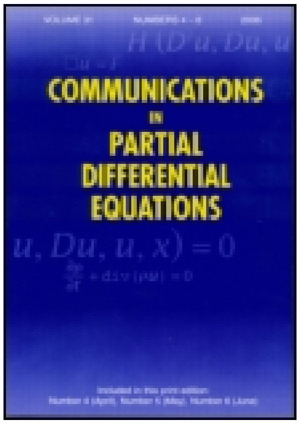

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Weak Solutions to Stochastic Wave Equations with Values in Riemannian Manifolds

Z. Brzeźniak [a] & M. Ondreját [b]

[a] Department of Mathematics, University of York, Heslington, UK

[b] Institute of Information Theory and Automation of the ASCR, Prague, Czech Republic

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Weak Solutions to Stochastic Wave Equations with Values in Riemannian Manifolds

Z. BRZEŹNIAK[1] AND M. ONDREJÁT[2]

[1]Department of Mathematics, University of York, Heslington, UK
[2]Institute of Information Theory and Automation of the ASCR, Prague, Czech Republic

Let M be a compact Riemannian manifold. We prove existence of a global weak solution of the stochastic wave equation $\mathbf{D}_t u_t = \mathbf{D}_x u_x + (X_u + \lambda_0(u)u_t + \lambda_1(u)u_x)\dot{W}$ *where X is a continuous vector field on M,* λ_0 *and* λ_1 *are continuous vector bundles homomorphisms from TM to TM, and W is a spatially homogeneous Wiener process on* $\mathbb{R}$ *with finite spectral measure. We use recently introduced general method of constructing weak solutions of SPDEs that does not rely on any martingale representation theorem.*

Keywords Geometric Wave Equation; Stochastic Wave Equation.

Mathematics Subject Classification Primary 60H15; Secondary 35R60, 58J65, 58E20, 35L70.

1. Introduction

Wave equations subject to random excitations have been intensively studied in the last forty years for their applications in physics, relativistic quantum mechanics or oceanography, see for instance [5–7, 10–13, 21, 23–26, 30, 32, 34–36]. Mathematical research has mostly considered stochastic wave equations with values in Euclidean spaces. However, many physical theories and models in modern physics such as harmonic gauges in general relativity, non-linear σ-models in particle systems, electro-vacuum Einstein equations or Yang-Mills field theory require the target space of the solutions to be a Riemannian manifold [18, 39]. Wave equations whose solutions take values in a Riemannian manifold are called *geometric wave equations* (GWEs).

Let us briefly outline the historical development of the theory of deterministic geometric wave equations (we refer the reader to nice surveys [39] and [41] for more details). Existence and uniqueness of global solutions to geometric wave equations is known to hold for an arbitrary target manifold provided the Minkowski space of

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Address correspondence to Zdzislaw Brzeźniak, Department of Mathematics, University of York, Heslington, York YO10 5DD, UK; E-mail: zdzislaw.brzezniak@york.ac.uk

the equation is $\mathbb{R}^{1+d}$ with $d = 1$ or $d = 2$, see [18, 19, 38, 43] or [9, 27]. In the case of $\mathbb{R}^{1+1}$, global solutions are known to exist either in the weak [43] or in the strong sense [18, 19, 22, 38] depending on the regularity of the initial conditions. In the case of $\mathbb{R}^{1+2}$, existence of global weak solutions was established in [9] and [27]. The case of $\mathbb{R}^{1+d}$ with $d \geq 3$ is more intriguing. Indeed, some interesting counterexamples were constructed which show that smooth solutions may explode in finite time and that weak solutions can be non-unique, see for example [8, 38, 39]. Existence (but not uniqueness) of global solutions on $\mathbb{R}^{1+d}$ for compact homogeneous targets was proved in [16].

Stochastic wave equations with values in Riemannian manifolds, also called *stochastic geometric wave equations* (SGWEs), see equation (1.1) below, were first studied by the present authors in [1] and [3]. In those papers we established existence and uniqueness of global strong solutions for SGWEs on the one-dimensional Minkowski space $\mathbb{R}^{1+1}$ when the target manifold M is an arbitrary compact Riemannian manifold. As usual, some additional technical assumptions on the coefficients were imposed. For instance, we assumed C^1-regularity linear growth of the diffusion coefficient, a finiteness of the moments up to order 2 of the spectral measure of the Wiener process. We considered the initial data $(u(0), u_t(0))$ from the space $H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}$ and we proved that there exists an $H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}$-valued continuous process $(u, \partial_t u)$ that is a solution to the SGWE (1.1). Finally, we proposed two natural definitions of an intrinsic and an extrinsic solution and we proved their equivalence.

In the subsequent paper [2] we investigated existence of solutions to SGWEs when the target manifold M is of a special form. To be precise, we showed existence (but not uniqueness) of a global weak solutions to SGWEs on a Minkowski space $\mathbb{R}^{1+d}$ taking values in a compact Riemannian homogeneous space (e.g., a sphere). In that paper we assumed that the diffusion coefficient is of the form $X_u + \lambda_0(u)u_t + \sum_{j=1}^{d} \lambda_j(u)u_{x_j}$, where X is a continuous vector field, λ_0 is a continuous function, $\lambda_1, \ldots, \lambda_d$ are continuous vector bundles homomorphisms from TM to TM and W is a spatially homogeneous Wiener process on $\mathbb{R}^d$ with a finite spectral measure. Comparing to our results proved in [1, 3], the assumptions on the spectral measure and on the space regularity of initial data were weakened. The price that had to be paid was a lower space-time regularity of the solution $(u, \partial_t u)$ which was only an $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$-valued weakly continuous process.

The aim of this paper is to substantially improve the results from [1–3]. We establish existence of a global solution on the Minkowski space $\mathbb{R}^{1+1}$ under weak regularity assumptions on the data (as in [2]) for a general target manifold M as in [1, 3]. Moreover, we weaken the assumptions on the spectral measure of the spatially homogeneous Wiener process by assuming that it is only finite.

In the main result of the present paper, i.e., Theorem 4.6, we generalize the work [2] in the one-dimensional $\mathbb{R}^{1+1}$ case, since we impose no restrictions on the target manifold.

This can be seen as an analogue of results from [43] for SGWE as far as the existence is concerned. One should point out that, besides existence, uniqueness of a solution was also proved in [43]. Yet, the length of the present paper and the relative complexity of the uniqueness problem made us decide to devote a separate paper to this subject.

To this end we assume that M is a compact Riemannian manifold (the assumption of compactness is imposed just for simplicity) and we consider the following SGWE

$$\mathbf{D}_t u_t = \mathbf{D}_x u_x + (X(u) + \lambda_0(u)u_t + \lambda_1(u)u_x)\dot{W} \tag{1.1}$$

with a random initial condition $(u(0,x,\omega), u_t(0,x,\omega)) = (u_0(\omega,x), v_0(\omega,x)) \in TM$. We assume that X is a continuous vector field on M, λ_0, λ_1 are continuous vector bundles homomorphisms from TM to TM and W is a spatially homogeneous Wiener process on $\mathbb{R}$ with a finite spectral measure, see Section 3 for details. By $\mathbf{D}$ we denote the connection on the pull-back bundle $u^{-1}TM$ induced by the Riemannian connection on M, see for instance [39]. Note however that deep understanding of the covariant derivative $\mathbf{D}$ is not necessary for reading this paper; see [3] where an attempt was made to present the theory in a self-contained way and the "acceleration" operators $\mathbf{D}_t\partial_t$ and $\mathbf{D}_x\partial_x$ were introduced in a reader-friendly way.

The equation (1.1) is written in a formal way and we showed in [1] that the two rigorous notions of a strong solution (intrinsic and extrinsic) are equivalent. Our proof relies on a use of the Nash embedding theorem [28] according to which M may be isometrically embedded into a certain Euclidean space $\mathbb{R}^n$. We show that in the setting of the present paper, the notions of a weak intrinsic and weak extrinsic solution coincide; see Theorem 4.3. Finally, in Theorem 4.6 we prove existence of a global weak solution of (1.1).

Our proof of the main theorem is based on a recently introduced general method of constructing weak solutions of SPDEs that does not rely on any kind of martingale representation theorem (cf. the yet unpublished paper [2] and [33]).

2. Notation and Conventions

- By $\mathbb{R}_+$ we denote the set $[0,\infty)$ and the set $\mathbb{N}$ of natural numbers begins with 1.
- $\mathscr{S}$ denotes the Schwartz space of smooth rapidly decreasing functions on $\mathbb{R}$ and $C_b(\mathbb{R}^d)$ denotes the space of all real continuous and bounded functions on $\mathbb{R}^d$.
- $\mathscr{S}'$ denotes the space of tempered distributions on $\mathbb{R}$.
- $\widehat{S}$ denotes the Fourier transform of a tempered distribution S.
- $\mathscr{B}(X)$ denotes the Borel σ-algebra on a topological space X.
- $\mathscr{L}(H,X)$ is the space of linear bounded operators from H to X.
- $\mathscr{J}_2(H,X)$ is the space of Hilbert-Schmidt operators between Hilbert spaces H and X.
- Whenever E is a vector class of functions defined on $\mathbb{R}$, E_{comp} will denote the subclass of those $f \in E$ which have compact support, for instance L^1_{comp}. An exception is made for C_0^k, $k \in \mathbb{N} \cup \{0,\infty\}$.
- A spectral measure on $\mathbb{R}$ is a positive symmetric tempered measure on $\mathbb{R}$.
- We use the standard convention $\inf \emptyset = \infty$.
- The Euclidean scalar product of vectors $\xi, \eta \in \mathbb{R}^n$ will be denoted either by $\xi \cdot \eta$ or $\langle \cdot, \cdot \rangle$.
- By *an approximation of identity* we mean a sequence (b_k) of $C_0^\infty(\mathbb{R})$-densities such that $\text{supp} b_k \subset \left(-\frac{1}{k}, \frac{1}{k}\right)$, for $k \in \mathbb{N}$.

3. Spatially Homogeneous Wiener Process

Following [35] and [4] we assume that μ is a finite symmetric Borel measure on $\mathbb{R}^d$ and we denote $\Gamma(x) = (2\pi)^{-d}\int_{\mathbb{R}} e^{-ix\cdot\xi}\mu(d\xi)$, $x \in \mathbb{R}^d$. We also assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)$ $t \geq 0$ is a stochastic basis. A spatially homogeneous $\mathbb{F}$-Wiener process with the *spectral measure* μ can be introduced in two equivalent ways. The first one is to consider a centered Gaussian jointly measurable random field $\{\mathcal{W}(t,x)\}_{t\geq 0, x\in\mathbb{R}^d}$ such that for every $x \in \mathbb{R}^d$, the process $\{\mathcal{W}(t,x)\}_{t\geq 0}$ is an $\mathbb{F}$-Wiener process with covariance $\Gamma(0)$, for all $0 \leq s < t$ the σ-algebra $\sigma\{\mathcal{W}(t,x) - \mathcal{W}(s,x)\}_{x\in\mathbb{R}^d}$ is independent of $\mathcal{F}_s$, for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ the function $\mathbb{R}^d \ni x \mapsto |\mathcal{W}(\cdot, x, \omega)|_{C[0,t]}$ is tempered and $\mathbb{E}\mathcal{W}(s,x)\mathcal{W}(t,y) = \min\{s,t\}\Gamma(x-y)$ for all $t, s \geq 0$ and $x, y \in \mathbb{R}^d$. The second one is to consider an $\mathcal{S}'(\mathbb{R}^d)$-valued $\mathbb{F}$-Wiener process satisfying $\mathbb{E}\{\langle W(s), \varphi_0\rangle\langle W(t), \varphi_1\rangle\} = \min\{s,t\}\langle\widehat{\varphi}_0, \widehat{\varphi}_1\rangle_{L^2(\mu)}$ for $t, s \geq 0$ and $\varphi_0, \varphi_1 \in \mathcal{S}(\mathbb{R}^d)$. The equivalence between these approaches, see e.g., [35, p. 190], follows from the fact that W can be recovered from $\mathcal{W}$ (and *vice versa*) by exploiting the formula $\langle W(t), \varphi\rangle = \int_{\mathbb{R}^d}\mathcal{W}(t,x)\varphi(x)dx$ for every $t \geq 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, a.s.. In our current paper we will use the second approach for the space dimension $d = 1$.

The following result, see Proposition 1.2 in [35] and Lemma 1 in [30], describes the reproducing kernel Hilbert space (RKHS) of a spatially homogeneous Wiener process and some of its properties.

Proposition 3.1. *Let W be an $\mathcal{S}'(\mathbb{R}^d)$-valued spatially homogeneous Wiener process with a finite spectral measure μ and let H_μ be the reproducing kernel Hilbert space of W. (More precisely, H_μ is the RKHS of the law of the $\mathcal{S}'$-valued Gaussian random variable $W(1)$). Then*

$$H_\mu = \{\widehat{\psi\mu} : \psi \in L^2_{\mathbb{C}}(\mathbb{R}^d, \mu), \overline{\psi(x)} = \psi(-x)\}, \quad \langle\widehat{\psi_0\mu}, \widehat{\psi_1\mu}\rangle_{H_\mu} = \langle\psi_0, \psi_1\rangle_{L^2(\mu)},$$

$H_\mu \hookrightarrow C_b(\mathbb{R}^d)$ continuously, $|\{H_\mu \ni \xi \mapsto \xi(x) \in \mathbb{C}\}|^2 = (2\pi)^{-d}\mu(\mathbb{R}^d)$ and

$$\|\xi \mapsto h\xi\|^2_{\mathcal{J}_2(H_\mu, L^2(\mathbb{R}^d))} = (2\pi)^{-d}\mu(\mathbb{R}^d)|h|^2_{L^2(\mathbb{R}^d)}, \quad h \in L^2(\mathbb{R}^d).$$

3.1. *Stochastic Integration*

If X and H are real separable Hilbert spaces and W is an H-cylindrical $\mathbb{F}$-Wiener process (i.e., with identity covariance) then the Itô integral $\int_0^\cdot h\,dW$ can be constructed as an X-valued continuous $\mathbb{F}$-local martingale provided that h is an $\mathbb{F}$-progressively measurable processes with paths in $L^2_{\text{loc}}(\mathbb{R}_+; \mathcal{J}_2(H, X))$ a.s., see for instance [14] for details.

A proof of the following proposition is based on the Garsia-Rodemich-Rumsey Lemma [17] and can be found for instance in Lemma 4 in [32].

