathscr{C}_{\rm LTI}$ with a Gumbel generator with parameter 1.2 (left side) and its corresponding PQD copula in $\mathscr{C}_{\rm LTI}$ (right side).

Proof. Let $f \in \Psi$, $C_f \in \mathscr{C}_{\rm LTI}$ and $A_f \in \mathscr{A}$. Notice that Z is a random variable whose support is $(0, 1)$. As a direct consequence of Theorem 3.1(a), we have

$$Z = \frac{C_f(X,Y)}{X} = f\left(\frac{f^{[-1]}(Y)}{X}\right) = f\left(\frac{f^{[-1]}(V)}{\frac{f^{[-1]}(V)}{f^{[-1]}(U)+f^{[-1]}(V)}}\right)$$
$$= f(f^{[-1]}(U) + f^{[-1]}(V)) = A_f(U, V).$$

Therefore due to Genest and Rivest (1993, Proposition 1.1) (see also Nelsen (2006, Theorem 4.3.4)), it follows that, for all $t \in (0, 1)$,

$$F_Z(t) = t - f^{[-1]}(t) f'(f^{[-1]}(t)^-),$$

and $X = \frac{f^{[-1]}(V)}{f^{[-1]}(U)+f^{[-1]}(V)}$ and $Z = A_f(U, V)$ are independent. □

Proposition 4.3 may be used to derive some inference procedure for copulas in $\mathscr{C}_{\rm LTI}$. In fact, roughly speaking, in the class $\mathscr{C}_{\rm LTI}$ the random variable Z previously defined plays the same role as the bivariate probability integral transformation $V = C(X, Y)$ plays in the case of Archimedean copulas (see Genest et al., 2006; Genest and Rivest, 1993; Wang and Wells, 2000).

To illustrate this, let $\mathbf{U}_1, \ldots, \mathbf{U}_n$ be bivariate vectors that are supposed to form a random sample from $C_f \in \mathscr{C}_{\rm LTI}$ (just for the sake of simplicity, consider here $f \in \Psi$). Obviously, such a sample can be derived by applying marginal empirical distribution functions to bivariate data (Genest and Favre, 2007). For each observation $\mathbf{U}_k = (U_{k1}, U_{k2})$ we could consider the quantity $Z_k = V_k/U_{k1}$, where $V_k = C_{f,n}(U_{k1}, U_{k2})$ is obtained from the empirical copula $C_{f,n}$ associated with the sample. The empirical distribution function of Z_k represents a nonparametric estimation of the distribution function given by (3).

By applying Proposition 4.3 the following statistical procedures can be implemented.

- If a copula C is assumed to be of the form (1), a nonparametric estimator of its generator could be derived from the pseudo-sample $Z_1, \ldots, Z_n$.
- In order to check H_0: *C is of the form* (1) against H_1: *C is not of the form* (1), a goodness-of-fit test can be constructed by using the fact that, under the null hypothesis, $(U_{k1})_k$ and $(Z_k)_k$ are independent.

5. Conclusions

Formula (1) introduces a new family of copulas that have some distinguishing properties:

- they are generated by means of univariate functions;
- they can capture non-exchangeable dependence structures;
- they can be easily simulated by using standard simulation algorithms for Archimedean copulas.

Moreover, such a class presents strong probabilistic similarities with the class of Archimedean copulas from theoretical and practical points of view.

The higher-dimensional characterization of the class $\mathscr{C}_{\rm LT}$ has been obtained by Jaworski (submitted for publication). Interestingly, multivariate copulas that are invariant under univariate truncation can be constructed by means of a suitable combination of the elements of $\mathscr{C}_{\rm LTI}$ on the basis of the vine method (Czado, 2010; Klüppelberg, 2010). However, such multivariate copulas do not seem to have the same nice features as in the bivariate case; for instance, they do not allow for a closed form and/or stochastic equivalence with the Archimedean class.

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