Akademie věd České republiky
Ústav teorie informace a automatizace

Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation

RESEARCH REPORT

ANDREA KARLOVÁ:

Stable distributions: On parametrizations of characteristic exponent

No. 2308 October 2011

ÚTIA AV ČR, P. O. Box 18, 182 08 Prague, Czech Republic
Telex: 122018 atom c, Fax: (+42) (2) 688 4903
E-mail: utia@utia.cas.cz

Stable Distributions:
On Parametrizations of Characteristic Exponent

Andrea Karlová

Contents

1 Introduction

In this report we investigate theory of stable distributions and their role in probability theory. We are interested in derivation of canonical measure, semigroup operator and mainly parametrizations of characteristic exponents. We finally introduce the new parametrization.

Let us start with a simple motiovation. Having a random variable X, its probability distribution F is characterized by functionals describing e.g. its shape, location or scale. In a very common problem, we have two random variables X_1, X_2, which probability distributions F_1, F_2 have different scale and location characteristics. We are interested whether the shape of their probability distributions is similar otherwise. A very simple procedure, allowing us to carry the comparison, is called normalization of random variable. It is a transformation shifting the expectations of the probability distributions to the same point of space, often origin of Cartesian System, and changing the scales to a certain unit. The transformed variable is in some sence "dimensionless", or expressed in dimensionless units. The most natural normalization is $X^* = (X - m)/\sigma$, where m is the first moment and σ^2 variance of probability distribution F of random variable X.

The pressumption on existence of finite first and second moments of considered probability distribution can be in some applications rather restricting. Depending on the nature of the modelled problem, it can be sometimes more convenient to consider transformation $(X - b)/a$, where $b \in \mathbb{R}$ corresponds to change of location, whereas $a > 0$ to units of measurement. The change of location can be perform by considering artificial centerings, such as truncated moments or median. The new distribution of random variable X^* in dimensionless units is related to original distribution $F(x)$ as $F(ax + b)$.

In this report we consider basic problems and theory connected with the investigation of collections of "dimensionless" random variables, which leads us to stable distributions. In the following section we start with definitions, preliminary consideration and more detailed outline of the report.

2 Definitions. Preliminaries

Consider two probability distributions F_1, F_2 on the real line with densities f_1, f_2. We will say that these two distributions F_1, F_2 are of the same type if $F_2(x) = F_1(ax + b), a > 0, b \in \mathbb{R}$, where a is a scaling factor and b a centering constant. For their densities holds $f_2(x) = af_1(ax + b)$. We understand that two distributions F_1, F_2 differ only by scale and location, it will be also denoted as $F_2 \stackrel{d}{=} aF_1 + b$.

Consider mutually independent random variables $X, X_1, X_2, \ldots$ with common distribution F and denote sum of n elements as $S_n = X_1 + \ldots + X_n$.

Definition 2.1. The distribution F is stable (in the broad sense) if for each n

there exists constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$S_n \stackrel{d}{=} a_n X + b_n \tag{2.1}$$

and F is not concentrated at one point. If $b_n = 0$ for all n, we call F strictly stable distribution.

Returning to concept of "dimensionless units", let us dwell on norming constants. The scaling constants for stable distributions are of the form $a_n = n^{1/\alpha}$, see Feller [1] VI.1.1., p.171. We will refer to α as the stability parameter of F. If F is stable in broad sense with stability parameter α, such that $\alpha \neq 1$, the centering constant b can be chosen such that $F(x+b)$ is strictly stable, see Feller [1] VI.1.2., p.172.

Consider two independent random variables X_1, X_2 with common distribution F_1, F_2. The distribution of their sum $X_1 + X_2$ is obtained as a convolution of distributions F_1 and F_2. We denote it as $F_1 * F_2$ and

$$P\{X_1 + X_2 \leq x\} = F_1 * F_2(x) = \int_{-\infty}^{\infty} F_1(x-y) F_2\{\mathrm{d}y\}. \tag{2.2}$$

If F_1, F_2 have densities, then the convolution of their densities is:

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-y) f_2(y) \mathrm{d}y.$$

The convolution operation is commutative and associative among distributions. The continuity property and differentiability is preserved by convolution, i.e. if F_2 is continuous, the distribution resulting from $F_1 * F_2$ remains continuous. If F_2 has density f_2 then $f_1 * f_2$ is a density of distribution $F_1 * F_2$. We refer to Feller [1] V.4.2, V.4.3, V.4.4, p.144-6, for Proofs.

The distribution of sum S_n of mutually independent random variables with common distribution F is then F^{n*}. We will call F^{n*} the n-th fold convolution of F with itself. For stable distributions, from condition from (2.1), follows that $F^{n*}(x) = F(a_n x + b_n)$, i.e. the n-th fold convolution of F with itself differs from F only by scale and location. The condition on stability of distribution implies that the distribution of sum of independent identically distributed random variables is of the same type as distribution of each random variable in the sum. Stable distribution belongs to larger group of probability distributions called infinitely divisible.

Definition 2.2. A distribution F is infinitely divisible if for each n there exists a distribution F_n such that $F = F_n^{n*}$.

It will turn out that limit distributions of sums of independent random variables are infinitely divisible. In the following text we will when necessary develop the theory firstly for infinitely divisible distributions and then consider properties of stable distribution as a special case. The main focus of this work is however on stable distributions and thus the study of infinitely divisible distribution is not aimed to be systematic, but rather complementary.

In the section I.2 we will develop the approach to central limit theorems via semigroups of operators associated to probability distributions. It will allow us to derive a canonical measure for stable distributions and a partial integro-differential operator generating the semigroup. Consider probability distribution F and abbreviate its expectation as

$$\mathbf{E}(u) = \int_{-\infty}^{\infty} u(x) F\{\mathrm{d}x\}. \tag{2.3}$$

The convolution of probability distribution F and a bounded point function $u(t,x) = u(t-x)$ is then

$$\mathbf{E}[u(t)] = \int_{-\infty}^{\infty} u(t-x) F\{\mathrm{d}x\}. \tag{2.4}$$