Proposition 3.2. *Assume that $p, r \in (2, \infty)$ and $\gamma \in (0, \frac{1}{2})$ satisfy $\gamma + \frac{1}{p} + \frac{1}{r} < \frac{1}{2}$. Then there exists a constant c_* such that for all separable Hilbert spaces U and K, every cylindrical Wiener process W on U and every progressively measurable process ψ with paths in $L^r_{\text{loc}}(\mathbb{R}_+; \mathcal{J}_2(U, K))$, the following inequality holds*

$$\mathbb{E}\left|\int_0^\cdot \psi(s)dW\right|^p_{C^\gamma([0,t];K)} \leq c_*\mathbb{E}\left(\int_0^t \|\psi(s)\|^r_{\mathcal{J}_2(U,K)}ds\right)^{\frac{p}{r}}, \quad t \geq 0.$$

4. Statements of the Main Results

Assumption 4.1. We assume that M is a d-dimensional compact submanifold in $\mathbb{R}^n$, X is a continuous vector field on M and λ_0, λ_1 are continuous vector bundles homomorphisms from TM to TM, i.e., $\lambda_i(p)$ are linear on T_pM for every $p \in M$ and $M \ni p \mapsto \lambda_i(p)Z(p) \in TM$ is continuous for every continuous vector field Z on M and $i \in \{0, 1\}$. We put $Y(p, \xi, \eta) := X(p) + \lambda_0(p)\xi + \lambda_1(p)\eta$, $p \in M$, $\xi, \eta \in T_pM$, where T_pM and N_pM denote the tangent and the normal space respectively at $p \in M$.

We will denote by TM and NM the tangent and the normal bundle of M, respectively. We will use the following notation.

- $A_p : T_pM \times T_pM \to N_pM$ is the second fundamental form of M in $\mathbb{R}^n$ at $p \in M$,
- For $k \geq 0$, $H^{k+1}_{loc} \times H^k_{loc}(TM)$ is the closed subset of the metric space $H^{k+1}_{loc}(\mathbb{R}; \mathbb{R}^n) \times H^k_{loc}(\mathbb{R}; \mathbb{R}^n)$ consisting of the elements (f, g) such that a.e., on $\mathbb{R}$, $(f, g) \in TM$.

Definition 4.2. Let W be a spatially homogeneous Wiener process with a finite spectral measure μ and let $z = (u, v)$ be an $\mathbb{F}$-adapted weakly continuous $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$-valued process. We say that z is an intrinsic solution to (1.1) iff, for all $\omega \in \Omega$, every smooth vector field Z on M and $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$,

$$\partial_t \langle u(\cdot, \omega), \varphi \rangle = \langle v(\cdot, \omega), \varphi \rangle \tag{4.1}$$

in the weak sense on $\mathbb{R}_+$, and the next equality holds almost surely for every $t \geq 0$,

$$\begin{aligned} \langle v(t) \cdot Z(u(t)), \varphi \rangle = \langle v(0) \cdot Z(u(0)), \varphi \rangle &- \int_0^t \langle u_x \cdot Z(u), \varphi_x \rangle ds \\ &+ \int_0^t \langle v \cdot (\nabla_v Z)|_u - u_x \cdot (\nabla_{u_x} Z)|_u, \varphi \rangle ds \\ &+ \int_0^t \langle [Y(u, v, u_x) \cdot Z(u)] dW, \varphi \rangle. \end{aligned} \tag{4.2}$$

We say that the process z is an extrinsic solution to (1.1) iff for every $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$, z satisfies the condition (4.1) and, instead of (4.2), the following equality holds a.s.,

$$\langle v(t) - v(0), \varphi \rangle = \int_0^t \langle Y(u, v, u_x) dW, \varphi \rangle - \int_0^t \langle u_x, \varphi_x \rangle ds + \int_0^t \langle \mathscr{A}(z), \varphi \rangle ds, \quad t \geq 0, \tag{4.3}$$

where $\mathscr{A}(z) = A_u(v, v) - A_u(u_x, u_x)$, $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(\mathbb{R})$ and

$$\int_0^t \langle g\, dW, \varphi \rangle := \int_0^t \{H_\mu \ni \xi \mapsto \langle \psi\xi, \varphi \rangle \in \mathbb{R}\} dW, \quad t \geq 0.$$

The next result describes the relationship between the two types of solutions.

Theorem 4.3. *An $\mathbb{F}$-adapted weakly continuous $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$-valued process z is an intrinsic solution to* (1.1) *if and only if it is an extrinsic solution to* (1.1).

Hence the following definition is well posed.

Definition 4.4. An intrinsic or an extrinsic solution to problem (1.1) is called a solution.

Definition 4.5. A function $L \in C(\mathbb{R}_+) \cap C^2(0, \infty)$ is called a good function iff $L' \geq 0$ and there exists $C_L > 0$ such that $tL'(t) + t^2 \max\{L''(t), 0\} \leq C_L L(t)$ for every $t > 0$.

We are now ready to formulate the main result of our paper. In the formulation of it we will use the following notation for $z = (u, v) \in H^1_{\text{loc}} \times L^2_{\text{loc}}(TM)$ and $T > 0$,

$$\mathbf{e}_{T,L}(t, z) = L(|X|^2_{C(M)} + |u|^2_{L^2(-T+t,T-t)} + |u_x|^2_{L^2(-T+t,T-t)} + |v|^2_{L^2(-T+t,T-t)}), \quad t \in [0, T]. \tag{4.4}$$

Theorem 4.6. *Let Θ be a Borel probability measure on $H^1_{\text{loc}} \times L^2_{\text{loc}}(TM)$ and let μ be a finite spectral measure on $\mathbb{R}$. Then there exists a completely filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, a spatially homogeneous $\mathbb{F}$-Wiener process W with spectral measure μ and a solution z such that Θ is equal to the law of $z(0)$ and for every good function L, there exists a constant c depending on C_L, λ_0, λ_1 and $\mu(\mathbb{R})$ such that for every $T > 0$ and $D \in \mathcal{B}(H^1_{\text{loc}} \times L^2_{\text{loc}}(TM))$, the following inequality is satisfied*

$$\mathbb{E}\mathbf{1}_D(z(0)) \sup_{s\in[0,t]} \mathbf{e}_{T,L}(s, z(s)) \leq 4e^{tc}\mathbb{E}\mathbf{1}_D(z(0))\mathbf{e}_{T,L}(0, z(0)), \quad t \in [0, T]. \tag{4.5}$$

5. Intermezzo Before the Proofs

In this section, before the reader proceeds to details of the proofs, we would like to present the main ideas of what will follow.

We will begin with approximating the data (z_0, W, Y) by more regular data (z_0^k, W^k, Y^k) so that the existence theorems from [1] can be applied to the approximating problem $\mathbf{D}_t u_t^k = \mathbf{D}_x u_x^k + Y^k(u^k, u_t^k, u_x^k)\dot{W}^k$, $z^k(0) = z_0^k$. The crucial fact is that the solutions $z^k = (u^k, u_t^k)$ satisfy *a uniform local energy inequality* from Theorem 6.6. Consequently the laws of the random variables (z^k, W^k) are tight (on an appropriate functional space) and therefore, by the Skorokhod–Jakubowski Theorem, there exists a subsequence (k_α) and random variables $(\mathbf{z}^\alpha, \mathbf{B}^\alpha)_{\alpha\in\mathbb{N}}$, $(\mathbf{z}, \mathbf{B})$ defined on the stochastic basis $([0, 1], \mathcal{B}[0, 1], \mathbb{B}, \text{Leb})$ such that $(z^{k_\alpha}, W^{k_\alpha}) = (\mathbf{z}^\alpha, \mathbf{B}^\alpha)$ in law and $(\mathbf{z}^\alpha, \mathbf{B}^\alpha)$ is convergent weakly to $(\mathbf{z}, \mathbf{B})$. Next we show that the process $\mathbf{B}$ is a spatially homogeneous Wiener process. Unfortunately, the convergence $\mathbf{z}^\alpha \to \mathbf{z}$ is too weak to imply the convergence of the nonlinear integrands in (4.1) or (4.3) and thus we cannot verify directly that $(\mathbf{z}, \mathbf{B})$ is a solution of (1.1).

This is the reason why we introduce a notion of a *pseudo-intrinsic equation* and prove in Lemma 9.13 that for every $l \in \mathbb{N}$ the mollified process $(b_l * \mathbf{z}, \mathbf{B})$ solves such an equation. Here $(b_l)_{l=1}^\infty$ is an approximation of identity as in section 2. The pseudo-intrinsic equation differs from (4.1) by a "small term" $\mathbf{Q}_{lm\gamma}$ which tends to 0 as $l \to \infty$. On the other hand, the convolution $b_l * \mathbf{z}^\alpha$ converges to $b_l * \mathbf{z}$ in a sufficiently strong sense. This in turn implies the convergence of the nonlinear integrands to the expected limits in the pseudo-intrinsic equations. Finally, by letting $l \to \infty$, since $b_l * \mathbf{z} \to \mathbf{z}$, we conclude that $(\mathbf{z}, \mathbf{B})$ is a solution of (1.1).

There are two important issues that should be pointed out. The first one is that we are using the Jakubowski generalization of the Skorokhod representation theorem [20] since laws of the processes z^k are tight in a non-metrizable space $\mathbb{L}$ of weakly continuous functions, see Section 8.1, whereas the classical Skorokhod theorem [40] requires the target space to be Polish.

The second one is that we do not use any result on the integral representations of martingales in order to prove that the limit process $(\mathbf{z}, \mathbf{B})$ is a solution. To explain this in a clearer way let us illustrate this trick on a scalar equation

$$dz = F(z)dt + G(z)dW \quad \text{approximated by } dz^k = F^k(z^k)dt + G^k(z^k)dW^k,$$

where (W^k) is a suitable approximation of the Wiener process W and we assume that the sequence (z^k) is tight on $C(\mathbb{R}_+)$. Then also the sequence (z^k, W^k) is tight on $C(\mathbb{R}_+) \times C(\mathbb{R}_+)$ and therefore there exists a subsequence (k_α) and a probability space with processes $(\mathbf{z}^\alpha, \mathbf{B}^\alpha)$, $(\mathbf{z}, \mathbf{B})$ such that $(z^{k_\alpha}, W^{k_\alpha}) = (\mathbf{z}^\alpha, \mathbf{B}^\alpha)$ in law and $(\mathbf{z}^\alpha, \mathbf{B}^\alpha)$ converge to $(\mathbf{z}, \mathbf{B})$ in $C(\mathbb{R}_+) \times C(\mathbb{R}_+)$. If we put $\mathscr{B}_t = \mathscr{F}_t^{(\mathbf{z},\mathbf{B})}$, $t \geq 0$, then $\mathbf{B}$ is a $\mathscr{B}$-Wiener process and the process $J = \mathbf{z} - \mathbf{z}(0) - \int_0^\cdot F(\mathbf{z})ds$ is a local $\mathscr{B}$-martingale satisfying $\langle J \rangle = \int_0^\cdot G^2(\mathbf{z})ds$, $\langle J, \mathbf{B} \rangle = \int_0^\cdot G(\mathbf{z})ds$. Hence we infer that $\langle J - \int_0^\cdot G(\mathbf{z})d\mathbf{B} \rangle = \langle J \rangle - 2\int_0^\cdot G(\mathbf{z})d\langle J, \mathbf{B} \rangle + \int_0^\cdot G^2(\mathbf{z})ds = 0$.

This method was recently developed by the authors and simultaneously implemented in the present paper, the yet unpublished paper [2] and in [33], mainly because we are not aware of any result on integral representation of martingales in neither Fréchet spaces nor, less generally, local Sobolev spaces available.

6. Approximation

Let us assume that $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$, is a complete filtered probability space. Let $z_0 = (u_0, v_0)$ be an $\mathscr{F}_0$-measurable $H^1_{\text{loc}} \times L^2_{\text{loc}}(TM)$-valued random variable whose law is Θ. Finally, let $(\beta^{ij} : i, j \in \mathbb{N})$ be i.i.d. standard $\mathbb{R}$-valued $\mathbb{F}$-Wiener processes.

6.1. *Approximation of the Initial Condition*

By Lemma A.2 we can find a sequence $z_0^k = (u_0^k, v_0^k)$ of $H^2_{\text{loc}} \times H^1_{\text{loc}}(TM)$-valued $(\mathscr{F}_0)$-simple random variables such that

$$z_0^k \to z_0 \text{ in } H^1_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \text{ on } \Omega$$

and there exist $C_M > 0$ such that for all $R > 0$ and $k \in \mathbb{N}$,

$$|z_0^k|_{H^1(-R,R) \times L^2(-R,R)} \leq C_M(R^{\frac{1}{2}} + |z_0|_{H^1(-R-1,R+1) \times L^2(-R-1,R+1)}) \text{ on } \Omega. \tag{6.1}$$

Remark 6.1. The approximation in $H^1_{\text{loc}} \times L^2_{\text{loc}}$ of the initial data z_0 by a sequence of $H^2_{\text{loc}} \times H^1_{\text{loc}}(TM)$-valued random variables would trivially follow from the density of $H^2_{\text{loc}} \times H^1_{\text{loc}}(TM)$ in $H^1_{\text{loc}} \times L^2_{\text{loc}}(TM)$ had we not required a sort of uniform approximation satisfying condition (6.1) which is not trivial and needs to be justified.

6.2. *Approximation of the Wiener Process*

It is well known, see [1], that there exist the unique strong solutions of stochastic geometric wave equations driven by spatially homogeneous Wiener processes with spectral measures having finite moments up to order 2. Since our assumptions on μ are much weaker, i.e., we only assume that μ is only a finite measure, a "localization" argument has to be employed. For this purpose we introduce a sequence (ν_k) of symmetric Borel measures defined by $\nu_k(V) = \mu(V \cap \widetilde{B}_k)$ for $k \in \mathbb{N}$, $V \in \mathscr{B}(\mathbb{R})$, where $\widetilde{B}_k := \{x \in \mathbb{R} : k-1 \leq |x| < k\}$ for $k \in \mathbb{N}$. We also introduce a corresponding sequence of Hilbert spaces $H_{\nu_k} := \{\widehat{\psi\nu_k} : \psi \in L^2_{\mathbb{C}}(\mathbb{R}, \nu_k), \overline{\psi(\cdot)} = \psi(-\cdot)\}$, $\langle \widehat{\psi_0\nu_k}, \widehat{\psi_1\nu_k}\rangle_{H_{\nu_k}} := \langle \psi_0, \psi_1\rangle_{L^2(\nu_k)}$. We write $J_k := \{1, \cdots, \dim(H_{\nu_k})\}$ if $\dim(H_{\nu_k}) < \infty$ or $J_k = \mathbb{N}$ otherwise. If $k \in \mathbb{N}$ then by $\{\xi_{kj} : j \in J_k\}$ we denote an orthonormal basis in H_{ν_k}. For each $k \in \mathbb{N}$ we consider a cylindrical spatially homogeneous $\mathbb{F}$-Wiener process $W^k(\varphi) = \sum_{i=1}^k \sum_{j \in J_i} \beta^{ij}\xi_{ij}(\varphi)$, $\varphi \in \mathscr{S}$ with reproducing kernel Hilbert space H_{ν_k} and with spectral measure $\mu_k := \sum_{i=1}^k \nu_i$.

