A Simple Decision Problem of a Market Maker

Martin Šmíd[1]

Abstract. We formulate a simple decision model of a market maker maximizing an utility from his consumption. We reduce the dimensionality of the problem to one. We find that, given our setting, the quotes set by the market maker depend on the inventory of the traded asset but not on the amount of cash held by the market maker.

Keywords: market microstructure, market makers, decision problem, probability constraints, stochastic optimization

JEL classification: C51, G10
AMS classification: 90C15

1 Introduction

One of the key roles in price formation at today's financial markets is played by market makers (MMs) who are obliged to set buying and selling qoutes and trade for the prices they set. Clearly, as other economic agents, MMs are profit maximizers. The economic analysis of their behaviour is, however, quite complicated since the decision problems they face are usually intractable (see [1], [2] or [3]).

In the present paper, we suggest a rather simple version of such a decision problem. In particular, we assume the MM to maximize (an utility) from his consumption while keeping the probability of the bankruptcy (i.e. running out of the money or the traded asset) at a prescribed, perhaps very small level. We do not give analytic solution of the problem but we reduce its dimensionality. As a result of our analysis, we find that the quotes depend on the inventory of the traded assets but they do not depend on the amount of cash held by the MM.

Even if our model is only single-period one, it does not suffer from the logic of "scorched earth", i.e. the today's actions do not steel from the future to a great extent. The reason for this is that, by keeping the probabilities of crossing zero by the inventory processes very small, the model tends to "keep a distance" from the boundaries. Nevertheless, dynamization of our model could be a promising direction of a further research.

The paper is organized as follows: after a definition of the setting, the model is formulated and partially solved. A short conclusion is finishing the paper.

2 The setting

Let there be two types of agents: the market makers posting quotes (the best bid and ask) and the (liquidity) traders.

We assume the market makers to be homogeneous, forming an oligopoly, so that they may be treated as a single representative agent who sets the quotes A and B (the best ask, bid, respectively) in order to maximize an utility from their consumption. Denote U the corresponding utility function and assume that it is strictly increasing.

In reaction to the quotes, the traders post market orders, which we assume to be unit for simplicity.

Assume the traders to post orders with an intensity depending solely on a distance of the corresponding quote to a fair price $\Pi \in \mathbb{R}$: In particular, the intensity of the arrival of buy orders is assumed to be

$$\kappa (A - \Pi) \geq 0.$$

[1]Institute of Information Theory and Automation,
Academy of Sciences of the Czech Republic, Dpt. of Econometrics, Pod Vodárenskou věží4, Praha 8, 182 08, e-mail: smid@utia.cas.cz

while the intensity of the sell orders' arrival is given by

$$\lambda(\Pi - B) \geq 0.$$

Quite naturally, assume that both κ and λ are continuous strictly decreasing, defined on $[0, H_\kappa]$, $[0, H_\lambda]$ respectively, for some $H_\kappa, H_\lambda > 0$ and that

$$\kappa(H_\kappa) = \lambda(H_\lambda) = 0$$

(i.e. nobody wants expensive asset and nobody wants to give anything for free).

3 The decision problem

Denote M and N the amount of money, traded asset, respectively, held by the MM and assume that the MM maximizes the (utility from his) consumption while keeping the probability of running out of the money or the commodity at a prescribed (small) level α i.e. he solves

$$V(M, N, \Pi) = \max_{A,B,C} U(C) \tag{1}$$

$$A, B, C \geq 0, \qquad B \leq A, \qquad \mathbb{P}[M + AX - BY - C \leq 0] \leq \alpha, \qquad \mathbb{P}[N + Y - X \leq 0] \leq \alpha$$
$$X \sim \mathrm{Po}(\kappa(A - \Pi)), \qquad Y \sim \mathrm{Po}(\lambda(\Pi - B)), \qquad X \perp\!\!\!\perp Y.$$

Here, X and Y are random variables counting numbers of buy market orders, sell market orders, respectively, which were posted given the quotes are A and B.

To simplify further analysis, we shall assume that the intensities κ and λ are high enough so that X and Y may be approximated by normal variables, namely

$$X \dot{\sim} \mathcal{N}(\kappa(A - \Pi), \kappa(A - \Pi)), \qquad Y \dot{\sim} \mathcal{N}(\lambda(\Pi - B), \lambda(\Pi - B)) \tag{2}$$

Note that, given this assumption,

$$N + Y - X \dot{\sim} \mathcal{N}(N + \lambda(\Pi - B) - \kappa(A - \Pi), \lambda(\Pi - B) + \kappa(A - \Pi)),$$
$$M + AX - BY - C \dot{\sim} \mathcal{N}\left(M + A\kappa(A - \Pi) - B\lambda(\Pi - B) - C, A^2\kappa(A - \Pi) + B^2\lambda(\Pi - B)\right)$$

so the constraints may be approximated by

$$\kappa(A - \Pi) - \lambda(\Pi - B) - N + q\sqrt{\kappa(A - \Pi) + \lambda(\Pi - B)} \leq 0 \tag{3}$$

$$B\lambda(\Pi - B) - A\kappa(A - \Pi) - M + C + q\sqrt{A^2\kappa(A - \Pi) + B^2\lambda(\Pi - B)} \leq 0 \tag{4}$$

where q is the α-quantile of the standard normal distribution.