With the probability distribution we associate the operator $\mathcal{F}$ from $C[-\infty,\infty]$ to $C[-\infty,\infty]$ such that integral in (2.4) is abbreviated as $\mathcal{F}u = F * u$. Having distributions F_1, F_2, we denote the operator associated with the convolution $F_1 * F_2$ as $\mathcal{F}_1\mathcal{F}_2$. We will understand it as a result of $\mathcal{F}_1$ operating on $\mathcal{F}_2 u$. The n-th fold convolution of F with itself, denoted as F^{n*}, has associated operator $\mathcal{F}^n$. The operators associated to probability distributions are positive, i.e. $u \geq 0$ implies $\mathcal{F}u \geq 0$. Further they inhere the property of convolution and thus they commute and are associative. To see this, it is very simple exercise, which uses Fubini Theorem:

$$\mathcal{F}_1\mathcal{F}_2(x) = \int_{-\infty}^{\infty} \mathcal{F}_2 u(x-y) F_1\{\mathrm{d}y\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x-y-z) F_2\{\mathrm{d}z\} F_1\{\mathrm{d}y\} =$$
$$= \int_{-\infty}^{\infty} \mathcal{F}_1 u(x-z) F_2\{\mathrm{d}z\} = \mathcal{F}_2\mathcal{F}_1(x),$$

which proves the commutativity. The associativity follows from Fubini Theorem and linearity of the integral. The operators have norm 1 and so

$$||\mathcal{F}u|| \leq ||u||. \tag{2.5}$$

In section I.3 we will show its importance in convergence of probability measures problems. The alternative approach is based on the Fourier transform of probability measure. As the measures have unit mass, the term 'characteristic function' is used.

Definition 2.3. For probability distribution F with a density f, the characteristic function of F is defined as:

$$\hat{F}(k) = \int_{-\infty}^{\infty} e^{ikx} F\{\mathrm{d}x\} = u(k) + iv(k) \tag{2.6}$$

where

$$u(k) = \int_{-\infty}^{\infty} \cos(kx) F\{\mathrm{d}x\}, \qquad v(k) = \int_{-\infty}^{\infty} \sin(kx) F\{\mathrm{d}x\}. \tag{2.7}$$

The ordinary Fourier transform of density f is defined as:

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \mathrm{d}x. \tag{2.8}$$

Of course, if $\hat{f}$ is integrable over $\mathbb{R}$, the Fourier inversion of density f exists, Feller [1] XV.3.3, p. 509:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) \mathrm{d}k. \tag{2.9}$$

If probability distribution F has finite second moments m_2 and let us denote m_1 as its first moment, then

$$\hat{F}'(0) = im_1, \qquad \hat{F}''(0) = -m_2. \tag{2.10}$$

Also if $\hat{F}''(0)$ exists, then second moment m_2 of probability distribution F is finite, see Feller [1], XV.4, p. 512. The characteristic function of probability distribution of sum of two independent random variables X_1, X_2 with common distributions F_1, F_2 is the product of their characteristic functions $\hat{F}_1 \hat{F}_2$. The probability distribution of sum S_n then has characteristic function $[\hat{F}(k)]^n$ and $F(ax+b)$ is transformed into $e^{ib}\hat{F}(ak)$. For strictly stable distributions we then have relation:

$$[\hat{F}(k)]^n = \hat{F}(a_n k) = \hat{F}(n^{1/\alpha} k). \tag{2.11}$$

For infinitely divisible distributions we define infinitely divisible characteristic function, which can be used as an alternative definition for infinitely divisible distributions:

Definition 2.4. A characteristic function $\hat{F}$ is infinitely divisible iff for each n there exists a characteristic function $\hat{F}_n$ such that

$$\hat{F}_n^n = \hat{F}. \tag{2.12}$$

We will use this transformation in section I.4 for deriving the general form of characteristic function of stable distribution. In section I.5 we will introduce different parametrizations of stable characteristic functions introduced by Zolotarev [3]. These play significant role in deriving forms of stable densities. In the text will be also useful other integral transformations, namely Mellin transform, Laplace transform, which we briefly introduce in section I.6. We finish this section by recalling the probability distributions which plays central role in the theory of infinitely divisible distributions and derivation of the central limit theorems. These will be find useful in further sections. Consider again mutually independent random variables $X_1, X_2, \ldots$ with common distribution F. Consider random variable N with common distribution G independent of each random variable X_n. We will call random variable $X = \sum_{k=1}^{N} X_k$ a random sum, the distribution of random sum will be called a compound distribution.

From law of total probability and due to independence of N with each X_n, we have for compound distribution:

$$P\{X=x\} = \sum_{n=0}^{\infty} P\{N=n\} P\Big\{\sum_{k=1}^{n} X_k = x\Big\}. \tag{2.13}$$

If random variable N has Poisson distribution with parameter $a > 0$, i.e. the probability, that there is n elements in random sum is:

$$P\{N=n\} = e^{-a}\frac{a^n}{n!}, \tag{2.14}$$

independent of each random variable X_n, then we will call the distribution of the random sum $X = \sum_{k=1}^{N} X_k$ compound Poisson distribution. Using (2.13) we have for random variable X with compound Poisson distribution:

$$P\{X \le x\} = e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}F^{n*}(x). \tag{2.15}$$

Obviously, if F is concentrated at point 1, i.e. Dirac measure assigning value 1 to point 1 and 0 otherwise, we have Poisson distribution with parameter a. As a simple exercise let us compute characteristic function of compound Poisson distribution H:

$$\hat{H}(k) = e^{-a}\int_{-\infty}^{\infty} e^{ikx}\sum_{n=0}^{\infty}\frac{a^n}{n!}F^{n*}\{\mathrm{d}x\} = e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}\int_{-\infty}^{\infty} e^{ikx}F^{n*}\{\mathrm{d}x\},$$

where we used linearity of integral and convergence of the sum. Using relation that $\widehat{F^{n*}} = \hat{F}^n$ we obtain characteristic function $\hat{H}$ of compound Poisson distribution:

$$\hat{H}(k) = e^{-a}\sum_{n=0}^{\infty}\frac{[a\hat{F}(k)]^n}{n!} = \exp\{a[\hat{F}(k)-1]\}. \tag{2.16}$$

3 Semigroups

Consider space of all continuous functions vanishing at infinity and define a family $\{u_t\}$ by $u(t,x) = u(t-x)$. Consider probability distributions $\{Q_t, t>0\}$ satisfying a convolution equation

$$Q_{t+s} = Q_t * Q_s.$$

The convolution operator associated to distribution Q_t is

$$\mathfrak{Q}(t)u(x) = \int_{-\infty}^{\infty} u(x-y)Q_t\{\mathrm{d}y\}, \tag{3.1}$$

and operators $\{\mathfrak{Q}(t)\}, t > 0$ form convolution semigroup, i.e. $\mathfrak{Q}(t+s) = \mathfrak{Q}(t)\mathfrak{Q}(s)$.

Denote $\mathbf{1}$ as identity operator. We will say that the convolution semigroup $\{\mathfrak{Q}(t)\}$ is continuous if $\mathfrak{Q}(h) \to \mathbf{1}$ for $h \to 0^+$, we define $\mathfrak{Q}(0) = \mathbf{1}$.