The following result, stating in particular, that each measure μ_k, $k \in \mathbb{N}$, satisfies the condition (2.3) from [1], is simple and hence its proofs will be omitted.

Lemma 6.2. *The system $\{\xi_{kj} : j \in J_k, k \in \mathbb{N}\}$ is an orthonormal basis in H_μ and*

$$\int_{\mathbb{R}} (1 + y^2)\mu_k(dy) < \infty, \quad k \in \mathbb{N}. \tag{6.2}$$

6.3. *Approximation of the Diffusion Coefficient Y*

Theorem 11.1 in [1] requires the diffusion coefficient to be of C^1-class and to satisfy the growth conditions (2.1)–(2.2) from therein. In order to apply this result we have to approximate our coefficient Y in a suitable way. The following result shows that this is possible.

Proposition 6.3. *There exist a compact K in $\mathbb{R}^n$ and the following objects:*

(i) *a sequence $(X^k)_{k=1}^\infty$ of smooth functions $X^k : \mathbb{R}^n \to \mathbb{R}^n$ with support in K,*
(ii) *a continuous function $\overline{X} : \mathbb{R}^n \to \mathbb{R}^n$ with support in K,*
(iii) *smooth functions $\lambda_i^k : \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$, $k \in \mathbb{N}$, $i \in \{0, 1\}$ with supports in K,*
(iv) *continuous functions $\bar{\lambda}_0, \bar{\lambda}_1 : \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ with support in K*

such that

- *for $k \in \mathbb{N}$ and $p \in M$ $X^k(p) \in T_pM$ and $\overline{X} = X$, and $\bar{\lambda}_0 = \lambda_0$, $\bar{\lambda}_1 = \lambda_1$ on M,*
- *$X^k \to \overline{X}$, $\lambda_0^k \to \bar{\lambda}_0$, $\lambda_1^k \to \bar{\lambda}_1$ uniformly on $\mathbb{R}^n$,*
- *for every $p \in M$ and $k \in \mathbb{N}$, $\lambda_0^k(p)$ and $\lambda_1^k(p)$ map T_pM into itself.*

In particular, $Y^k(p, \xi, \eta) = X^k(p) + \lambda_0^k(p)\xi + \lambda_1^k(p)\eta$, $p \in M$, $\xi, \eta \in T_pM$ satisfy the conditions (2.1) and (2.2) from [1] for every $k \in \mathbb{N}$ and a map $\overline{Y}$ defined by

$$\overline{Y}(q, \xi, \eta) = \overline{X}(q) + \bar{\lambda}_0(q)\xi + \bar{\lambda}_1(q)\eta, \quad q, \xi, \eta \in \mathbb{R}^n \tag{6.3}$$

is an extension of the map Y.

Proof. Let V and P be respectively the neighbourhood of M and the function from Lemma A.1. We define a vector field $\widetilde{X}$ on V by $\widetilde{X}(q) = X(P(q))$ for $q \in V$. Obviously, $\widetilde{X}$ is an extension of the vector field X. Next, by employing the partition of unity, we can find a compactly supported continuous function $X^\circ : \mathbb{R}^n \to \mathbb{R}^n$ such that the restriction of X° to M equals X.

Now, let $(b_k)_{k=1}^{\infty}$ be an approximation of identity on $\mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a compactly supported C^∞-function such that for $p \in M$, π_p is the orthogonal projection from $\mathbb{R}^n$ to T_pM. Let us define a sequence $(X^k)_{k\in\mathbb{N}^*}$ of compactly supported C^∞-functions by $X^k = \pi \circ (b_k * X^\circ)$. Obviously the restriction of each X^k to M is a smooth vector field on M. Moreover, X^k converges to $\overline{X} := \pi \circ X^\circ$ as $k \to \infty$, uniformly on $\mathbb{R}^n$.

The construction of the approximation of the functions λ_0 and λ_1 can be done in a fully analogous manner. If $j \in \{0, 1\}$, then a function B_j defined by $B_j : M \ni p \mapsto \lambda_j(p)\pi_p \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ a continuous function on M. Obviously, B_j can be extended to a compactly supported continuous $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$-valued function on $\mathbb{R}^n$, denoted again by B_j. Finally, we set $\lambda_j^k := \pi \circ B_j^k$ and $\bar{\lambda}_j := \pi \circ B_j$, where $(B_j^k)_{k=1}^{\infty}$ is a sequence of $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$-valued compactly supported C^∞-functions such that B^k converges uniformly to B on $\mathbb{R}^n$. □

6.4. *Solutions of the Approximating Problems*

It has been shown in [1, Theorem 11.1], that for each $k \in \mathbb{N}$ there exists an $\mathbb{F}$-adapted $H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}(TM)$-valued continuous process $z^k = (u^k, v^k)$ such that $z^k(0) = z_0^k$ a.s.,

- every path of the process u^k belongs to $C^1(\mathbb{R}_+, H^1_{\mathrm{loc}}(\mathbb{R}))$,
- $\frac{du^k}{dt}(t, \omega) = v^k(t, \omega)$ in $H^1_{\mathrm{loc}}(\mathbb{R})$ for every $(t, \omega) \in \mathbb{R}_+ \times \Omega$

and, for every $t \geq 0$ and $R > 0$, the following equality is satisfied in $L^2((-R, R); \mathbb{R}^n)$ a.s.

$$v^k(t) = v_0^k + \int_0^t [u_{xx}^k - A_{u^k}(v^k, v^k) + A_{u^k}(u_x^k, u_x^k)]ds + \int_0^t Y^k(u^k, v^k, u_x^k)dW^k. \quad (6.4)$$

Remark 6.4. By [1, Theorem 11.1] a strong solutions to problem (6.4) exists if the spectral measure μ_k satisfies condition (6.2) and the diffusion coefficient Y^k satisfies the growth and smoothness conditions (2.1)–(2.2) from therein.

Remark 6.5. By [1, Theorem 12.1], the process $z^k = (u^k, v^k)$ satisfies the *extrinsic* equation (6.4) for every $t \geq 0$ and $R > 0$ in $L^2((-R, R); \mathbb{R}^n)$ almost surely if and only if it satisfies, for all $t \geq 0$ and smooth vector field Z on M, almost surely, the following intrinsic equation

$$\langle v^k(t), Z(u^k(t))\rangle_{\mathbb{R}^n} = \langle v_0^k, Z(u^k(0))\rangle_{\mathbb{R}^n} + \int_0^t \langle Y^k(u^k(s), v^k(s), u_x^k(s)), Z(u^k(s))\rangle_{\mathbb{R}^n} dW^k$$
$$+ \int_0^t [\langle u_{xx}^k(s), Z(u^k(s))\rangle_{\mathbb{R}^n} + \langle v^k(s), \nabla_{v^k(s)} Z|_{u^k(s)}\rangle_{\mathbb{R}^n}]ds. \quad (6.5)$$

In the following we will show that the approximating processes (u^k, v^k) and X^k from Proposition 6.3 satisfy a local energy inequality with

$$\mathbf{e}_{k,L,T}(t, u, v) = L(|X^k|^2_{C(M)} + |u|^2_{L^2(-T+t,T-t)} + |u_x|^2_{L^2(-T+t,T-t)} + |v|^2_{L^2(-T+t,T-t)}).$$

Theorem 6.6 (Local Energy Inequality). *Assume that $z^k = (u^k, v^k)$ are the processes introduced in Section* 6.4. *Assume that L is a good function. Then there exists a constant c_* depending on C_L, λ_0, λ_1 and $\mu(\mathbb{R})$ such that for every $T > 0$, $D \in \mathscr{B}(H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM))$ and $k \in \mathbb{N}$, the following inequality holds*

$$\mathbb{E} \sup_{s \in [0,t]} \mathbf{1}_D(z^k(0)) \mathbf{e}_{k,L,T}(s, z^k(s)) \leq 4e^{tc_*} \mathbb{E} \mathbf{1}_D(z^k(0)) \mathbf{e}_{k,L,T}(0, z^k(0)), \quad t \in [0, T].$$

Proof of Theorem 6.6. Let us fix $T > 0$ and $\varepsilon > 0$. Define a process M_ε by

$$M_\varepsilon(t) = \int_0^t \frac{2L'(2\varepsilon + \mathbf{e}_{k,\mathrm{id},T}(s))}{\varepsilon + L(2\varepsilon + \mathbf{e}_{k,\mathrm{id},T}(s))} \times \langle v^k(s), Y^k(u^k(s), v^k(s), u^k_x(s) dW^k_s) \rangle_{L^2(s-T, T-s)}, \quad t \in [0, T].$$

Let us choose a $C_0^\infty(\mathbb{R})$-function ψ such that $\psi = 1$ on $(-T, T)$. Then, by [33, Proposition 8.1] applied to the processes $Z = (U, V) = (\psi u^k, \psi v^k)$, $\alpha := [A_{u^k}(u^k_x, u^k_x) - A_{u^k}(v^k, v^k)]\psi - u^k \psi_{xx} - 2\psi_x u^k_x$, $\beta := \psi Y^k(u^k, v^k, u^k_x)$ and $F(y) := (|X^k|^2_{C(M)} + |y|^2)/2$, we infer that, for every $t \in [0, T]$,

$$\log(\varepsilon + L(2\varepsilon + \mathbf{e}_{k,\mathrm{id},T}(t, z^k(t)))) \leq \log(\varepsilon + L(2\varepsilon + \mathbf{e}_{k,\mathrm{id},T}(0, z^k(0)))) + ct + M_\varepsilon(t) - \frac{1}{2}\langle M_\varepsilon \rangle_t.$$

Let us take any $\delta \in \mathbb{R}$ and an arbitrary $\mathscr{F}_0$-measurable non-negative random variable Q and put $N_{Q,\delta} = Q^\delta \exp\{\delta M_\varepsilon - \delta^2 \langle M_\varepsilon \rangle / 2\}$. Since we can find a positive number κ depending only on C_L, λ_0, λ_1 and $\mu(\mathbb{R})$, such that $\langle M_\varepsilon \rangle_t \leq \kappa t$, $t \in [0, T]$, by the Doob inequality, the following inequality holds

$$\mathbb{E} \sup_{r \in [0,t]} N_{Q,1}(r) \leq \mathbb{E} \sup_{r \in [0,t]} N^2_{Q,\frac{1}{2}}(r) \leq 4\mathbb{E} N^2_{Q,\frac{1}{2}}(t) \leq 4e^{\kappa t/4} \mathbb{E} N_{Q,1}(t) \leq 4e^{\kappa t/4} \mathbb{E} Q, \quad t \in [0, T].$$

The proof is accomplished by applying the Fatou Lemma when we let ε to $\searrow 0$. □

7. Pseudointrinsic Equation

We will see later in this paper that we can find a subsequence of $z^k = (u^k, v^k)$ that converges (on another probability space) to a limit $z = (u, v)$ in the locally uniform weak topology of $H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$. Unfortunately, this convergence is too weak to allow us to pass directly to a limit neither in the extrinsic equation (6.4) nor in the intrinsic equation (6.5), where quadratic nonlinearities appear. Thus, to see that we really have a solution, we must resolve this difficulty by a forced strengthening of weak convergence to strong convergence which is done by convoluting the solutions z^k with a smooth density b. Obviously, the limit $b * z$ solves a different, yet better tractable "*pseudointrinsic*" equation (as this approach is only applicable for the intrinsic equation (6.5)), see Lemma 7.1 and Lemma 9.13.

Lemma 7.1. *Let us assume that b is a C_0^∞ symmetric density on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$, $Z : \mathbb{R}^n \to \mathbb{R}^n$ are C_0^∞-class functions. Then for each $k \in \mathbb{N}$, the process z^k constructed* *in Section* 6.4 *satisfies, for every* $t \geq 0$, *a.s.*,

$$\begin{aligned}&\langle v^k(t) \cdot Z(b * u^k(t)), \varphi\rangle \\ &\quad = \langle v_0^k \cdot Z(b * u_0^k), \varphi\rangle + \int_0^t \langle [Y^k(u^k, v^k, u_x^k) \cdot Z(b * u^k)] dW^k, \varphi\rangle \\ &\quad - \int_0^t \langle u_x^k \cdot Z(b * u^k), \varphi_x\rangle ds + \int_0^t \langle v^k \cdot Z'_{b*u^k}(b * v^k) - u_x^k \cdot Z'_{b*u^k}(b * u_x^k), \varphi\rangle ds \\ &\quad + \int_0^t \langle [A_{u^k}(u_x^k, u_x^k) - A_{u^k}(v^k, v^k)] \cdot Z(b * u^k), \varphi\rangle ds. \qquad (7.1)\end{aligned}$$

Proof of Lemma 7.1. Let us choose a positive real number $r > 0$ such that both φ and b have their support in $(-r, r)$ and let us put $R = 3r$. Next let us put for any $h \in L^1_{loc}$, $\bar{h}(x) := \int_{-R}^R h(y) b(x - y) dy$ and define

$$B : K := [L^2((-R, R); \mathbb{R}^n)]^2 \ni w = (w_1, w_2) \mapsto \int_{-R}^R w_2(x) \cdot Z(\bar{w}_1(x)) \varphi(x) dx.$$

Since obviously the function B is of C^2 class and $\bar{h} = b * h$ on $(-r, r)$, for $h \in L^1_{loc}$, the result follows from the Itô formula [14, Theorem 4.17] applied to the process z^k. □

8. Tightness of the Sequence of Approximating Solutions

Let us fix $m \in \mathbb{N}$ and $r > 0$. Let us define a set $S_{m,r}$ by the following formula

$$S_{m,r} := \{z_0 \in H^1_{loc} \times L^2_{loc}(TM) : |z_0|_{H^1(-2m-1,2m+1)\times L^2(-2m-1,2m+1)} \leq r\}. \qquad (8.1)$$

It follows from (6.1), see Lemma A.2, that there exists a constant $C_{m,r} > 0$ such that $S_{m,r} \subseteq \bigcap_{k=1}^\infty \{z_0 : |z_0^k|_{H^1(-2m,2m)\times L^2(-2m,2m)} \leq C_{m,r}\}$. Hence by Theorem 6.6 we infer that

$$C_{m,r,q} := \sup_{k\in\mathbb{N}} \mathbb{E}\Big[\mathbf{1}_{S_{m,r}}(z_0^k) \sup_{t\in[0,m]} |z^k(t)|^q_{H^1(-m,m)\times L^2(-m,m)}\Big] < \infty, \quad q \in (0, \infty). \qquad (8.2)$$

Hence, by applying the Chebyshev inequality we infer that

$$\mathbb{P}\left(\left\{\mathbf{1}_{S_{m,r}}(z_0^k)|z^k|^q_{L^\infty(0,m;H^1(-m,m)\times L^2(-m,m))} > \frac{1}{\delta}\right\}\right) \leq C_{m,r,q}\delta, \quad \delta > 0. \qquad (8.3)$$

In the following two sections we will deal with the tightness of the sequence $(z^k)_{k\in\mathbb{N}}$ and of some auxiliary processes. We begin with the former.