Proposition 1. *For problem (1) with the approximation (2), the following is true:*
(i) An optimal solution exists for any $M \geq 0, N \geq 0, \Pi \in \mathbb{R}^+$.
(ii) In optimum, (4) is fulfilled with "=".
(iii) In optimum, either $\lambda(\Pi - B) = 0$ or (3) is fulfilled with "=".
(iv) If $\hat{A}, \hat{B}, \hat{C}$ is the optimal solution of (1) then $\kappa(\hat{A} - \Pi), \lambda(\Pi - \hat{B})$ is an optimal solution of

$$v(N, \Pi) = \max_{K,L \geq 0} \left[A(K)K - B(L)L - q\sqrt{A(K)^2K + B(L)^2L}\right] \tag{5}$$

$$B(L) \leq A(K),$$
$$K - L - N + q\sqrt{K + L} \leq 0 \tag{6}$$

and $\hat{C} = v(N, \Pi) + M$. Here,

$$A(K) = A(K, \Pi) = \Pi + \kappa^{-1}(K), \qquad B(L) = B(L, \Pi) = \Pi - \lambda^{-1}(L).$$

(v) If $(\hat{K}, \hat{L})$ is the optimal solution of (5) then either $\hat{L} = 0$ or

$$\hat{L} = \Lambda(\hat{K}), \qquad \Lambda(K) = K - N + \frac{q^2}{2} + \frac{q}{2}\sqrt{q_\alpha^2 + 8K - 4N}$$

(vi) Denote

$$\eta(K) = \left[A(K)\left(K - q\sqrt{K}\right)\right], \qquad K_0 = N + \frac{q^2}{2}\left(1 - \sqrt{\frac{4N}{q^2}+1}\right), \qquad K_L = \kappa(B(0)).$$

If the optimal solution $\hat{K}$ *of the problem*

$$\tilde{v}(N,\Pi) = \max_K \eta(K), \qquad 0 \le K \le K_0 \wedge K_L, \tag{7}$$

coincides with the solution of

$$\dot{v}(\Pi) = \max_K \eta(K), \qquad 0 \le K \le \infty \tag{8}$$

then $(\hat{K}, 0)$ *is the solution of (5).*
(vii) If $(\hat{K},\hat{L})$, $\hat{L} > 0$ *is an optimal solution of (5) then* $\hat{K}$ *is an optimal solution of*

$$\dot{v}(N,\Pi) = \max_{K \ge 0}\left[A(K)K - B(\Lambda(K))\Lambda(K) - q\sqrt{A(K)^2K + B(\Lambda(K))^2\Lambda(K)}\right], \tag{9}$$

$$B(\Lambda(K)) \le A(K), \qquad K \ge \frac{N}{2} - \frac{q}{8}$$

and $\hat{L} = \Lambda(\hat{K})$.

Proof. (i) A strategy $A = \Pi + H_\kappa, B = \Pi - H_\lambda, C = 0$, producing $X \equiv 0$, $Y \equiv 0$, is clearly feasible. The existence of optimal solution then follows from the continuity of κ and λ.
(ii) If $\hat{A}, \hat{B}, \hat{C}$ were optimal and (4) was fulfilled with "$<$" for $A = \hat{A}$, $B = \hat{B}$, $C = \hat{C}$ then there would exist $\bar{C} > \hat{C}$ still fulfilling (4) which is a contradiction with the optimality of $\hat{A}, \hat{B}, \hat{C}$.
(iii) If $\hat{A}, \hat{B}, \hat{C}$ were optimal with $\hat{L} > 0$ (with the consequence that $B > \Pi - H_\lambda$) and if (3) held with "$<$" for $A = \hat{A}$, $B = \hat{B}$ then, from the continuity of the intensities, there would exist $\Delta > 0$ such that (3) is fulfilled for

$$A = \hat{A}, \qquad B = \hat{B} - \Delta, \tag{10}$$

for which, however, $AX - BY$ is less both in expectation and in variance than the same variable given $A = \hat{A}$, $B = \hat{B}$ which implies that (4) is fulfilled with $<$ given (10) yielding the existence of a feasible $\bar{C} > \hat{C}$, i.e. a contradiction to the optimality of $\hat{A}, \hat{B}, \hat{C}$.
(iv) Consider a problem

$$\max_{A,B \ge 0} g(A,B) \tag{11}$$

$$g(A,B) = A\kappa(A - \Pi) - B\lambda(\Pi - B) - q\sqrt{A^2\kappa(A-\Pi) + B^2\lambda(\Pi - B)}$$