Let us dwell longer on the semigroup continuity. Because operators are transition, $||\mathfrak{Q}(t)u|| \leq ||u||$. Natural question to ask is what are the properties of the semigroup $\{\mathfrak{Q}(t)\}$ with respect to time. For $h > 0$ we get:

$$||\mathfrak{Q}(t+h)u - \mathfrak{Q}(t)u|| = ||\mathfrak{Q}(t)[\mathfrak{Q}(h) - \mathbf{1}]u|| \leq ||(\mathfrak{Q}(h) - \mathbf{1})u||$$

By making h small, from continuity of the semigroup the left-hand side can be estimated by ε, independently of t. The continuous semigroup is then uniformly continuous. Considered inequality also implies that we learn the most about behaving of semigroup from its behaviour on the neighbourhood of 0. The following definition brings some enlightment.

Definition 3.1. An operator $\mathfrak{A}$ from C^∞ to C^∞ is the generator of the convolution semigroup $\mathfrak{Q}(t)$ if

$$\frac{1}{h}[\mathfrak{Q}(h) - \mathbf{1}] \to \mathfrak{A} \text{ for } h \to 0^+. \tag{3.2}$$

Being given the probability distribution, the form of generator is obviously a question.

Let us define a measure Ω_t by its density as:

$$\Omega_t\{\mathrm{d}x\} = t^{-1}x^2 Q_t\{\mathrm{d}x\}. \tag{3.3}$$

In case Q_t has finite second moments, Ω_t is finite on the whole real line. It corresponds to a probability distribution induced by its second moments. The assumption of finiteness of second moments is however unnecessary. In the sequel we do not assume that Ω_t is finite on $\mathbb{R}$. We will assume that $\Omega_t\{I\} < \infty$ for every finite interval I. The following definition provides a clear summary.

Definition 3.2. A measure Ω on the real line is called canonical if $\Omega\{I\}$ is finite for every finite interval I and if integrals

$$M^+(x) = \int_x^\infty y^{-2}\Omega\{\mathrm{d}y\} \qquad\qquad M^-(-x) = \int_{-\infty}^{-x} y^{-2}\Omega\{\mathrm{d}y\}$$

converges for every $x > 0$.

For the reason of having the distributions in semigroup connected to one location, we need to center them. As the introduction of canonical measure allows us to relax assumption on finiteness of the moments, we further introduce the artificial centering:

Definition 3.3. We call a real function τ_s a truncation function, if τ_s is continuous, monotone, and for $s > 0$ arbitrary but fixed assigns values $\tau_s(x) = x$ for $|x| \leq s$ and $\pm s$ otherwise.

This allows us to define truncated moments of probability distribution Q_t as:

$$\mathbf{E}[\tau_s] = \int_{-\infty}^{\infty} \tau_s(x) Q_t\{\mathrm{d}x\} = \int_{-s}^{s} x Q_t\{\mathrm{d}x\}. \tag{3.4}$$

Thus we do not require x to be integrable with respect to each Q_t over whole real line, but only on finite intervals, where the integrals will always exist. In fact, we can choose as truncation function any bounded continuous function, which is twice differentiable near the origin and $\tau(0) = \tau''(0) = 0, \tau'(0) = 1$. The assumption on truncation function being bounded guarantees integrability. The continuity is rather strong and can be replaced by continuity around the origin.

We can now carry on the investigation of the form of generator of semigroup $\{\mathfrak{Q}(t)\}$:

$$\frac{\mathfrak{Q}(t) - \mathbf{1}}{t} u(x) = \frac{1}{t} \int_{-\infty}^{\infty} [u(x-y) - u(x)] Q_t\{\mathrm{d}y\} = \int_{-\infty}^{\infty} \frac{u(x-y) - u(x)}{y^2} \Omega_t\{\mathrm{d}y\},$$

where relation (3.3) ensures that y^{-2} is integrable with respect to Ω_t over any domain excluding origin. We further incorporate the centering:

$$b_t = \frac{1}{t} \int_{-\infty}^{\infty} \tau_s(x) Q_t\{\mathrm{d}x\} = \int_{-\infty}^{\infty} \tau_s(x) x^{-2} \Omega_t\{\mathrm{d}x\}. \tag{3.5}$$

The assumption on truncation function $\tau(0) = 0$ is essential for existence of finite value b_t, because y^{-2} is not necessarily integrable with respect to Ω_t. Putting it all together we arrive to:

$$\frac{\mathfrak{Q}(t) - \mathbf{1}}{t} u(x) = \int_{-\infty}^{\infty} \frac{u(x-y) - u(x) - \tau_s(x) u'(x)}{y^2} \Omega_t\{\mathrm{d}y\} + b_t u'(x). \tag{3.6}$$

The choice of truncation function influences only the location shifts of the semigroup and so we denote:

$$\mathfrak{A}^\tau u(x) = \int_{-\infty}^{\infty} \frac{u(x-y) - u(x) - \tau_s(y) u'(x)}{y^2} \Omega\{\mathrm{d}y\},$$

where we replaced Ω_t by any canonical measure Ω. The integral makes sense for any canonical measure.

Because we replaced the probability distribution Q_t by canonical measure Ω_t, it is obviously essential to ask if Ω_t converges to some Ω for $t \to 0^+$, what are assumptions on convergence of canonical measures and what is the relation of the limit measure Ω to generator of the semigroup. The following theorem from Feller [1], Theorem IX.5.1, p.302, gives an answer:

Theorem 3.1. *A continuous convolution semigroup* $\{\mathfrak{Q}(t)\}$ *has a generator* $\mathfrak{A}$ *of the form*

$$\mathfrak{A} u = \mathfrak{A}^\tau u + b u'$$

where

$$\mathfrak{A}^{\tau}u(x) = \int_{-\infty}^{\infty} \frac{u(x-y) - u(x) - \tau_s(y)u'(x)}{y^2}\Omega\{\mathrm{d}y\}$$

and where Ω *is a canonical measure. Conversely, every operator* $\mathfrak{A}$ *of this form generates a continuous convolution semigroup* $\{\mathfrak{Q}(t)\}$*. The measure* Ω *is unique, and*

$$\Omega_t\{I\} \to \Omega\{I\} \text{ for } t \to 0^+$$

for infinite intervals I *and*

$$\frac{1 - Q_t(x)}{t} \longrightarrow M^+(x) \qquad \frac{Q_t(-x)}{t} \longrightarrow M^-(-x).$$

Let us now derive the generator of stable semigroup. Consider probability distribution Q_t and divide interval $[0, t]$ into n equidistant partitions, where n is integer. Then from semigroup property $Q_t = Q_{t/n}^{n*}$ Recalling the definition **??**, $Q_t(x) = F(a_t x + b_t)$. Let us summarize it in the definition of stable semigroup:

Definition 3.4. A semigroup $\{\mathfrak{Q}(t)\}$ is called stable if its distributions are of the form

$$Q_t(x) = F(a_t(x - b_t)) \tag{3.7}$$

where $b_t \in \mathbb{R}, a_t > 0$ are continuous functions of t, such that as $t \to 0^+ :\ a_t \to \infty$ and $b_t \to 0$. Function F is a fixed probability distribution which is not concentrated at one point.