8.1. *Tightness of the Sequence* $(z^k)_{k\in\mathbb{N}}$ *on* $\mathbb{L}$

As in Appendix B we set

$$\mathbb{L} = \mathbb{L}^1 \oplus \mathbb{L}^0 := C_w(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R})) \oplus C_w(\mathbb{R}_+; L^2_{\mathrm{loc}}(\mathbb{R})),$$

where $\mathbb{L}^k$, $k = 0, 1$, is a locally convex topological vector space of weakly continuous $H^k_{\mathrm{loc}}(\mathbb{R})$-valued functions defined on $\mathbb{R}_+$. Hence $\mathbb{L}$ is a locally convex topological vector space of weakly continuous $H^1_{loc}(\mathbb{R}) \times L^2_{loc}(\mathbb{R})$-valued functions defined on $\mathbb{R}_+$. Some useful properties of these spaces are discussed in Appendix B.

Lemma 8.1. *The sequence of laws of* (z^k) *constructed in Section* 6.4 *is tight on* $\mathbb{L}$.

Proof. Let us introduce the following open subsets of $\mathbb{L}^i$, with $a > 0$ and $m \in \mathbb{N}$,

$$J^i_m(a) = \left\{ h \in \mathbb{L}^i : \sup_{t\in[0,m]} |h(t)|_{H^i(-m,m)} > a \right\},$$
$$K^i_m(a) = \left\{ h \in \mathbb{L}^i : \sup_{0\le s<t\le m} \left[\frac{|h(t) - h(s)|_{H^{-1}(-m,m)}}{(t-s)^{\frac{1}{8}}} \right] > a \right\}.$$

The inequalities below follow from the Sobolev embedding theorems and Lemma 3.1.

$$|h|_{H^{-1}(-m,m)} \le (2m)^{\frac{1}{2}} |h|_{L^1(-m,m)}, \quad h \in L^1(-m,m), \tag{8.4}$$
$$\|\{\xi \mapsto h\xi\}\|_{\mathcal{J}_2(H_{\mu_k}, H^{-1}(-m,m))} \le c_\circ [\mu(\mathbb{R})]^{\frac{1}{2}} |h|_{L^2(-m,m)}, \quad h \in L^2(-m,m), \quad k \in \mathbb{N}. \tag{8.5}$$

Let us fix $\varepsilon > 0$. Since z_0 is an $H^1_{loc} \times L^2_{loc}(TM)$-valued random variable, for every $m \in \mathbb{N}$ we can find a number $r_m > 0$ such that $\mathbb{P}(S_{m,r_m}) > 1 - \frac{\varepsilon}{2\cdot 8^m}$. With the numbers $C_{m,r_m,q}$, $q = 1, 2$ having been defined in formula (8.2), we put

$$\begin{aligned}
\alpha_m &:= \varepsilon^{-1} \cdot 6 \cdot 8^m \cdot m^{\frac{7}{8}} [C_{m,r_m,1} + (8m)^{\frac{1}{2}} \cdot C_A \cdot C_{m,r_m,2}] \\
&\quad + [\varepsilon^{-1} \cdot 2 \cdot 8^m \cdot m \cdot \beta_m \cdot (1 + C_{m,r_m,8})]^{\frac{1}{8}}, \\
\beta_m &:= 3^{15} c_* C_Y^8 c_\circ^8 [\mu(\mathbb{R})]^4 (2m)^5, \quad C_A := \sup\{|A_p(\xi,\xi)| : |\xi|_{T_pM} = 1, p \in M\}, \\
C_Y &:= \sup\left\{ \frac{|Y^k(p,\xi,\eta)|}{1 + |\xi| + |\eta|} : k \in \mathbb{N}, \xi, \eta \in T_pM, p \in M \right\}.
\end{aligned}$$

Since

$$\mathbb{P}[u^k \in J^1_m(\alpha_m)] \le \mathbb{P}(\Omega \backslash S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{t\in[0,m]} |u^k(t)|_{H^1(-m,m)} > \alpha_m \right] \le \varepsilon 8^{-m}, \tag{8.6}$$
$$\mathbb{P}[v^k \in J^0_m(\alpha_m)] \le \mathbb{P}(\Omega \backslash S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{t\in[0,m]} |v^k(t)|_{L^2(-m,m)} > \alpha_m \right] \le \varepsilon 8^{-m}, \tag{8.7}$$

by taking into account inequality (8.8) and $|h|_{H^{-1}(-m,m)} \le |h|_{L^2(-m,m)}$, we infer that

$$\begin{aligned}
\mathbb{P}[u^k \in K^1_m(\alpha_m)] &\le \mathbb{P}(\Omega \backslash S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{0\le s<t\le m} \frac{\int_s^t |v^k(r)|_{L^2(-m,m)} dr}{(t-s)^{\frac{1}{8}}} > \alpha_m \right] \\
&\le \mathbb{P}(\Omega \backslash S_{m,r_m}) + \mathbb{P}\left[m^{\frac{7}{8}} \mathbf{1}_{S_{m,r_m}} \sup_{t\in[0,m]} |v^k(t)|_{L^2(-m,m)} > \alpha_m \right] \le \varepsilon 8^{-m}. \quad (8.8)
\end{aligned}$$

In order to prove an analogous estimate for the term $\mathbb{P}[v^k \in K_m^0(\alpha_m)]$ we define the following three auxiliary $L^2(-R,R)$-valued, for every $R>0$, processes:

$$I_1^k = \int_0^\cdot u_{xx}^k\,ds, \quad I_2^k = \int_0^\cdot [A_{u^k}(u_x^k, u_x^k) - A_{u^k}(v^k, v^k)]\,ds, \quad I_3^k = \int_0^\cdot Y^k(u^k, v^k, u_x^k)\,dW^k.$$

By inequality $|\partial_{xx}h|_{H^{-1}(-m,m)} \le |h|_{H^1(-m,m)}$ the process I_1^k satisfies the following inequality,

$$\begin{aligned}\mathbb{P}\left[I_1^k \in K_m^0\left(\frac{\alpha_m}{3}\right)\right] &\le \mathbb{P}(\Omega\setminus S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{0\le s<t\le m} \frac{\int_s^t |u_{xx}^k(r)|_{H^{-1}(-m,m)}dr}{(t-s)^{\frac18}} > \frac{\alpha_m}{3}\right] \\ &\le \mathbb{P}(\Omega\setminus S_{m,r_m}) + \mathbb{P}\left[m^{\frac78}\mathbf{1}_{S_{m,r_m}} \sup_{t\in[0,m]} |u_x^k(t)|_{L^2(-m,m)} > \frac{\alpha_m}{3}\right] \le \varepsilon 8^{-m}.\end{aligned} \tag{8.9}$$

Moreover, since $\mathbb{P}[I_2^k \in K_m^0(\alpha_m/3)]$ is smaller or equal than

$$\mathbb{P}(\Omega\setminus S_{m,r_m}) + \mathbb{P}\left[\mathbf{1}_{S_{m,r_m}} \sup_{0\le s<t\le m} \frac{\int_s^t |A_{u^k}(u_x^k, u_x^k) - A_{u^k}(v^k, v^k)|_{H^{-1}(-m,m)}dr}{(t-s)^{\frac18}} > \frac{\alpha_m}{3}\right],$$

by applying inequality (8.4), we infer that

$$\begin{aligned}&\mathbb{P}\left[I_2^k \in K_m^0\left(\frac{\alpha_m}{3}\right)\right] \\ &\quad\le \mathbb{P}\left[(8m)^{\frac12} C_A \mathbf{1}_{S_{m,r_m}} \sup_{0\le s<t\le m} \frac{\int_s^t |z^k(r)|^2_{H^1(-m,m)\times L^2(-m,m)}dr}{(t-s)^{\frac18}} > \frac{\alpha_m}{3}\right] \\ &\quad\quad + \frac{\varepsilon}{2\cdot 8^m} \le \frac{\varepsilon}{2\cdot 8^m} + \mathbb{P}\left[m^{\frac78}(8m)^{\frac12} C_A \mathbf{1}_{S_{m,r_m}} \sup_{t\in[0,m]} |z^k(t)|^2_{H^1(-m,m)} > \frac{\alpha_m}{3}\right] \le \frac{\varepsilon}{8^m}.\end{aligned} \tag{8.10}$$

Finally, by Proposition 3.2 we infer that

$$\begin{aligned}\mathbb{P}\left[I_3^k \in K_m^0\left(\frac{\alpha_m}{3}\right)\right] &\le \mathbb{P}(\Omega\setminus S_{m,r_m}) + \mathbb{P}\left[|\mathbf{1}_{S_{m,r_m}} I_3^k|_{C^{\frac18}([0,m];H^{-1}(-m,m))} > \frac{\alpha_m}{3}\right] \le \frac{\varepsilon}{2\cdot 8^m} \\ &\quad + \frac{3^8 c_*}{\alpha_m^8}\mathbb{E}\int_0^m \mathbf{1}_{S_{m,r_m}} |Y^k(u^k(s), v^k(s), u_x^k(s))|^8_{\mathcal{J}_2(H_{\mu_k}, H^{-1}(-m,m))} ds \le \frac{\varepsilon}{8^m}.\end{aligned} \tag{8.11}$$

Indeed, by (8.5) we have

$$\begin{aligned}|Y^k(u^k, v^k, \partial_x u^k)|^8_{\mathcal{J}_2(H_{\mu_k}, H^{-1}(-m,m))} &\le c_\circ^8[\mu(\mathbb{R})]^4 |Y^k(u^k, v^k, \partial_x u^k)|^8_{L^2(-m,m)} \\ &\le 3^7 C_Y^8 c_\circ^8 [\mu(\mathbb{R})]^4[(2m)^4 + |\partial_x u^k|^8_{L^2(-m,m)} + |v^k|^8_{L^2(-m,m)}] \\ &\le 3^7 C_Y^8 c_\circ^8 [\mu(\mathbb{R})]^4 (2m)^4 (1 + |z^k|^8_{H^1(-m,m)\times L^2(-m,m)}).\end{aligned}$$

The estimates (8.9)–(8.11) imply that

$$\mathbb{P}[v^k \in K_m^0(\alpha_m)] \le \sum_{j=1}^{3} \mathbb{P}\left[I_j^k \in K_m^0\left(\frac{\alpha_m}{3}\right)\right] \le \frac{3\varepsilon}{8^m}. \tag{8.12}$$

On the other hand, by Proposition B.1 the set C_ε defined below is compact in $\mathbb{L}$,

$$C_\varepsilon = \left\{ \bigcap_{m=1}^{\infty} [\mathbb{L}^1 \backslash (J_m^1(\alpha_m) \cup K_m^1(\alpha_m))] \right\} \times \left\{ \bigcap_{m=1}^{\infty} [\mathbb{L}^0 \backslash (J_m^0(\alpha_m) \cup K_m^0(\alpha_m))] \right\},$$

and, by inequalities (8.6)–(8.8) and (8.12) we infer that $\mathbb{P}[z^k \in C_\varepsilon] \ge 1 - \varepsilon$ for $k \in \mathbb{N}$. This concludes the proof of Lemma 8.1. □

8.2. *Tightness of the Auxiliary Processes*

In Section 7 we introduced the pseudointrinsic equation (7.1) in order to avoid the lack of convergence when passing to the limit in the intrinsic equation (6.5). However, there are other terms present in equation (7.1), denoted in what follows by $Q^k_{b,\varphi,Z}$, that might not converge to the corresponding limit term. Luckily these terms on one hand form a tight sequence and on the other are "uniformly small". This is made precise in the following Lemma 8.3. We begin with a useful, but somehow nonstandard, notation.

Notation 8.2. *If $a, b \in \mathbb{R}$ are such that $a < b$, then by ${}_0\mathrm{Lip}[a, b]$ we will denote a Banach space of all Lipschitz continuous functions $h : [a, b] \to \mathbb{R}$ such that $h(a) = 0$, equipped with a norm $|\cdot|_{{}_0\mathrm{Lip}[a,b]}$ defined by $|h|_{{}_0\mathrm{Lip}[a,b]} = \sup_{a\le s<t\le b}(t - s)^{-1}|h(t) - h(s)|$ for $h \in {}_0\mathrm{Lip}[a, b]$.*

Lemma 8.3. *Let b be a C_0^∞-class symmetric density on $\mathbb{R}$ with support in $(-1, 1)$, φ a smooth real function on $\mathbb{R}$ with support in $(-r, r)$, $Z : \mathbb{R}^n \to \mathbb{R}^n$ be a C_0^∞-class function such that $Z(p) \in T_pM$, $p \in M$. Assume that $(z^k)_{k=1}^\infty$ is the sequence of processes constructed in Section 6.4. Denote by $Q^k_{b,\varphi,Z}$, where $k \in \mathbb{N}$, a process defined by the following formula*

$$Q^k_{b,\varphi,Z}(t) = \int_0^t \langle [A_{u^k(s)}(u_x^k(s), u_x^k(s)) - A_{u^k(s)}(v^k(s), v^k(s))] \cdot Z \circ (b * u^k(s)), \varphi\rangle_{L^2(\mathbb{R})} ds, \quad t \in \mathbb{R}_+.$$

Then the sequence $(Q^k_{b,\varphi,Z} : k \in \mathbb{N})$ is tight on $C(\mathbb{R}_+)$. Moreover, there exists a constant ζ depending on A, Z and φ such that

$$|Q^k_{b,\varphi,Z}|_{{}_0\mathrm{Lip}[0,l]} \le \zeta |z^k|^2_{L^\infty((0,l);H^1(-r,r)\times L^2(-r,r))} |b * u^k - u^k|_{L^\infty((0,l);L^\infty(-r,r))} \tag{8.13}$$

$$\le \zeta |z^k|^2_{L^\infty((0,l);H^1(-r-1,r+1)\times L^2(-r-1,r+1))}. \tag{8.14}$$