$$B \le A, \qquad \kappa(A - \Pi) - \lambda(\Pi - B) - N + q\sqrt{\kappa(A-\Pi) + \lambda(\Pi - B)} \le 0$$

and note that g is the negative of the LHS of (4) without M and C hence, by (ii), for $\hat{A}, \hat{B}, \hat{C}$ optimal to (1) it has to hold that

$$g(\hat{A}, \hat{B}) + M - \hat{C} = 0 \tag{12}$$

We show that $\hat{A}, \hat{B}$ is then the optimal solution of (11): Indeed, if $\hat{A}, \hat{B}$ were not optimal to (11) there would exist $\bar{A}, \bar{B}$, $\bar{B} \le^{-}$, fulfilling (3) such that $g(\bar{A}, \bar{B}) > g(\hat{A}, \hat{B})$. However, then $\bar{A}, \bar{B}, \bar{C}$, $\bar{C} = \hat{C} + (g(\bar{A}, \bar{B}) - g(\hat{A}, \hat{B})) > \hat{C}$ would fulfil both (3) and (4) which is a contradiction to the optimality of $\hat{A}, \hat{B}, \hat{C}$. Finally, note that (11) is equivalent to (5).
(v) Let $\hat{K} \ge 0$,$\hat{L} > 0$ be optimal. By (iii), (3) holds with "=" hence $\hat{L}$ has to solve the equation

$$(\hat{K} - L - N)^2 = q^2(L + \hat{K})$$

whose solutions are

$$L_{1,2} = \hat{K} - N + \frac{q^2}{2} \pm q\sqrt{\frac{q^2}{4} + 2\hat{K} - N}$$

A trivial calculation shows that $\hat{K} - L_1 - N < 0$ which proves that λ_1 is indeed a solution of (3) with "=". Once $\hat{K} - L_2 - N > 0$, L_2 cannot be a solution of (3) with "=" hence L_1 is unique. Assume now that $\hat{K} - L_2 - N \le 0$ i.e. both L_1 and L_2 are candidates for the value of $\hat{L}$. Denote $f(L) = \frac{\hat{K}-L-N}{\sqrt{\hat{K}+L}}$

and note that (3) holds with the equality iff $f(L) = -q$. The fact that $f'(L) = \frac{1}{2(\hat{K}+L)^{3/2}}\left(N - 3\hat{K} - L\right)$ proves that f is possibly increasing starting form zero up to some threshold and decreasing starting from the threshold hence, necessarily, f is increasing in L_2 and decreasing in L_1which implies, however, that $(\hat{K}, 0)$ fulfils (6) and, by arguments similar to those from (iii), the strategy $\hat{K}, \hat{L}$ is dominated by $(\hat{K}, 0)$ if $\hat{L} = L_2$. Therefore, it has to be $\hat{L} = L_1$.
(vi) Note first that (7) coincides with (5) with an additional constraint $L = 0$. Further, since the objective function of (5) is decreasing in L, any solution if (5) is dominated by $(\hat{K}, 0)$ given that $\hat{K}$ is a solution of (8). If $\hat{K}$, in addition, satisfies the conditions of (7), necessarily $(\hat{K}$,0) is the solution of (5).
(vii) The assertion follows from (v). □

The Proposition, we have just proved, gives us a directions for solving the problem (5) (and, consequently, (1)). The procedure is as follows

1. Solve (7) and (8). If the optimal solution $\hat{K}_0$ of (7) coincides with that of (8) then $\left(\hat{K}_0, 0\right)$ is the optimal solution of (5) and we may stop the procedure.
2. If the solutions of (7) and (8) differ, solve (9) and check the objective value reached its solution $(\hat{K}, \hat{L})$ with that of $(\hat{K}_0, 0)$. The solution with the higher objective value is the solution of (5).

Note also that, quite surprisingly, the optimal quotes depend only on the inventory of the traded asset but not on the inventory of money.

4 Conclusion

We have formulated a simple but realistic decision problem of a market maker and we reduced its solution to one-dimensional problems. Our result may be useful in further analysis of the microstructure effects at financial markets.

Acknowledgements

This work was supported by grants No. 402/09/0965, P402/10/1610 and P402/10/0956 of the Czech Science Foundation. I thank an anonymous referee for a correction of several my errors.

References

[1] B. Biais, L. Glosten, and Ch. Spatt. Market microstructure: A survey of microfoundations, empirical results, and policy implications. *Journal of Financial Markets*, 8:217–264, 2005.

[2] A. Madhavan. Market microstructure: A survey. *Journal of Financial Markets*, 3:205–258, 2000.

[3] A. Madhavan, and S. Schmidt. An analysis of changes in specialist inventories and qoutations. *J. of Finance*, vol. XLVIII (5). 1993