The assumption on continuity of functions a_t, b_t is important for continuity of stable semigroup. In what follows our aim is to determine the form of canonical measure Ω. Using the results of previous theorem we obtain for $x > 0$:

$$\frac{1 - F(a_t(x - b_t))}{t} \longrightarrow M^+(x), \qquad \frac{F(a_t(-x - b_t))}{t} \longrightarrow M^-(-x). \tag{3.8}$$

We observe that for t sufficiently small $F(a_t(x - b_t))$ behaves like $F(a_t x)$. It is because $b_t \to 0$ for $t \to 0$ and F is monotone. Further for t approaching origin $a_t \to \infty$ and so $F(a_t) \to 1$, thus $1 - F(a_t) \approx t$ for ct sufficiently small. We can now rewrite (3.8) as:

$$\frac{1 - F(a_t x)}{1 - F(a_t)} \longrightarrow c_1 M^+(x), \qquad \frac{F(-a_t x)}{1 - F(a_t)} \longrightarrow c_2 M^-(-x), \tag{3.9}$$

where $x > 0$. Consider now $x_1, x_2 > 0$ such that $x = x_1 x_2$. Then:

$$\frac{1 - F(a_t x_1 x_2)}{1 - F(a_t)} = \frac{1 - F(a_t x_1 x_2)}{1 - F(a_t x_1)} \frac{1 - F(a_t x_1)}{1 - F(a_t)}. \tag{3.10}$$

From the definition of canonical measure relation (3.10) converges to finite positive limit $M^+(x) = M^+(x_1x_2)$. Because also:

$$\frac{1 - F(a_t x_1 x_2)}{1 - F(a_t x_1)} \longrightarrow M^+(x_2), \qquad \frac{F(a_t x_1)}{1 - F(a_t)} \longrightarrow M^+(x_1), \tag{3.11}$$

we obtain functional equation $M^+(x_1x_2) = M^+(x_1)M^+(x_2)$. Transform $x = e^z$ and define $m(z) := M^+(e^z)$. We see that limit satisfies the logarithmic version of Cauchy functional equation:

$$m(z_1 + z_2) = m(z_1)m(z_2). \tag{3.12}$$

Let us solve the equation. We look for the solutions that are bounded in finite intervals. Obviously from (3.12) follows $m(z) = m(z/2)m(z/2)$ and thus $m(z) \geq 0$ for all z. Assume first that $m(z) = 0$ for some z, then $m(z/2) = 0$ and by induction $m(z/2^n) = 0$. Obviously also $m(nz) = m^n(z)$, thus if $m(z_1) = 0$, then $m(z) = 0$ for any $z > z_1$ and so $m(z) = 0$ for any $z > 0$. We obtained one solution of equation (3.12), that is identical 0 for all $z > 0$, i.e. the limit in (3.9) vanishes identically. Let us consider only strictly positive solutions. Solutions of type $e^{\alpha z}$ for $\alpha \in \mathbb{R}$ are good candidates. Consider $m(1) = e^{-\alpha}$ and define $g(z) = e^{\alpha z}m(z)$. Because $g(z_1 + z_2) = g(z_1)g(z_2)$ for any z_1, z_2 and $g(1) = 1$, we would like to prove that $g(z) = 1$ for all $z > 0$. For any positive integers m, n we have: $g(m/n) = g^{m/n}(1) = 1$. Thus $g(y) = 1$ for any y, where y is rational number. By considering $s = z + y$, where z is arbitrary real and y rational number. Then $g(y) = g(y - s)g(s) = g(y - s)$ for arbitrary y and thus $g(z) = 1$ for any real z. We have thus showed that the second solution is of the form $m^+(z) = e^{-\alpha_1 z}$ for all z and some constant α_1. Substituting back we obtain $M^+(x) = x^{-\alpha_1}$.

Repeating the same procedure for left tail in (3.9), we conclude that either limits M^+, M^- vanish identically or these are of the following form:

$$M^+(x) = c_1 x^{-\alpha_1}, \qquad M^-(-x) = c_2 x^{-\alpha_2}, \tag{3.13}$$

where $x > 0$ and $c_1, c_2 > 0$. To show that $\alpha_1 = \alpha_2$ consider sum of tails and use relation (3.9):

$$\frac{1 - F(a_t x) + F(-a_t x)}{1 - F(a_t)} \longrightarrow M^+(x) + M^-(-x).$$

Because the same argumentation as in (3.10) holds even for sum of tails, we again obtain for sum $M^+(x) + M^-(-x)$ the solution of logarithmic version of Cauchy functional equation in the form: $x^{-\alpha}$. Comparing this result with (3.13), we conclude that $\alpha = \alpha_1 = \alpha_2$. Finally, we determine canonical measure Ω. Because for $x > 0$:

$$M^+(x) = \int_x^\infty y^{-2}\Omega\{\mathrm{d}y\} = c_1 x^{-\alpha}, \qquad M^-(-x) = \int_{-\infty}^{-x} y^{-2}\Omega\{\mathrm{d}y\} = c_2 x^{-\alpha},$$

a density of canonical measure Ω is $\Omega\{\mathrm{d}x\} = \alpha c_1 x^{1-\alpha}$ for $x > 0$ and $\Omega\{\mathrm{d}x\} = \alpha c_2 |x|^{1-\alpha}$ for $x < 0$. Because integrals $M^+(x), M^-(-x)$ are supposed to be finite for every $x > 0$, we conclude that $\alpha > 0$. Further for $x > 0$, we have

$$\Omega\{(-x,x)\} = \frac{\alpha}{2-\alpha}(c_1 + c_2)x^{2-\alpha} = Cx^{2-\alpha},$$

for some $C > 0$. To fulfil condition that canonical measure Ω assigns to every finite interval I finite value, we obtain the second condition $\alpha < 2$. We conclude that $0 < \alpha < 2$.