Proof. Since $u^k(s) \in M$, both $A_{u^k(s)}(u_x^k(s), u_x^k(s))$ and $A_{u^k(s)}(v^k(s), v^k(s))$ belong to $N_{u^k(s)}M$ and $Z(u^k(s))$ belongs $T_{u^k(s)}M$, and Z is globally Lipschitz. Since we can find a constant $\zeta' > 0$ such that

$$\begin{aligned}|Q^k_{b,\varphi,Z}|_{{}_0\mathrm{Lip}[0,l]} &\leq |\varphi|_{L^\infty(\mathbb{R})}|Z\circ(b*u^k) - Z\circ u^k|_{L^\infty((0,l),L^\infty(-r,r))}\\ &\quad\times |A_{u^k}(u^k_x, u^k_x) - A_{u^k}(v^k, v^k)|_{L^\infty((0,l),L^1(-r,r))}\\ &\leq \zeta'|z^k|^2_{L^\infty((0,l);H^1(-r,r)\times L^2(-r,r))}|b*u^k - u^k|_{L^\infty((0,l);L^\infty(-r,r))}\\ &\leq \zeta' r_M|z^k|^2_{L^\infty((0,l);H^1(-r-1,r+1)\times L^2(-r-1,r+1))},\end{aligned}$$

where $r_M := \sup_{x\in M}|x|$. Hence the inequalities (8.13), (8.14) follow. In order to deal with the tightness part of the lemma let us fix $\varepsilon > 0$ and define $m_l = \min\{m \in \mathbb{N} : m \geq r+1, m \geq l\}$, $l \in \mathbb{N}$. Then we define a sequence $(J^l)_{l=1}^\infty$ of subsets of $C(\mathbb{R}_+)$ by

$$J^l = \{h \in C(\mathbb{R}_+) : h(0) = 0, |h|_{{}_0\mathrm{Lip}[0,l]} \leq \varepsilon^{-1}\cdot 3^l\cdot\zeta\cdot C_{m_l,r_l,2}\}, \quad l \in \mathbb{N}.$$

As before, using the notation (8.1), for every $l \in \mathbb{N}$ we can find $r_l > 0$ so that $\mathbb{P}(S_{m_l,r_l}) > 1 - \frac{\varepsilon}{3^l}$. Then by (8.3) we have the following inequalities

$$\begin{aligned}\mathbb{P}[Q^k_{b,\varphi,Z} \notin J^l] &\leq \mathbb{P}(\Omega\backslash S_{m_l,r_l}) + \mathbb{P}[\mathbf{1}_{S_{m_l,r_l}}|Q^k_{b,\varphi,Z}|_{{}_0\mathrm{Lip}[0,l]} > \varepsilon^{-1}\cdot 3^l\cdot\zeta\cdot C_{m_l,r_l,2}]\\ &\leq \varepsilon 3^{-l} + \mathbb{P}[\mathbf{1}_{S_{m_l,r_l}}|z^k|^2_{L^\infty((0,m_l);H^1(-m_l,m_l)\times L^2(-m_l,m_l))} > \varepsilon^{-1}\cdot 3^l\cdot C_{m_l,r_l,2}]\\ &\leq 2\varepsilon 3^{-l}.\end{aligned}$$

Hence $\mathbb{P}[Q^k_{b,\varphi,Z} \in \bigcap_{l=1}^\infty J^l] \geq 1-\varepsilon$. Since the set $\bigcap_{l=1}^\infty J^l$ is a compact subset of $C(\mathbb{R}_+)$, as follows from the Arzela-Ascoli Theorem, in view of the Prokhorov Theorem the proof of Lemma 8.1 is concluded. □

9. The Skorokhod Representation Theorem

Let us consider the following objects.

- A smooth symmetric density b on $\mathbb{R}$ with support in $(-1,1)$, $b_l = lb(l\cdot)$, $l \in \mathbb{N}$,
- the sequence (z^k) of processes constructed in Section 6.4,
- a family $(\beta^{ij})_{i,j\in\mathbb{N}}$ of i.i.d. Brownian Motions used in Section 6.2,
- the orthonormal bases $(\xi_{ij})_{i\in\mathbb{N},j\in J_i}$ of H_{v_i} introduced Section 6.2,
- $Q^k_{b,\varphi,Z}$ the processes from Lemma 8.3,
- (φ_m) the sequence in $C^\infty(\mathbb{R})$ with supports in $(-r_m, r_m)$ from Proposition D.1,
- the smooth vector fields $(Z^1,\ldots,Z^N)$ satisfying (A.1),
- the spaces $\mathbb{L}^k$ introduced in Section B and the extension $\overline{Y}$ of Y from (6.3).

Remark 9.1. Using Proposition A.1, each Z^i can be extended to a C_0^∞-mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, denoted again by Z^i.

Let us recall that by Lemmata 8.1 and 8.3, the sequence of laws of

$$(u^k_0, v^k_0, u^k, v^k, (\beta^{ij})_{i,j}, (Q^k_{b_l,\varphi_m,Z^\gamma})_{l,m\in\mathbb{N},\gamma\in\{1,\ldots,N\}})_{k\in\mathbb{N}}$$

is tight on the space $H^1_{\mathrm{loc}}(\mathbb{R})\times L^2_{\mathrm{loc}}(\mathbb{R})\times\mathbb{L}^1\times\mathbb{L}^0\times[C(\mathbb{R}_+)]^{\mathbb{N}^2}\times[C(\mathbb{R}_+]^{\mathbb{N}^2\times\{1,\ldots,N\}}$. Moreover, as remarked in Section 6.1, the sequence (u^k_0, v^k_0) converges in $H^1_{\mathrm{loc}}(\mathbb{R})\times$

$L^2_{\mathrm{loc}}(\mathbb{R})$ to z_0 on Ω. Hence, by the Skorokhod-Jakubowski Theorem C.1, there exists a subsequence (k_α) and the following Borel measurable maps with σ-compact range, with $[0, 1]$ being equipped with the Borel σ-algebra and the Lebesgue measure denoted by Leb (the integration with respect to the measure Leb will be denoted by the old symbol $\mathbb{E}$)

- $\mathbf{u} : [0, 1] \to \mathbb{L}^1$, $\mathbf{u}^\alpha : [0, 1] \to C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R}))$, $\alpha \in \mathbb{N}$,
- $\mathbf{v} : [0, 1] \to \mathbb{L}^0$, $\mathbf{v}^\alpha : [0, 1] \to C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))$, $\alpha \in \mathbb{N}$,
- $\mathbf{B}^\alpha_{ij} : [0, 1] \to C(\mathbb{R}_+)$, $\mathbf{B}_{ij} : [0, 1] \to C(\mathbb{R}_+)$, $\alpha, i, j \in \mathbb{N}$,
- $\mathbf{Q}^\alpha_{lm\gamma} : [0, 1] \to C(\mathbb{R}_+)$, $\mathbf{Q}_{lm\gamma} : [0, 1] \to C(\mathbb{R}_+)$, $\alpha, l, m \in \mathbb{N}$, $\gamma \in \{1, \ldots, N\}$,

such that for each $\alpha \in \mathbb{N}$, the laws on the Borel σ-algebra of

$$C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R})) \times C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R})) \times [C(\mathbb{R}_+)]^{\mathbb{N}^2} \times [C(\mathbb{R}_+)]^{\mathbb{N}^2 \times \{1,\ldots,N\}},$$

of $(u^{k_\alpha}, v^{k_\alpha}, (\beta^{ij})_{i,j\in\mathbb{N}}, (Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma})_{l,m\in\mathbb{N},\gamma\in\{1,\ldots,N\}})$ under the probability measure $\mathbb{P}$ and of $\big(\mathbf{u}^\alpha, \mathbf{v}^\alpha, (\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}, (\mathbf{Q}^\alpha_{lm\gamma})_{l,m\in\mathbb{N},\gamma\in\{1,\ldots,N\}}\big)$ under the Lebesgue measure Leb are equal, and

- $(\mathbf{u}^\alpha(0), \mathbf{v}^\alpha(0))$ converges in $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}$ to $(\mathbf{u}(0), \mathbf{v}(0))$ on $[0, 1]$,
- $\mathbf{z}^\alpha = (\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ converges in $\mathbb{L}^1 \times \mathbb{L}^0$ to $\mathbf{z} = (\mathbf{u}, \mathbf{v})$ on $[0, 1]$,
- $\mathbf{B}^\alpha_{ij}$ converges in $C(\mathbb{R}_+)$ to $\mathbf{B}_{ij}$ on $[0, 1]$ for every $i, j \in \mathbb{N}$,
- $\mathbf{Q}^\alpha_{ml\gamma}$ converges in $C(\mathbb{R}_+)$ to $\mathbf{Q}_{ml\gamma}$ on $[0, 1]$ for every $m, l \in \mathbb{N}$, $\gamma \in \{1, \ldots, N\}$.

Remark 9.2. In fact, the Skorokhod–Jakubowski theorem implies only that $\mathbf{u}^\alpha$ and $\mathbf{v}^\alpha$ are $\mathbb{L}^1$ and, respectively $\mathbb{L}^0$-valued random variables. However, since the embeddings $C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R})) \hookrightarrow \mathbb{L}^1$ and $C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R})) \hookrightarrow \mathbb{L}^0$ are continuous, in view of Proposition C.2, we infer that these sets are Borel subsets of $\mathbb{L}^0$ and $\mathbb{L}^1$ respectively and that

$$\mathrm{Leb}(\{\mathbf{u}^\alpha \in C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R}))\}) = \mathbb{P}\{u^{k_\alpha} \in C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R}))\}) = 1,$$
$$\mathrm{Leb}(\{\mathbf{v}^\alpha \in C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))\}) = \mathbb{P}\{v^{k_\alpha} \in C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))\}) = 1.$$

Hence we may assume that for every $\alpha \in \mathbb{N}$, $\mathbf{u}^\alpha$, respectively $\mathbf{v}^\alpha$, is a random variable with values in $C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R}))$, respectively $C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))$.

Notation 9.3. *By $\mathscr{B}_t$, where $t \geq 0$, we will denote the σ-algebra on* $[0, 1]$ *generated by the random variables* $\mathbf{v}(0)$, $\mathbf{u}(s)$, $\mathbf{B}_{ij}(s)$ *and* $\mathbf{Q}_{ml\gamma}(s)$ *for* $s \in [0, t]$, $i, j, m, l \in \mathbb{N}$ *and* $\gamma \in \{1, \ldots, N\}$. *By* $\mathbb{B}$ *we will denote the filtration* $(\mathscr{B}_t)_{t\geq 0}$. *Denote finally by* $\bar{\mathbb{B}} = (\bar{\mathscr{B}}(t))_{t\geq 0}$ *the natural augmentation of the filtration* $\mathbb{B} = (\mathscr{B}(t))_{t\geq 0}$.

Let us point out here that, as in [42], in view of [15, p. 75], in order to show that a process is a $\bar{\mathbb{B}}$-martingale it is enough to show that it is a $\mathbb{B}$-martingale.

9.1. *Uniform Local Energy and Other Inequalities*

The following results are an immediate consequence of Theorem 6.6 and the equality of the laws of $\mathbf{z}^\alpha$ and z^{k_α} on the Borel σ-algebra over $C(\mathbb{R}_+; H^2_{\mathrm{loc}}(\mathbb{R})) \times C(\mathbb{R}_+; H^1_{\mathrm{loc}}(\mathbb{R}))$. Define sets first, for $m \in \mathbb{N}$, $r > 0$,

$$B_{m,r} = \{z \in H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R}) : |z|_{H^1(-m,m)\times L^2(-m,m)} < r\}.$$

Corollary 9.4. *Let T, L, D and c_* be the same as in Theorem* 6.6. *Then*

$$\mathbb{E}\mathbf{1}_D(\mathbf{z}^\alpha(0)) \sup_{s\in[0,t]} \mathbf{e}_{k_\alpha,L,T}(s,\mathbf{z}^\alpha(s)) \le 4e^{tc_*}\mathbb{E}\mathbf{1}_D(\mathbf{z}^\alpha(0))\mathbf{e}_{k_\alpha,L,T}(0,\mathbf{z}^\alpha(0)), \quad t\in[0,T]. \quad (9.1)$$

In particular, for every $m\in\mathbb{N}$, $r>0$ and $q\in(0,\infty)$,

$$\mathbf{C}_{m,r,q} := \sup_{\alpha\in\mathbb{N}} \mathbb{E}\left[\mathbf{1}_{B_{2m,r}}(\mathbf{z}^\alpha(0)) \sup_{t\in[0,m]} |\mathbf{z}^\alpha(t)|^q_{H^1(-m,m)\times L^2(-m,m)}\right] < \infty. \quad (9.2)$$

Corollary 9.5. *Let T, L, D and c_* be the same as in Theorem* 6.6. *Then*

$$\mathbb{E}\mathbf{1}_D(\mathbf{z}(0)) \sup_{s\in[0,t]} \mathbf{e}_{T,L}(s,\mathbf{z}(s)) \le 4e^{tc_*}\mathbb{E}\mathbf{1}_D(\mathbf{z}(0))\mathbf{e}_{T,L}(0,\mathbf{z}(0)),\ t\in[0,T], \quad (9.3)$$

where $\mathbf{e}_{T,L}$ was defined in (4.4). *In particular, for all $m\in\mathbb{N}$, $q>0$ and $r>0$,*

$$\mathbb{E}\left[\mathbf{1}_{B_{2m,r}}(\mathbf{z}(0)) \sup_{t\in[0,m]} |\mathbf{z}(t)|^q_{H^1(-m,m)\times L^2(-m,m)}\right] \le \mathbf{C}_{m,r,q}. \quad (9.4)$$

Proof of Corollary 9.5. Let us fix $T, L>0$. Let us fix $r>0$ such that $\mathrm{Leb}(\{\mathbf{z}(0)\in\partial B_{T,r}\})=0$ and assume that $D\subseteq H^1_{loc}\times L^2_{loc}$ is an open set such that $\mathrm{Leb}(\{\mathbf{z}(0)\in\partial D\})=0$. By Corollary 9.4 applied to $D\cap B_{T,r}$, we obtain by the Fatou Lemma and [37, 2.8 a,c] that

$$\begin{aligned}\mathbb{E}\mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}(0)) \sup_{s\in[0,t]} \mathbf{e}_{L,T}(s,\mathbf{z}(s)) &\le \liminf_{\alpha\to\infty} \mathbb{E}\mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}^\alpha(0)) \sup_{s\in[0,t]} \mathbf{e}_{k_\alpha,L,T}(s,\mathbf{z}^\alpha(s))\\ &\le 4e^{tc_*}\mathbb{E}\limsup_{\alpha\to\infty} \mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}^\alpha(0))\mathbf{e}_{k_\alpha,L,T}(0,\mathbf{z}^\alpha(0))\\ &\le 4e^{tc_*}\mathbb{E}\mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}(0))\mathbf{e}_{L,T}(0,\mathbf{z}(0)), \quad t\in[0,T],\end{aligned}$$

as the integrands on the right hand side of the first line are uniformly bounded in $\alpha\in\mathbb{N}$. Since open sets D satisfying $\mathrm{Leb}(\{\mathbf{z}(0)\in\partial D\})=0$ form a basis of topology on the separable metric space $H^1_{loc}\times L^2_{loc}$, we infer that for every open set D, and consequently for every G_δ-set D,

$$\mathbb{E}\mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}(0)) \sup_{s\in[0,t]} \mathbf{e}_{L,T}(s,\mathbf{z}(s)) \le 4e^{tc_*}\mathbb{E}\mathbf{1}_{D\cap B_{T,r}}(\mathbf{z}(0))\mathbf{e}_{L,T}(0,\mathbf{z}(0)), \quad t\in[0,T]. \quad (9.5)$$