Limits $M^+(x), M^-(-x)$ determine the measure Ω up to possible atom at origin. Such an atom exists only if both $M^+(x), M^-(-x)$ vanish identically and Ω is concentrated at the origin. In that case, using Taylor expansion for $u \in \mathcal{C}_b^3(\mathbb{R})$:

$$u(x-y) = u(x) + yu'(x) + \frac{1}{2}y^2u''(x) + \frac{1}{6}y^3u'''(x - \theta y),$$

and so the generator $\mathfrak{A}$ from Theorem 3.1 takes the form

$$\mathfrak{A}u(x) = c\lim_{y\to 0}\frac{u(x-y) - u(x) - yu'(x)}{y^2} = \frac{1}{2}cu''(x). \tag{3.14}$$

This is however generator of the semigroup of normal distributions with variance ct. We conclude that for $\alpha = 2$, and limits $M^+(x), M^-(-x)$ vanishing identically, we obtained normal distribution.

Definition 3.5. The stable canonical measure is defined as:

$$\Omega\{(-x,x)\} = \begin{cases} Cp_1x^{2-\alpha} & \text{for } x > 0, \\ Cp_2|x|^{2-\alpha} & \text{for } x < 0. \end{cases}$$

Using Theorem 3.1, we can now formulate the operator generating the stable semigroup $\{\mathfrak{Q}(t)\}$ as:

$$\mathfrak{A} = c_1\mathfrak{A}_1 + c_2\mathfrak{A}_2 + b\frac{\mathrm{d}}{\mathrm{d}x}, \tag{3.15}$$

where operator $\mathfrak{A}_1$ describes the contribution on positive axis, whereas $\mathfrak{A}_2$ on negative.

For $0 < \alpha < 1$ we have:

$$\mathfrak{A}_1u(x) = \int_0^\infty \frac{u(x-y) - u(x)}{y^{\alpha+1}}\mathrm{d}y, \tag{3.16}$$

$$\mathfrak{A}_2u(x) = \int_{-\infty}^0 \frac{u(x-y) - u(x)}{|y|^{\alpha+1}}\mathrm{d}y. \tag{3.17}$$

For $1 < \alpha < 2$, the first moments exists and so we use the natural centering to zero expectation:

$$\mathfrak{A}_1u(x) = \int_0^\infty \frac{u(x-y) - u(x) - yu'(x)}{y^{\alpha+1}}\mathrm{d}y, \tag{3.18}$$

$$\mathfrak{A}_2u(x) = \int_{-\infty}^0 \frac{u(x-y) - u(x) - yu'(x)}{|y|^{\alpha+1}}\mathrm{d}y. \tag{3.19}$$

Finally for $\alpha = 1$, the first moments do not exists and thus we use truncation function τ_s as artificial centering:

$$\mathfrak{A}_1 u(x) = \int_0^\infty \frac{u(x-y) - u(x) - \tau_s(y)u'(x)}{y^2} \mathrm{d}y, \tag{3.20}$$

$$\mathfrak{A}_2 u(x) = \int_{-\infty}^0 \frac{u(x-y) - u(x) - \tau_s(y)u'(x)}{|y|^2} \mathrm{d}y. \tag{3.21}$$

The following corollary is simple consequence of previous derivation, yet its benefit is tremendous. It explains the role of constants c_1, c_2 in the generator of the stable semigroup. In fact, as we will see later, constants c_1, c_2 significantly influence the shape and scale of stable distribution.

Corollary 3.1. *For $0 < \alpha < 2$ and $c_1 \geq 0, c_2 \geq 0, c_1 + c_2 > 0$ there exists exactly one stable distribution function F such that as $x \to \infty$:*

$$x^\alpha [1 - F(x)] \to c_1, \qquad x^\alpha F(-x) \to c_2. \tag{3.22}$$

For $\alpha = 2$, $c_1 = c_2 = 0$ and stable distribution function F corresponds to normal distribution.

4 Characteristic Function

In virtue of previous section, consider a sequence of probability distributions $\{F_n\}$. We ask now what is the relation between a proper convergence of probability distributions and a convergence of their characteristic functions. For the sequence $\{F_n\}$ of probability distributions to converge properly to a probability distribution F, it is necessary and sufficient that the sequence of their characteristic functions $\{\hat{F}_n\}$ converges pointwise to a limit $\hat{F}$ and $\hat{F}$ is continuous at the origin, see Continuity theorem in Feller [1] XV.3.2, p.508. When the limit $\{\hat{F}\}$ of sequence of characteristic functions is continuous everywhere on the real line and satisfies condition (2.12), the limit distribution is infinitely divisible. The next theorem, Feller [1] XVII.1.1, p.555, gives condition for continuity of the limit.

Theorem 4.1. *Let $\{\hat{F}_n\}$ be a sequence of characteristic functions. For existence of a continuous limit*

$$\hat{F}(k) = \lim_{n\to\infty} \hat{F}_n^n(k) \tag{4.1}$$

it is necessary and sufficient that

$$n[\hat{F}_n(k) - 1] \longrightarrow \psi(k) \tag{4.2}$$

where ψ is a continuous function. Then

$$\hat{F}(k) = e^{\psi(k)} \tag{4.3}$$

We will call $\psi(k)$ a characteristic exponent of infinitely divisible distribution. The theorem is crucial for the theory of infinitely divisible distributions with useful consequences. One of them is that the class of infinitely divisible distributions coincides with the class of limit distributions of compound Poisson distributions, see Feller [1] XVII.1.3, p.557. We will show later how we can approximate stable distributions by a sequence of compound Poisson distributions. It remains to determine the form of characteristic exponent. We firstly show the derivation of canonical form of characteristic exponent for infinitely divisible distributions to see the connection to canonical measures introduced in Definition 3.2. As we already determined the canonical measure for stable distributions, we will use it for deriving the canonical form of the characteristic exponent of the stable characteristic function.

Consider sequence of compound Poisson distributions F_n with characteristic functions $\exp\{a_n(\hat{G}_n - 1)\}$, where $a_n > 0$. Denote m_n the first moment of probability distribution G_n and note that m_n does not need to be finite. It is easy exercise to see, that compound Poisson distribution has a first moment equal to $a_n m_n$. As we are investigating the convergence of random variables and it is useful to investigate them in dimensionless units, we add the artificial centering real valued constants b_n to characteristic functions in the considered sequence of compound Poisson processes, so the characteristic exponents of this sequence are of the following form:

$$\psi_n(k) = a_n[\hat{G}_n(k) - 1 - ikb_n]. \tag{4.4}$$

Let us dwell on the choice of centering constant. Whenever the first moments of probability distributions $\{G_n\}$ are finite, it is obviously reasonable choice to center to zero. In general case, Feller in [1], discussion on p.558, Lemma XVII.2.1, p.559, argues that reasonable choice of b_n is such that $\mathcal{I}\hat{F}_n(1) = 0$, i.e. ψ_n is a real function at point 1 and so from Definition 2.3:

$$b_n = \int_{-\infty}^{\infty} \sin x G_n\{\mathrm{d}x\}. \tag{4.5}$$

The argumentation proceeds as follows: consider sequence $\psi_n(k)$ from (5.4), with b_n chosen such that $\psi_n \to \psi$ and ψ is continuous. Consider real valued functions u_n, v_n, u, v, such that $\psi_n(k) = u_n(k) + iv_n(k)$, $\psi(k) = u(k) + iv(k)$, respectively. Because b_n is chosen such that $u_n(k) + iv_n(k) \to u(k) + iv(k)$, then $u_n(k) = a_n[\int_{-\infty}^{\infty} \cos(kx) G_n\{\mathrm{d}x\} - 1]$ converges to $u(k)$ and $v_n(k) = a_n[\int_{-\infty}^{\infty} \sin(kx) G_n\{\mathrm{d}x\} - b_n k]$ converges to $v(k)$. For choice $k = 1$, we have for imaginary part of $\psi_n(1)$ $a_n[\int_{-\infty}^{\infty} \sin x G_n\{\mathrm{d}x\} - b_n] \to v(1)$. We can subtract $ikv_n(1)$ from $\psi_n(k)$ and obtain $a_n[\hat{G}_n(k) - 1 - ik\int_{-\infty}^{\infty} \sin x G_n\{\mathrm{d}x\}]$, which converges to $\psi(k) - ikv(1)$. Thus we see, that if we choose b_n as in (5.5), our choice of centering function influences only the location of the limiting distribution. As we integrate bounded functions in (5.5), the integral always exists and the centering is always possible. Rewriting (5.4) as

$$\psi_n(k) = a_n \int_{-\infty}^{\infty} [e^{ikx} - 1 - ik\sin x] G_n\{\mathrm{d}x\}, \tag{4.6}$$

by using L'Hospital Theorem we see that integrand in $\psi_n(k)$ behaves like $-\frac{1}{2}k^2x^2$ at the neighbourhood of origin.

In virtue of Theorem 5.1, we seek for conditions under which sequence of characteristic exponents ψ_n converges to continuous limit ψ. Let us define measure:

$$\Omega_n\{\mathrm{d}x\} = a_n x^2 G_n\{\mathrm{d}x\}. \tag{4.7}$$

If F_n have finite second moments, Ω_n defines a finite measure with a total mass $\mu_n = a_n \int_{-\infty}^{\infty} x^2 G_n\{\mathrm{d}x\}$ on the whole real line. Making use of (2.10), we also have $\mu_n = -\psi_n''(0)$. By assuming that $\psi_n \to \psi$ and also $\psi_n'' \to \psi''$, we have $\mu_n \to -\psi''(0)$ and so

$$\frac{a_n}{\mu_n}\int_{-\infty}^{\infty} e^{ikx} x^2 G_n\{\mathrm{d}x\} \longrightarrow \frac{\psi''(k)}{\psi''(0)}.$$

Thus $-\psi''(k) = \int_{-\infty}^{\infty} e^{ikx}\Omega\{\mathrm{d}x\}$. It can be easily verified that

$$\psi(k) = \int_{-\infty}^{\infty} \left[\frac{e^{ikx}}{x^2} + c_1 k + c_2\right]\Omega\{\mathrm{d}x\},$$

From condition $\psi(0) = 0$, we determine $c_2 = -x^{-2}$. Second condition $\psi(1) = \int_{-\infty}^{\infty} \cos x G\{\mathrm{d}x\}$ implies $c_1 = -i\frac{\sin x}{x^2}$. Contrary to first condition, which is basic property of characteristic function and needs to be always satisfied, the second condition can be chosen differently. As we see, the choice of the second condition determines the choice of centering.

The good candidate for canonical form of a characteristic exponent is:

$$\psi(k) = \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2}\Omega\{\mathrm{d}x\}. \tag{4.8}$$

The measures Ω does not necessarily need to be finite and can be replaced by canonical measure defined in Definition 3.2. For any canonical measure Ω and function ψ defined in (5.8), $\exp\{\psi\}$ is a unique representation of characteristic function of infinitely divisible distribution, see Feller [1], Lemma XVII.2.2, XVII.2.3, p.560-561 for Proofs.

Consider sequence of canonical measures Ω_n, we then describe the characteristic exponents of $\{\hat{F}_n\}$ as

$$\psi_n(k) = \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2}\Omega_n\{\mathrm{d}x\} + ib_n k. \tag{4.9}$$

The following theorem summarizes our investigation:

Theorem 4.2. *Let Ω_n is a canonical measure and ψ_n is defined in (5.9). In order that ψ_n tends to continuous limit ψ it is necessary and sufficient that there exist a canonical measure Ω, such that $\Omega_n \to \Omega$ properly and $b_n \to b$. Then ψ is given by:*

$$\psi(k) = \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2}\Omega\{\mathrm{d}x\} + ibk. \tag{4.10}$$

Let us now focus on the characteristic exponent of stable distributions. In section I.2 we derived the form of canonical measure for stable distributions. Let us now use this result: canonical measure connected with stable distribution is for $x > 0$: $\Omega\{(0, x)\} = Cp_1 x^{2-\alpha}$ and $\Omega\{(-x, 0)\} = Cp_2 x^{2-\alpha}$, where $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$.