Finally, inequality (9.5) holds for every Borel set D by regularity of the law of $\mathbf{z}(0)$. In the last step, we let $r\nearrow\infty$ running over $r>0$ satisfying $\mathrm{Leb}(\{\mathbf{z}(0)\in\partial B_{T,r}\})=0$. □

Corollary 9.6. *For every $m\in\mathbb{N}$, there exists a constant ζ°_m such that for every $\gamma\in\{1,\dots,N\}$, $T>0$, $\kappa\in[\max\{r_m+1,T\},\infty)\cap\mathbb{N}$, $\alpha, l\in\mathbb{N}$ and for every $r>0$,*

$$\mathbb{E}\{\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))|\mathbf{Q}_{lm\gamma}|_{{}_0\mathrm{Lip}[0,T]}\} \le \zeta^\circ_m \mathbf{C}^{\frac{1}{2}}_{\kappa,r,4}\{\mathbb{E}[\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))|b_l*\mathbf{u}-\mathbf{u}|^2_{L^\infty((0,T);L^\infty(-r_m,r_m))}]\}^{\frac{1}{2}}.$$

Proof of Corollary 9.6. Since the laws on $\mathcal{B}(C(\mathbb{R}_+; H^2_{\text{loc}}(\mathbb{R}) \times H^1_{\text{loc}}(\mathbb{R}) \times \mathbb{R}))$ of the random variables $(u^{k_\alpha}, v^{k_\alpha}, Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma})$ and $(\mathbf{u}^\alpha, \mathbf{v}^\alpha, \mathbf{Q}^\alpha_{lm\gamma})$ are equal, by (8.13) we infer that for every $\gamma \in \{1, \ldots, N\}$ and $\alpha, l \in \mathbb{N}$ almost surely,

$$|\mathbf{Q}^\alpha_{lm\gamma}|_{_0\text{Lip}[0,T]} \leq \zeta^\circ_m |\mathbf{z}^\alpha|^2_{L^\infty((0,T);H^1(-r_m,r_m)\times L^2(-r_m,r_m))} |b_l * \mathbf{u}^\alpha - \mathbf{u}^\alpha|_{L^\infty((0,T);L^\infty(-r_m,r_m))}.$$

The inequality (9.2), with $\mathbf{I} = \mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))$, now implies that

$$\mathbb{E}\{\mathbf{I}|\mathbf{Q}^\alpha_{lm\gamma}|_{_0\text{Lip}[0,T]}\} \leq \zeta^\circ_m \mathbf{C}^{\frac{1}{2}}_{\kappa,r,4} \{\mathbb{E}[\mathbf{I}|b_l * \mathbf{u}^\alpha - \mathbf{u}^\alpha|^2_{L^\infty((0,T);L^\infty(-r_m,r_m))}]\}^{\frac{1}{2}}. \tag{9.6}$$

Since the weak convergence in $H^1_{\text{loc}}(\mathbb{R})$ implies the strong convergence in $L^\infty_{\text{loc}}(\mathbb{R})$ we infer that on $[0, 1]$

$$\lim_{\alpha\to\infty} |b_l * \mathbf{u}^\alpha - \mathbf{u}^\alpha|_{L^\infty((0,T);L^\infty(-r_m,r_m))} = |b_l * \mathbf{u} - \mathbf{u}|_{L^\infty((0,T);L^\infty(-r_m,r_m))}$$

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))|b_l * \mathbf{u}^\alpha - \mathbf{u}^\alpha|^4_{L^\infty((0,T);L^\infty(-r_m,r_m))}] \\
&\quad \leq 16\mathbb{E}[\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))|\mathbf{u}^\alpha|^4_{L^\infty((0,\kappa);L^\infty(-\kappa,\kappa))}] \\
&\quad \leq c_\kappa \mathbb{E}[\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))|\mathbf{u}^\alpha|^4_{L^\infty((0,\kappa);H^1(-\kappa,\kappa))}] \leq c_\kappa \mathbf{C}_{\kappa,r,4}.
\end{aligned}$$

If $r > 0$ is such that $\text{Leb}(\{z(0) : |\mathbf{z}(0)|_{H^1(-2\kappa,2\kappa)\times L^2(-2\kappa,2\kappa)} = r\}) = 0$, the final result follows by letting $\alpha \to \infty$ in inequality (9.6) and applying the Fatou Lemma. Since the set of such numbers r is dense in $(0, \infty)$, the case of an exceptional r follows from the monotonicity. □

9.2. *Identification of the Random Variables on* [0, 1]

In this whole section we assume that the sequence (φ_m) is as in Proposition D.1.

Lemma 9.7. *There exists a set* $\widehat{\Omega} \subset [0, 1]$ *of full* Leb*-measure such that for every* $\hat{\omega} \in \widehat{\Omega}$, *and for all* $R > 0$ *and* $t \geq 0$, *the equality* $\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t \mathbf{v}(s)ds$ *holds in* $L^2(-R, R)$.

Proof. Let us note that the sequence (φ_m) separate points of $L^1_{\text{loc}}(\mathbb{R})$. Hence, it is enough to find set $\widehat{\Omega} \subset [0, 1]$ such that $\text{Leb}(\widehat{\Omega}) = 1$ and for every $\hat{\omega} \in \widehat{\Omega}$,

$$\langle u(\cdot), \varphi_m \rangle_{L^2(\mathbb{R})} - \langle u(0), \varphi_m \rangle_{L^2(\mathbb{R})} - \int_0^\cdot \langle v(s), \varphi_m \rangle_{L^2(\mathbb{R})} ds \quad \text{in } C(\mathbb{R}_+). \tag{9.7}$$

For this aim we introduce the following continuous mappings $B_m : \mathbb{L}^1 \times \mathbb{L}^0 \to C(\mathbb{R}_+)$

$$B_m(u, v) = \langle u(\cdot), \varphi_m \rangle_{L^2(\mathbb{R})} - \langle u(0), \varphi_m \rangle_{L^2(\mathbb{R})} - \int_0^\cdot \langle v(s), \varphi_m \rangle_{L^2(\mathbb{R})} ds.$$

Since, for every $\alpha \in \mathbb{N}$, the laws on the Borel σ-algebra on $\mathbb{L}^1 \times \mathbb{L}^0$ of $(u^{k_\alpha}, v^{k_\alpha})$ and $(\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ are equal and $(\mathbf{u}^\alpha, \mathbf{v}^\alpha)$ converges in $\mathbb{L}^1 \times \mathbb{L}^0$ to $(\mathbf{u}, \mathbf{v})$ on $[0, 1]$, $B_m(u^{k_\alpha}, v^{k_\alpha}) = 0$ Leb almost surely and $B_m(\mathbf{u}, \mathbf{v}) = \lim_{\alpha\to\infty} B_m(\mathbf{u}^\alpha, \mathbf{v}^\alpha) = 0$ Leb-a.s., we infer that $B_m(\mathbf{u}^\alpha, \mathbf{v}^\alpha) = 0$ Leb almost surely. This completes the proof of (9.7) and so the result follows. □

Corollary 9.8. *The process* $\mathbf{v}$ *has* $L^2_{\mathrm{loc}}(\mathbb{R})$*-valued weakly continuous paths and is* $\bar{\mathbb{B}}$*-adapted.*

Proof. Let $t > 0$, $m \in \mathbb{N}$ and $j \in \mathbb{N}$. Then $a_j(t) = j\big(\langle \mathbf{u}(t), \varphi_m\rangle_{L^2(\mathbb{R})} - \langle \mathbf{u}(t - j^{-1})_+, \varphi_m\rangle_{L^2(\mathbb{R})}\big)$ is $\mathscr{B}_t$-measurable and $a_j(t) \to \langle \mathbf{v}(t), \varphi_m\rangle_{L^2(\mathbb{R})}$ Leb-almost surely by Lemma 9.7. Hence $\langle \mathbf{v}(t), \varphi_m\rangle_{L^2(\mathbb{R})}$ is $\mathscr{B}_t$-measurable. Finally, (φ_m) generates $\mathscr{B}(L^2_{\mathrm{loc}}(\mathbb{R}))$ by Proposition D.1. □

Lemma 9.9. *For every* $t \geq 0$, $\mathbf{z}(t) \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ Leb*-almost surely.*

Proof. Notice that the $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ is a closed subspace of the space $H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$ under the weak convergence of sequences. Therefore, since for all $\alpha \in \mathbb{N}$ and $t \geq 0$, $z^{k_\alpha}(t)$ has the same law as $\mathbf{z}^\alpha(t)$ and $\mathbf{z}^\alpha(t) \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ a.s., we infer that $\mathbf{z}(t) \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ a.s. Since the paths of the process $\mathbf{z}$ are $H^1_{\mathrm{loc}}(\mathbb{R}) \times L^2_{\mathrm{loc}}(\mathbb{R})$-valued weakly continuous, we can exchange the order of "*a.s.*" and "$t \geq 0$". This concludes the proof. □

Lemma 9.10. *The processes* $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$ *are independent standard* $\mathbb{B}$*-Wiener processes.*

Proof. Let us observe that for every α the laws of $\mathscr{B}[C(\mathbb{R}_+)]^{\mathbb{N}^2}$-valued random variables $(\beta^{ij})_{i,j\in\mathbb{N}}$ and $(\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}$ are equal. Moreover, $(\mathbf{B}^\alpha_{ij})_{i,j\in\mathbb{N}}$ converges in $[C(\mathbb{R}_+)]^{\mathbb{N}^2}$ as $\alpha \to \infty$ to $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$. Therefore we infer that the processes $(\mathbf{B}_{ij})_{i,j\in\mathbb{N}}$ are independent for all $0 \leq t_0 < t_1$, the random variable $(t_1 - t_0)^{-\frac{1}{2}}[\mathbf{B}_{ij}(t_1) - \mathbf{B}_{ij}(t_0)]$ is $N(0, 1)$. Let us fix $\kappa \in \mathbb{N}$ and $0 \leq r_1 \leq \cdots \leq r_\kappa \leq t_0 < t_1$. Put $\bar{\kappa} = \kappa + \kappa^2 + \kappa^3(N + 1)$ and consider the following $\mathbb{R}^{\bar{\kappa}}$-valued random variables,

$$
\begin{aligned}
O^k &= (\langle v_0^k, \varphi_m\rangle_{L^2(\mathbb{R})}, \langle u^k(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}, \beta^{ij}(r_\delta), Q^k_{b_l,\varphi_m,Z^\gamma}(r_\delta))_{\max\{i,j,l,m,\delta\}\leq\kappa,\gamma\leq N}\\
\mathbf{O}^\alpha &= (\langle \mathbf{v}^\alpha(0), \varphi_m\rangle_{L^2(\mathbb{R})}, \langle \mathbf{u}^\alpha(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}, \mathbf{B}^\alpha_{ij}(r_\delta), \mathbf{Q}^\alpha_{lm\gamma}(r_\delta))_{\max\{i,j,l,m,\delta\}\leq\kappa,\gamma\leq N}\\
\mathbf{O} &= (\langle \mathbf{v}(0), \varphi_m\rangle_{L^2(\mathbb{R})}, \langle \mathbf{u}(r_\delta), \varphi_m\rangle_{L^2(\mathbb{R})}, \mathbf{B}_{ij}(r_\delta), \mathbf{Q}_{lm\gamma}(r_\delta)))_{\max\{i,j,l,m,\delta\}\leq\kappa,\gamma\leq N}.
\end{aligned}
\tag{9.8}
$$

The law of O^{k_α} under $\mathbb{P}$ coincides with the law of $\mathbf{O}^\alpha$ under Leb for every α and moreover $\mathbf{O}^\alpha$ converges to $\mathbf{O}$ on $[0, 1]$. Hence, if $g_0 \in C_b(\mathbb{R}^{\bar{\kappa}})$ and $g_1 \in C_b(\mathbb{R})$, then for every $I, J \in \mathbb{N}$ we have the following sequence of equalities:

$$
\begin{aligned}
&[\mathbb{E}g_0(\mathbf{O})][\mathbb{E}g_1(\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0))]\\
&\quad= \lim_{\alpha\to\infty}[\mathbb{E}g_0(\mathbf{O}^\alpha)][\mathbb{E}g_1(\mathbf{B}^\alpha_{IJ}(t_1) - \mathbf{B}^\alpha_{IJ}(t_0))]\\
&\quad= \lim_{\alpha\to\infty}[\mathbb{E}g_0(O^{k_\alpha})][\mathbb{E}g_1(\beta^{IJ}(t_1) - \beta^{IJ}(t_0))] = \lim_{\alpha\to\infty}\mathbb{E}[g_0(O^{k_\alpha})g_1(\beta^{IJ}(t_1) - \beta^{IJ}(t_0))]\\
&\quad= \lim_{\alpha\to\infty}\mathbb{E}[g_0(\mathbf{O}^\alpha)g_1(\mathbf{B}^\alpha_{IJ}(t_1) - \mathbf{B}^\alpha_{IJ}(t_0))] = \mathbb{E}[g_0(\mathbf{O})g_1(\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0))].
\end{aligned}
$$

Hence we infer that for all $I, J \in \mathbb{N}$, the random variable $\mathbf{B}_{IJ}(t_1) - \mathbf{B}_{IJ}(t_0)$ is independent from the σ-algebra $\mathscr{B}_{t_0}$. The proof is complete. □

Remark 9.11. The process $\mathbf{W}(\varphi) = \sum_{i=1}^\infty \sum_{j\in J_i} \mathbf{B}_{ij}\xi_{ij}(\varphi)$, $\varphi \in \mathscr{S}$ is a spatially homogeneous $\mathbb{B}$-Wiener process with the spectral measure μ.