As $\Omega\{\mathrm{d}x\} = C(2-\alpha)p_1 x^{1-\alpha}\mathrm{d}x$ for $x > 0$, then

$$\psi(k) = C(2-\alpha)p_1 \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2} x^{1-\alpha} 1_{\{x>0\}}\mathrm{d}x. \tag{4.11}$$

For $x < 0$ we have $\Omega\{\mathrm{d}x\} = C(2-\alpha)p_2|x|^{1-\alpha}\mathrm{d}x$ and so the characteristic exponents rewrites as:

$$\psi(k) = C(2-\alpha)p_2 \int_{-\infty}^{\infty} \frac{e^{ikx} - 1 - ik\sin x}{x^2} |x|^{1-\alpha} 1_{\{x<0\}}\mathrm{d}x. \tag{4.12}$$

Consider case when $0 < \alpha < 1$. The integral

$$\int_0^{\infty} \frac{e^{ikx} - 1}{x^{\alpha+1}}\mathrm{d}x \tag{4.13}$$

converges for all x, thus we omit centering. To see that integral converges, use Taylor expansion in 0 and use boundedness of e^{ikx} and vanishing of $x^{-\alpha-1}$ at infinity. To derive form of characteristic exponent we assume first that $k > 0$. Because e^{ikx} is univalent in any strip $(c, c + 2\pi n)$ for some real c and integer n and there exists analytic continuation of integrand on strip $(0, i\varepsilon)$ for $\varepsilon > 0$, we compute $\psi(k)$ as $\lim_{\varepsilon\to 0^+} \psi(k + i\varepsilon)$:

$$\begin{aligned}
\psi(k) &= \lim_{\varepsilon\to 0^+} C(2-\alpha)p_1 \int_0^{\infty} \frac{e^{-(\varepsilon-ik)x} - 1}{x^{\alpha+1}}\mathrm{d}x = \\
&= -C(2-\alpha)p_1 \lim_{\varepsilon\to 0^+} \frac{\varepsilon - ik}{\alpha} \int_0^{\infty} e^{-(\varepsilon-ik)x} x^{-\alpha}\mathrm{d}x = \\
&= -C(2-\alpha)p_1 \frac{\Gamma(1-\alpha)}{\alpha} \lim_{\varepsilon\to 0^+} (\varepsilon - ik)^{\alpha},
\end{aligned}$$

where we used integration by parts and substitution. Let us compute the limit. Because $(\varepsilon - ik)^{\alpha} = (\varepsilon^2 + k^2)^{\alpha/2} e^{i\theta\varepsilon}$ with $\tan\theta = -\frac{\varepsilon}{k}$ and for $\varepsilon \to 0^+$ argument θ tends to $-\pi/2$, we have:

$$\lim_{\varepsilon\to 0^+} (\varepsilon - ik)^{\alpha} = \lim_{\varepsilon\to 0^+} (\varepsilon^2 + k^2)^{\alpha/2} e^{i\theta\varepsilon} = k^{\alpha} e^{-i\alpha\pi/2}. \tag{4.14}$$

Then for $k > 0$ we obtain:

$$\psi(k) = -Cp_1 \frac{2-\alpha}{\alpha}\Gamma(1-\alpha) k^{\alpha} e^{-i\alpha\pi/2} = Cp_1 \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} k^{\alpha} e^{-i\alpha\pi/2}.$$

To threat the case when $k < 0$ we consider $\psi(-k)$ and keep $k > 0$. Because $\psi(-k)$ lead us to integral:

$$\int_0^{\infty} \frac{e^{-ikx} - 1}{x^{\alpha+1}}\mathrm{d}x,$$

which is complex conjugate to (5.13) and so we conclude that $\psi(k) = \overline{\psi(-k)}$. Thus for $x > 0$ we obtained:

$$\psi(k) = Cp_1 \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} |k|^\alpha e^{-i\,\mathrm{sgn}(k)\alpha\pi/2}. \tag{4.15}$$

For $x < 0$ we have convergent integral:

$$\int_0^\infty \frac{e^{-ikx} - 1}{x^{\alpha+1}} \mathrm{d}x$$

and extend integrand on strip $(-i\varepsilon, 0)$ for $\varepsilon > 0$. We proceed analogically as for $x > 0$ and obtain:

$$\psi(k) = Cp_2 \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} |k|^\alpha e^{i\,\mathrm{sgn}(k)\alpha\pi/2}. \tag{4.16}$$

Finally, we express the general form for characteristic exponent of stable distribution as:

$$\psi(k) = C \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} |k|^\alpha \Big[p_1 e^{-i\,\mathrm{sgn}(k)\alpha\pi/2} + p_2 e^{i\,\mathrm{sgn}(k)\alpha\pi/2} \Big]. \tag{4.17}$$

Using Euler formulas we can further reformulate later as:

$$\begin{aligned} \psi(k) &= C \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} |k|^\alpha \Big[\cos(\alpha\pi/2) - i\,\mathrm{sgn}(k)(p_1 - p_2)\sin(\alpha\pi/2) \Big] = \\ &= C \frac{\Gamma(3-\alpha)\cos(\pi\alpha/2)}{\alpha(\alpha-1)} |k|^\alpha \Big[1 - i\,\mathrm{sgn}(k)(p_1 - p_2)\tan(\pi\alpha/2) \Big]. \end{aligned} \tag{4.18}$$

Consider case when $1 < \alpha < 2$. Because first moments exist, we use centering to 0 and obtain:

$$\psi(k) = C(2-\alpha)p_1 \int_0^\infty \frac{e^{ikx} - 1 - ikx}{x^{\alpha+1}} \mathrm{d}x + C(2-\alpha)p_2 \int_{-\infty}^0 \frac{e^{ikx} - 1 - ikx}{|x|^{\alpha+1}} \mathrm{d}x.$$

Assume first that $k > 0$. The integral:

$$\int_0^\infty \frac{e^{ikx} - 1 - ikx}{x^{\alpha+1}} \mathrm{d}x$$

converges for all x. Contrary to previous case, in case we omit centering, the integral does not converge. We consider analytic contiuantion of integrand on the strip $(0, i\varepsilon)$ and compute limit $\psi(k) = \lim_{\varepsilon\to 0^+} \psi(k + i\varepsilon)$:

$$\begin{aligned} \psi(k) &= \lim_{\varepsilon\to 0^+} C(2-\alpha)p_1 \int_0^\infty \frac{e^{-(\varepsilon - ik)x} - 1 + (\varepsilon - ik)x}{x^{\alpha+1}} \mathrm{d}x = \\ &= -C(2-\alpha)p_1 \lim_{\varepsilon\to 0^+} \frac{\varepsilon - ik}{\alpha} \int_0^\infty \frac{e^{-(\varepsilon - ik)x} - 1}{x^\alpha} \mathrm{d}x = \\ &= C(2-\alpha)p_1 \lim_{\varepsilon\to 0^+} \frac{(\varepsilon - ik)^2}{\alpha(\alpha-1)} \int_0^\infty e^{-(\varepsilon - ik)x} x^{-\alpha+1} \mathrm{d}x = \\ &= C(2-\alpha)p_1 \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} \lim_{\varepsilon\to 0^+} (\varepsilon - ik)^\alpha, \end{aligned}$$

where we used twice integration by parts and substitution. Using (5.14) and fact that $\psi(-k)$ equals complex conjugate to $\psi(k)$ we arrive to same formula as in case $0 < \alpha < 1$:

$$\psi(k) = Cp_1 \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} |k|^\alpha e^{-i\,\mathrm{sgn}(k)\alpha\pi/2}. \tag{4.19}$$

Analogically, we compute situation for $x < 0$ and finally obtain:

$$\psi(k) = C\frac{\Gamma(3-\alpha)\cos(\pi\alpha/2)}{\alpha(\alpha-1)} |k|^\alpha \Big[1 - i\,\mathrm{sgn}(k)(p_1-p_2)\tan(\pi\alpha/2)\Big]. \tag{4.20}$$

5 Parametrizations of Characteristic Exponent

Zolotarev introduced and studied different parametrizations of the characteristic exponent of stable characteristic functions. Motivation for this effort stems from the fact that different parametrizations allows one to study different properties of stable distributions. In this section we will investigate these parametrizations and inspired by these we introduce the new one which will be systematically used in the rest of the text.

We will assume that distributions are strictly stable, because obviously the shift in location can be easily performed by adding ibk into characteristic exponent $\psi(k)$.

Let us start with discussion. We firstly show different way of argumentation on how to determine the characteristic exponent of stable distribution. Recall relation (2.11) and Theorem (5.1), then for the characteristic exponent of stable distribution holds following relation:

$$n\psi(k) = \psi(n^{1/\alpha}k). \tag{5.1}$$

As the relation above is a special example of Cauchy functional equation which we solved in section I.2, the good candidate for the solution is:

$$\psi(k) = -zk^\alpha, \tag{5.2}$$

where $z \in \mathbb{C}$ is a constant. To verify that $\hat{F}(k) = e^{-zk^\alpha}$ is a characteristic function, we need to check that $\hat{F}(0) = 1$, $|\hat{F}(k)| \le 1$ for all k and that $\hat{F}(k)$ is continuous everywhere. First and third property is obviously satisfied. The second condition holds only if $\mathcal{R}z > 0$. Denote the real part of z as c and represent z as $z = c(1+id)$, then

$$\psi(k) = -c|k|^\alpha - icd\,\mathrm{sgn}\,k|k|^\alpha. \tag{5.3}$$

If $d = 0$, $\psi(k)$ is real-valued function and the stable distribution is symmetric. Let us introduce kernel of asymmetry $\omega(k;\alpha,d)$ and rewrite $\psi(k)$ as:

$$\psi(k) = -c|k|^\alpha - ic\omega(k;\alpha,d). \tag{5.4}$$

The asymmetry kernel is then:

$$\omega(k;\alpha,d) = d\,\mathrm{sgn}\,k|k|^{\alpha}. \tag{5.5}$$

To show good reason for introducing asymmetry kernel, consider representation of constant z from (6.2) in polar form: $z = re^{i\theta}$ for $r > 0$. To guarantee boundedness of $\hat{F}(k)$ we have restriction: $-\pi/2 < \theta < \pi/2$. The asymmetry kernel is then:

$$\omega(k;\alpha,\theta) = \mathrm{sgn}(k) r |k|^{\alpha} \sin\theta. \tag{5.6}$$

The transformation between these parametrizations is obviously:

$$r = c\sqrt{1+d^2},$$
$$\theta = \arctan d.$$

Let us investigate the formula from previous section. We see that

$$c = C\frac{\Gamma(3-\alpha)\cos(\pi\alpha/2)}{\alpha(\alpha-1)}, \tag{5.7}$$
$$d = (p_1 - p_2)\tan(\pi\alpha/2). \tag{5.8}$$

Zolotarev refers to this parametrization as parametrization A:

$$\psi^{A}(k) = -\lambda|k|^{\alpha} + i\lambda\beta k|k|^{\alpha-1}\tan(\pi\alpha/2), \tag{5.9}$$

where obviously $\lambda = c$ corresponds to scale of the distribution, $\beta = p_1 - p_2$ influences the skewness of the distribution. Characteristic exponent under parametrization A is often used as an alternative definition of stable distribution.

Transforming into polar form:

$$r = C\frac{\Gamma(3-\alpha)\cos(\pi\alpha/2)}{\alpha(\alpha-1)}\sqrt{1+\frac{c_1-c_2}{c_1+c_2}\tan(\alpha\pi/2)}$$

We refer to this parametrization as form B:

$$\psi^{B}(k) = i\lambda b k - |k|^{\alpha} e^{-i\,\mathrm{sgn}(k)\beta}.$$

Parameters $\alpha, \beta, \lambda, b$ are considered under parametrization B. For this reason we will determine the transformation of parameters $\alpha, \beta, \lambda, b$. Finally we would like to introduce new parametrization, which we will use in the following and we hope it will find its popularity. The proposed parametrization is very practical and logical as will be explained later. Consider numbers p_1, p_2, such that $0 \leq p_1, p_2 \leq 1$ and $p_1 + p_2 = 1$. Then we parametrize z in (6.2) as $z = ce^{i\alpha(p_1-p_2)\frac{\pi}{2}}$ and so:

$$\psi(k) = -c|k|^{\alpha} e^{i\alpha(p_1-p_2)\frac{\pi}{2}\,\mathrm{sgn}\,k}. \tag{5.10}$$

Then the asymmetry kernel rewrites as:

$$\omega(k;\alpha,p_1)=\operatorname{sgn}(k)\sin((\alpha(p_1-p_2)\frac{\pi}{2}). \tag{5.11}$$

Let us denote this parametrization as $\psi^P(k)$, where P stands for practical. Yet, the last statement may be seen as obscure, it will be shown in following chapter that $p_2 = P\{X \leq 0\}$ where random variable X has stable probability function.

6 Discussion. Summary

Putting Ω, M^+, M^- into context with Lévy measure as in Sato. In literature Lévy measure is alternativelly defined as ... It corresponds to $\nu(x) = y^{-2}\Omega\{\mathrm{d}y\}$.

Historical comment of Lévy Khinchine and choice of measures, comment on Felle choice of measure M: deriving of stable charact. exp.: difference between Feller and Zolotarev.

Semigroup approach: infinitely divisible distributions are limits of compound poisson distribution. Add comment on trajectories ...

References

[1] Feller W. (1971). *An Introduction to Probability Theory and Its Applications, Vol. 2, 2nd Edn.*, Wiley.

[2] Orey S. (1968). On Continuity Properties of Infinitely Divisible Distribution Functions. *Annals of Mathematical Statistics*, Vol. 39, No 3, pp.936-937

[3] Zolotarev V.M. (1986). *One-dimensional stable distributions.* Amer. Math. Soc., Providence, RI.