Recalling that $\overline{Y}$ is an extension of Y defined in formula (6.3) let us define functions $U^l = b_l * u$, $V^l = b_l * v$ and $\mathcal{V}_{lm\gamma}$, $\mathcal{V}^k_{lm\gamma}$, $\mathcal{V}^\infty_{lm\gamma}$, $\mathcal{V}^{ijk}_{lm\gamma}$, $\mathcal{V}^{ij\infty}_{lm\gamma}$ from $\mathbb{L}^1 \times \mathbb{L}^0$ to $C(\mathbb{R}_+)$ by

$$\mathcal{V}_{lm\gamma}(z)(t) = \langle v(t) \cdot Z^\gamma(U^l(t)), \varphi_m\rangle - \langle v(0) \cdot Z^\gamma(U^l(0)), \varphi_m\rangle + \int_0^t \langle u_x \cdot (Z^\gamma)'_{U^l} U^l_x, \varphi_m\rangle ds$$

$$+ \int_0^t \langle u_x \cdot Z^\gamma(U^l), \partial_x \varphi_m\rangle ds - \int_0^t \langle v \cdot (Z^\gamma)'_{U^l} V^l, \varphi_m\rangle ds$$

$$\mathcal{V}^k_{lm\gamma}(z)(t) = \sum_{i=1}^k \sum_{j\in J_i} \int_0^t \langle [Y^k(u, v, u_x) \cdot Z^\gamma(U^l)]\xi_{ij}, \varphi_m\rangle^2 ds$$

$$\mathcal{V}^\infty_{lm\gamma}(z)(t) = \sum_{i=1}^\infty \sum_{j\in J_i} \int_0^t \langle [\overline{Y}(u, v, u_x) \cdot Z^\gamma(U^l)]\xi_{ij}, \varphi_m\rangle^2 ds$$

$$\mathcal{V}^{ijk}_{lm\gamma}(z)(t) = \mathbf{1}_{[i\le k]\cap[j\in J_i]} \int_0^t \langle [Y^k(u, v, u_x) \cdot Z^\gamma(U^l)]\xi_{ij}, \varphi_m\rangle ds,$$

$$\mathcal{V}^{ij\infty}_{lm\gamma}(z)(t) = \mathbf{1}_{[j\in J_i]} \int_0^t \langle [\overline{Y}(u, v, u_x) \cdot Z^\gamma(U^l)]\xi_{ij}, \varphi_m\rangle ds.$$

Lemma 9.12. *Assume that $k, l, m \in \mathbb{N}$ and $\gamma \in \{1, \ldots, N\}$. Then the functions $\mathcal{V}_{lm\gamma}$, $\mathcal{V}^k_{lm\gamma}$, $\mathcal{V}^\infty_{lm\gamma}$, $\mathcal{V}^{ijk}_{lm\gamma}$ and $\mathcal{V}^{ij\infty}_{lm\gamma}$ are sequentially continuous mappings from $\mathbb{L}^1 \times \mathbb{L}^0$ to $C(\mathbb{R}_+)$ and, if z^k converges to z in $\mathbb{L}^1 \times \mathbb{L}^0$, then $\mathcal{V}^k_{lm\gamma}(z^k)$ converges to $\mathcal{V}^\infty_{lm\gamma}(z)$ and $\mathcal{V}^{ijk}_{lm\gamma}(z^k)$ converges to $\mathcal{V}^{ij\infty}_{lm\gamma}(z)$ in $C(\mathbb{R}_+)$.*

Proof. It is enough to apply the Lebesgue Dominated Convergence Theorem. Indeed, if $z^k = (u^k, v^k)$ converges to $z = (u, v)$ in $\mathbb{L}^1 \times \mathbb{L}^0$ then, for every $R > 0$,

$$\lim_{k\to\infty} \left(\sup_{t\in[0,R]} |b_l * v^k(t) - b_l * v(t)|_{C([-R,R])} + \sup_{t\in[0,R]} |u^k(t) - u(t)|_{C([-R,R])} \right) = 0,$$

if h^k converge to h uniformly on $[0, R] \times [-R, R]$ then

$$\lim_{k\to\infty} \sup_{t\in[0,R]} |\langle v^k(t), h^k(t)\rangle_{L^2(-R,R)} - \langle v(t), h(t)\rangle_{L^2(-R,R)}| = 0$$

and, by Proposition 3.1,

$$\sum_{i=1}^\infty \sum_{j\in J_i} |\xi_{ij}(x)|^2 = \frac{1}{2\pi} \sum_{i=1}^\infty \nu_i(\mathbb{R}) = \frac{1}{2\pi}\mu(\mathbb{R}), \quad x \in \mathbb{R}. \tag{9.9}$$

□

Lemma 9.13 (Pseudointrinsic Equation). *Under the above assumptions,*

$$\langle \mathbf{v}(t) \cdot Z^\gamma(\mathbf{u}^l(t)), \varphi_m\rangle = \mathbf{Q}_{lm\gamma}(t) - \int_0^t [\langle \mathbf{u}_x \cdot (Z^\gamma)'_{\mathbf{u}^l} \mathbf{u}^l_x, \varphi_m\rangle + \langle \mathbf{u}_x \cdot Z^\gamma(\mathbf{u}^l), \partial_x \varphi_m\rangle] ds$$

$$+ \langle \mathbf{v}_0 \cdot Z^\gamma(\mathbf{u}^l_0), \varphi_m\rangle + \int_0^t \langle \mathbf{v} \cdot (Z^\gamma)'_{\mathbf{u}^l} \mathbf{v}^l, \varphi_m\rangle ds$$

$$+ \int_0^t \langle [Y(\mathbf{z}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u}^l)] d\mathbf{W}, \varphi_m\rangle$$

holds for every $t \geq 0$, $m, l \in \mathbb{N}$, $\gamma \leq N$ *a.s., where* $(\mathbf{u}^l, \mathbf{v}^l) = b_l * \mathbf{z}$ *and* $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$.

Remark 9.14. The equation in Lemma 9.13 is similar but not identical to the pseudointrinsic equation (7.1). For instance, the residual term $\mathbf{Q}_{lm\gamma}$ is not an indefinite integral here.

Proof. Let us begin with fixing $T > 0$, $m, l \in \mathbb{N}$, $\gamma \in \{1, \ldots, N\}$, $\mathbb{N} \ni \kappa \geq \max\{t, r_m + 1\}$. Let us take $0 \leq t_0 < t_1 \leq T$. Then we observe that using the notation of Corollary 9.4 and Corollary 9.5, the following equality

$$\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t) - Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t) = \int_0^t \langle [Y^{k_\alpha}(u^{k_\alpha}, v^{k_\alpha}, u^{k_\alpha}_x) \cdot Z^\gamma(b_l * u^{k_\alpha})]dW^{k_\alpha}, \varphi_m \rangle_{L^2(\mathbb{R})}$$

holds for every $\alpha \in \mathbb{N}$ and $t \geq 0$ a.s. by (7.1). Since

$$\sup_{t\in[0,\kappa]} \{|\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t)| + |\mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})(t)| + |\mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})(t)| + |Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t)|\}$$

is bounded by $c_{\kappa,m}[1 + |z^{k_\alpha}|^2_{L^\infty((0,\kappa);H^1(-\kappa,\kappa)\times L^2(-\kappa,\kappa))}]$ by (8.14) and (9.9), we infer that also

$$\sup_{t\in[0,\kappa]} \{|\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t)| + |\mathcal{V}^{k_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t)| + |\mathcal{V}^{ijk_\alpha}_{lm\gamma}(\mathbf{z}^\alpha)(t)| + |\mathbf{Q}^\alpha_{lm\gamma}(t)|\}$$

is bounded by $c_{\kappa,m}[1 + |\mathbf{z}^\alpha|^2_{L^\infty((0,\kappa);H^1(-\kappa,\kappa)\times L^2(-\kappa,\kappa))}]$ a.s. and, by Theorem 6.6, that for every $r > 0$ and $i, j, \alpha \in \mathbb{N}$ the processes

$$\mathbf{1}_{B_{2\kappa,r}}(z^{k_\alpha}_0)[\mathcal{V}_{lm\gamma}(z^{k_\alpha}) - Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}], \quad \mathbf{1}_{B_{2\kappa,r}}(z^{k_\alpha}_0)\{[\mathcal{V}_{lm\gamma}(z^{k_\alpha}) - Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}]^2 - \mathcal{V}^{k_\alpha}_{lm\gamma}(z^{k_\alpha})\},$$
$$\mathbf{1}_{B_{2\kappa,r}}(z^{k_\alpha}_0)\{[\mathcal{V}_{lm\gamma}(z^{k_\alpha}) - Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}]\beta^{ij} - \mathcal{V}^{ijk_\alpha}_{lm\gamma}(z^{k_\alpha})\},$$

are $\mathbb{F}$-martingales on $[0, \kappa]$. Hence, with the same notation as in (9.8), we have $\Pi_0 = \Pi_1$ where, for $i \in \{0, 1\}$,

$$\begin{aligned}
\Pi_i &= \mathbb{E}g_0(\mathbf{O})\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}(0))[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_i) - \mathbf{Q}_{lm\gamma}(t_i)] \\
&= \lim_{\alpha\to\infty} \mathbb{E}g_0(\mathbf{O}^\alpha)\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}^\alpha(0))[\mathcal{V}_{lm\gamma}(\mathbf{z}^\alpha)(t_i) - \mathbf{Q}^\alpha_{lm\gamma}(t_i)] \\
&= \lim_{\alpha\to\infty} \mathbb{E}g_0(O^{k_\alpha})\mathbf{1}_{B_{2\kappa,r}}(z^{k_\alpha}_0)[\mathcal{V}_{lm\gamma}(z^{k_\alpha})(t_i) - Q^{k_\alpha}_{b_l,\varphi_m,Z^\gamma}(t_i)]
\end{aligned}$$

and analogously, by Lemma 9.12 and the Lebesgue DC Theorem as (9.2) holds,

$$\begin{aligned}
&\mathbb{E}g_0(\mathbf{O})\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}(0))\{[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_1) - \mathbf{Q}_{lm\gamma}(t_1)]^2 - \mathcal{V}^\infty_{lm\gamma}(\mathbf{z})(t_1)\} \\
&\quad = \mathbb{E}, g_0(\mathbf{O})\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}(0))\{[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_0) - \mathbf{Q}_{lm\gamma}(t_0)]^2 - \mathcal{V}^\infty_{lm\gamma}(\mathbf{z})(t_0)\}, \\
&\mathbb{E}g_0(\mathbf{O})\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}(0))\{[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_1) - \mathbf{Q}_{lm\gamma}(t_1)]\mathbf{B}_{ij}(t_1) - \mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})(t_1)\} \\
&\quad = \mathbb{E}g_0(\mathbf{O})\mathbf{1}_{B_{2\kappa,r}}(\mathbf{z}(0))\{[\mathcal{V}_{lm\gamma}(\mathbf{z})(t_0) - \mathbf{Q}_{lm\gamma}(t_0)]\mathbf{B}_{ij}(t_0) - \mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})(t_0)\}
\end{aligned}$$

holds for every $r > 0$ such that $\mathrm{Leb}(\{|\mathbf{z}(0)|_{H^1(-2\kappa,2\kappa)\times L^2(-2\kappa,2\kappa)} = r\}) = 0$. In particular,

$$\mathcal{V}_{lm\gamma}(\mathbf{z}) - \mathbf{Q}_{lm\gamma}, \quad [\mathcal{V}_{lm\gamma}(\mathbf{z}) - \mathbf{Q}_{lm\gamma}]^2 - \mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z}) \quad \text{and} \quad [\mathcal{V}_{lm\gamma}(\mathbf{z}) - \mathbf{Q}_{lm\gamma}]\mathbf{B}_{ij} - \mathcal{V}^{ij\infty}_{lm\gamma}(\mathbf{z})$$

are local $\mathbb{B}$-martingales for every $l, m, i, j \in \mathbb{N}$, $\gamma \in \{1, \ldots, N\}$. Hence, using the equality $\langle M^1 - M^2\rangle = \langle M^1\rangle - 2\langle M^1, M^2\rangle + \langle M^2\rangle$ for the quadratic variation, we infer that

$$\Big\langle \mathcal{V}_{lm\gamma}(\mathbf{z}) - \mathbf{Q}_{lm\gamma} - \int_0^{\cdot} \langle [\overline{Y}(\mathbf{z}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u}^l)]d\mathbf{W}, \varphi_m\rangle_{L^2(\mathbb{R})}\Big\rangle$$
$$= 2\mathcal{V}^{\infty}_{lm\gamma}(\mathbf{z}) - 2\sum_{i=1}^{\infty}\sum_{j\in J_i}\int_0^{\cdot} \langle [\overline{Y}(\mathbf{z}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u}^l)]\xi_{ij}, \varphi_m\rangle^2_{L^2(\mathbb{R})} ds = 0.$$

The result now follows from equality $\overline{Y}(\mathbf{u}, \mathbf{v}, \mathbf{u}_x) = Y(\mathbf{u}, \mathbf{v}, \mathbf{u}_x)$ due to Lemma 9.9. □

Lemma 9.15. *In the framework described above the following identity holds for every $t \geq 0$, $m \in \mathbb{N}$ and $\gamma \in \{1, \ldots, N\}$ almost surely,*

$$\begin{aligned}\langle \mathbf{v}(t) \cdot Z^\gamma(\mathbf{u}(t)), \varphi_m\rangle &= \langle \mathbf{v}(0) \cdot Z^\gamma(\mathbf{u}(0)), \varphi_m\rangle + \int_0^t \langle [Y(\mathbf{u}, \mathbf{v}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u})]d\mathbf{W}, \varphi_m\rangle \\ &\quad + \int_0^t [\langle \mathbf{v} \cdot (\nabla_{\mathbf{v}} Z^\gamma)|_{\mathbf{u}}, \varphi_m\rangle \\ &\quad - \langle \mathbf{u}_x \cdot Z^\gamma(\mathbf{u}), \partial_x\varphi_m\rangle - \langle \mathbf{u}_x \cdot (\nabla_{\mathbf{u}_x} Z^\gamma)|_{\mathbf{u}}, \varphi_m\rangle]ds. \qquad (9.10)\end{aligned}$$

Proof. Let us fix $R > 0$. Since $\lim_{l\to\infty}[\sup_{t\in[0,R]} |b_l * \mathbf{u}(t) - \mathbf{u}(t)|_{C([-R,R])}] = 0$, we get

$$\lim_{l\to\infty}[\sup_{t\in[0,R]} |Z^\gamma(b_l * \mathbf{u}(t)) - Z^\gamma(\mathbf{u}(t))|_{C([-R,R])}] = 0 \quad \text{on } [0, 1], \quad \gamma \in \{1, \ldots, N\}.$$

On the other hand, for every $(t, \omega) \in \mathbb{R}_+ \times [0, 1]$, the sequences $\mathbf{u}_x * b_l$ and $\mathbf{v} * b_l$ converge in $L^2_{\mathrm{loc}}(\mathbb{R})$ to $\mathbf{u}_x$ and $\mathbf{v}$ respectively. Moreover, by equality (9.9), for every $T > 0$, $m \in \mathbb{N}$ and $\gamma \in \{1, \ldots, N\}$, the following inequality holds on $[0, 1]$

$$\begin{aligned}&\lim_{l\to\infty}\sum_{i=1}^{\infty}\sum_{j\in J_i}\int_0^T \langle [Y(\mathbf{u}(s), \mathbf{v}(s), \mathbf{u}_x(s))\xi_{ij}] \cdot [Z^\gamma(\mathbf{u}(s) * b_l) - Z^\gamma(\mathbf{u}(s))], \varphi_m\rangle^2_{L^2(\mathbb{R})} ds \\ &\leq \lim_{l\to\infty} \frac{r_m\mu(\mathbb{R})}{\pi} |\varphi_m|^2_{L^\infty(\mathbb{R})} \sup_{t\in[0,T]} |Z^\gamma(b_l * u(t)) - Z^\gamma(u(t))|^2_{C([-r_m,r_m])} \\ &\quad \times \sup_{t\in[0,T]} |Y(\mathbf{u}(t), \mathbf{v}(t), \mathbf{u}_x(t))|^2_{L^2(-r_m,r_m)} = 0.\end{aligned}$$

Hence, by (for instance) Proposition 4.1 in [31], locally uniformly in t, in probability,

$$\lim_{l\to\infty}\int_0^t \langle [Y(\mathbf{u}, \mathbf{v}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u} * b_l)]d\mathbf{W}, \varphi_m\rangle_{L^2(\mathbb{R})} = \int_0^t \langle [Y(\mathbf{u}, \mathbf{v}, \mathbf{u}_x) \cdot Z^\gamma(\mathbf{u})]d\mathbf{W}, \varphi_m\rangle_{L^2(\mathbb{R})}.$$

Finally, by Corollary 9.6 and inequality (9.4) we infer that for all $T > 0$, $m \in \mathbb{N}$ and $\gamma \in \{1, \cdots, N\}$, $\lim_{l\to\infty} |\mathbf{Q}_{lm\gamma}|_{{}_0\mathrm{Lip}[0,T]} = 0$ in probability. Therefore, the result follows by applying Lemma 9.13. □

10. The Relationship between an Intrinsic and an Extrinsic Solution

The aim of this short section is to prove that the notions of an intrinsic and an extrinsic solutions are, roughly speaking, equivalent.

Lemma 10.1. *Let $z = (u, v)$ be an adapted weakly continuous $H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$-valued process such that for every $\omega \in \Omega$ and every $\varphi \in L^2_{\mathrm{comp}}(\mathbb{R})$, $\frac{d}{dt}\langle u(\cdot, \omega), \varphi\rangle_{L^2(\mathbb{R})} = \langle v(\cdot, \omega), \varphi\rangle_{L^2(\mathbb{R})}$ in the weak sense on $\mathbb{R}_+$. Let W be a spatially homogeneous Wiener process with a finite spectral measure μ. Then the following three statements are equivalent.*

(i) (z, W) *satisfies* (9.10) *a.s. for every* $t \geq 0$, $m \in \mathbb{N}$ *and* $\{Z^\gamma\}_{\gamma=1}^N$ *satisfying* (A.1).
(ii) (z, W) *satisfies the intrinsic equation* (4.2).
(iii) (z, W) *satisfies the extrinsic equation* (4.3).

Proof. If the condition (*i*) is satisfied then by Proposition D.1 the equation (4.2) is satisfied almost surely for every $t \geq 0$, $\gamma \in \{1, \ldots, N\}$ and every $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$. Let (b_l) be an approximation of identity and define the following processes, where $t \geq 0$,

$$h_l^\gamma(t) = b_l * [v(t) \cdot Z^\gamma(u(t))]$$
$$H_l^\gamma(t) = [\partial_x b_l] * [u_x(t) \cdot Z^\gamma(u(t))] - b_l * [u_x(t) \cdot \nabla_{u_x(t)} Z^\gamma|_{u(t)}] + b_l * [v(t) \cdot \nabla_{v(t)} Z^\gamma|_{u(t)}]$$
$$g_l^\gamma(t)\xi = b_l * \{[Y(u(t), v(t), u_x(t)) \cdot Z^\gamma(u(t))]\xi\}, \quad \xi \in H_\mu.$$

Since by Lemma 3.1,

$$\begin{aligned}\sup_{t\in[0,R]} |g_l^\gamma(t)|_{\mathcal{J}_2(H_\mu, H^m(-R,R))} &\leq c_{b_l} \sup_{t\in[0,R]} |Y(z(t), u_x(t)) \cdot Z^\gamma(u(t))|_{\mathcal{J}_2(H_\mu, L^2(-1-R,R+1))} \\ &= c_{b_l}[\mu(\mathbb{R})]^{\frac{1}{2}} \sup_{t\in[0,R]} |Y(z(t), u_x(t)) \cdot Z^\gamma(u(t))|_{L^2(-1-R,R+1)} \\ &\leq c_{l,\gamma,R,\mu,Y}[1 + \sup_{t\in[0,R]} |z(t)|_{H^1(-1-R,R+1)\times L^2(-1-R,R+1)}] \\ &\leq c_{l,\gamma,R,\mu,Y,z}(\omega) < \infty,\end{aligned}$$

we infer that

$$h_l^\gamma(t) = h_l^\gamma(0) + \int_0^t H_l^\gamma(s)ds + \int_0^t g_l^\gamma(s)dW, \quad t \in [0, T)$$

where the integrals converge in every $H^m(-R, R)$ for any $R > 0$, $m \in \mathbb{N}$. On the other hand, since $Z^\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is a C_0^∞-class function, for every $t \geq 0$

$$Z^\gamma(u(t)) = Z^\gamma(u(0)) + \int_0^t (Z^\gamma)'(u(s))v(s)ds, \quad \text{in } L^2(-R, R), R > 0.$$

Since for every $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$ the map $H^1(-R, R) \times L^2((-R, R); \mathbb{R}^n) \ni (u, v) \mapsto \int_{\mathbb{R}} u(x)v(x)\varphi(x)dx \in \mathbb{R}$ is of C^2-class, in view of the Itô formula, see for instance

Theorem 4.17 in [14], we infer that for every $t \geq 0$ and $R > 0$, almost surely,

$$\begin{aligned}\langle h_l^\gamma(t) Z^\gamma(u(t)), \varphi\rangle_{L^2(\mathbb{R})} = \langle h_l^\gamma(t) Z^\gamma(u(0)), \varphi\rangle_{L^2(\mathbb{R})} &+ \int_0^t \langle h_l^\gamma(s)(Z^\gamma)'(u(s))v(s), \varphi\rangle_{L^2(\mathbb{R})} ds \\ &+ \int_0^t \langle H_l^\gamma(s) Z^\gamma(u(s)), \varphi\rangle_{L^2(\mathbb{R})} ds \\ &+ \int_0^t \langle [g_l^\gamma(s)dW] Z^\gamma(u(s)), \varphi\rangle_{L^2(\mathbb{R})} ds.\end{aligned}$$

Next, since h_l^γ converges in L^2_{loc} to $v(t) \cdot Z^\gamma(u(t))$ and

$$\begin{aligned}\langle \{[\partial_x b_l] * [u_x \cdot Z^\gamma(u)]\} Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} &= \langle \partial_x\{b_l * [u_x \cdot Z^\gamma(u)]\} Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} \\ &= -\langle \{b_l * [u_x \cdot Z^\gamma(u)]\}(Z^\gamma)'(u)u_x, \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad - \langle \{b_l * [u_x \cdot Z^\gamma(u)]\} Z^\gamma(u), \varphi_x\rangle_{L^2(\mathbb{R})}\end{aligned}$$

we infer that $\langle H_l^\gamma Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})}$ converges to

$$\begin{aligned}&-\langle [u_x \cdot Z^\gamma(u)](Z^\gamma)'(u)u_x, \varphi\rangle_{L^2(\mathbb{R})} - \langle [u_x \cdot Z^\gamma(u)] Z^\gamma(u), \varphi_x\rangle_{L^2(\mathbb{R})} \\ &-\langle [u_x \cdot \nabla_{u_x} Z^\gamma|_u] Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})} + \langle [v \cdot \nabla_v Z^\gamma|_u] Z^\gamma(u), \varphi\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

Finally, by the Lebesgue Dominated Convergence Theorem and identity (9.9),

$$\lim_{l\to\infty} \sum_{i=1}^\infty \sum_{j\in J_i} \int_0^t \langle \{b_l * [w(s)\xi_{ij}] - w(s)\xi_{ij}\} Z^\gamma(u(s)), \varphi\rangle^2_{L^2(\mathbb{R})} ds = 0$$

where $w = Y(u, v, u_x) \cdot Z^\gamma(u)$. Hence, again by Proposition 4.1 in [31], the following equality holds a.s., for every $t \geq 0$ and $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$.

$$\begin{aligned}&\langle [v(t) \cdot Z^\gamma(u(t))] Z^\gamma(u(t)), \varphi\rangle \\ &= \int_0^t \langle [Y(u, v, u_x) \cdot Z^\gamma(u)] Z^\gamma(u) dW, \varphi\rangle \\ &\quad + \langle [v(0) \cdot Z^\gamma(u(0))] Z^\gamma(u(0)), \varphi\rangle - \int_0^t \langle [u_x \cdot Z^\gamma(u)](Z^\gamma)'(u)u_x, \varphi\rangle ds \\ &\quad + \int_0^t [\langle [v \cdot Z^\gamma(u)](Z^\gamma)'(u)v, \varphi\rangle + \langle [v \cdot \nabla_v Z^\gamma|_u] Z^\gamma(u), \varphi\rangle] ds \\ &\quad - \int_0^t [\langle [u_x \cdot Z^\gamma(u)] Z^\gamma(u), \varphi_x\rangle + \langle [u_x \cdot \nabla_{u_x} Z^\gamma|_u] Z^\gamma(u), \varphi\rangle] ds.\end{aligned}$$

Now by the equality on p. 479 in [1] the vector fields (Z^γ) satisfy

$$A_p(\xi, \xi) = \sum_{\gamma=1}^N [(\xi \cdot \nabla_\xi Z^\gamma|_p) Z^\gamma(p) + (\xi \cdot Z^\gamma(p)) \nabla_\xi Z^\gamma|_p], \quad p \in M, \quad \xi \in T_pM.$$

Hence, for $t \geq 0$, $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$,

$$\langle v(t) - v(0), \varphi\rangle = \int_0^t \langle Y(u, v, u_x) dW, \varphi\rangle + \int_0^t [\langle A_u(v, v) - A_u(u_x, u_x), \varphi\rangle - \langle u_x, \varphi_x\rangle] ds$$

and so (4.3) holds. In consequence, condition (i) implies condition (iii).

To prove that condition (iii) implies condition (ii), we define the following four processes

$$u_l(t) = b_l * u(t), \quad a_l(t) = b_l * [A_{u(t)}(v(t), v(t)) - A_{u(t)}(u_x(t), u_x(t))],$$
$$v_l(t) = b_l * v(t), \quad g_l(t)\xi = b_l * [Y(u(t), v(t), u_x(t))\xi], \quad \xi \in H_\mu, \quad t \geq 0.$$

Proceeding analogously as in the first part of the proof, the following equality holds in $H^m(-R, R)$ whenever $l, m \in \mathbb{N}$ and $R > 0$,

$$v_l(t) = v_l(0) + \int_0^t [\partial_{xx} u_l(s) + a_l(s)]ds + \int_0^t g_l(s)dW, t \geq 0.$$

Hence, by the Itô formula, for all $l \in \mathbb{N}$ and $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$,

$$\begin{aligned}\langle v_l(t) \cdot Z(u(t)), \varphi\rangle_{L^2(\mathbb{R})} &= \langle v_l(0) \cdot Z(u(0)), \varphi\rangle_{L^2(\mathbb{R})} + \int_0^t \langle [g_l(s)dW] \cdot Z(u(s)), \varphi\rangle_{L^2(\mathbb{R})} \\ &\quad + \int_0^t \langle [\partial_{xx} u_l(s) + a_l(s)] \cdot Z(u(s)), \varphi\rangle_{L^2(\mathbb{R})} ds \\ &\quad + \int_0^t \langle v_l(s) \cdot \nabla_{v(s)} Z|_{u(s)}, \varphi\rangle_{L^2(\mathbb{R})} ds\end{aligned}$$

holds a.s. for every $t \geq 0$. Since

$$\langle \partial_{xx} u_l \cdot Z(u), \varphi\rangle_{L^2(\mathbb{R})} = -\langle \partial_x u_l \cdot \nabla_{\partial_x u} Z|_u, \varphi\rangle_{L^2(\mathbb{R})} - \langle \partial_x u_l \cdot Z(u), \partial_x \varphi\rangle_{L^2(\mathbb{R})},$$

$v_l(t)$ and $\partial_x u_l(t)$ converge in $L^2_{loc}(\mathbb{R})$ to $v(t)$ and $\partial_x u(t)$, $a_l(t)$ converges in $L^1_{loc}(\mathbb{R})$ to $A_{u(t)}(v(t), v(t)) - A_{u(t)}(u_x(t), u_x(t))$ and

$$\lim_{l\to\infty} \sum_{i=1}^{\infty} \sum_{j \in J_i} |\langle [g_l \xi_{ij} - Y(z, u_x)\xi_{ij}] \cdot Z(u), \varphi\rangle_{L^2(\mathbb{R})}|^2_{\mathcal{J}_2(H_\mu, \mathbb{R})} = 0,$$

by the Lebesgue DC Theorem and Proposition 4.1 in [31],

$$\begin{aligned}\langle v(t) \cdot Z(u(t)), \varphi\rangle &= \langle v_0 \cdot Z(u_0), \varphi\rangle + \int_0^t \langle v \cdot \nabla_v Z|_u + [A_u(v, v) - A_u(u_x, u_x)] \cdot Z(u), \varphi\rangle ds \\ &\quad - \int_0^t [\langle u_x \cdot \nabla_{u_x} Z|_u, \varphi\rangle + \langle u_x \cdot Z(u), \varphi_x\rangle] ds \\ &\quad + \int_0^t \langle [Y(u, v, u_x) \cdot Z(u)]dW, \varphi\rangle, \quad \text{a.s.}\end{aligned}$$

for every $t \geq 0$ and every $\varphi \in H^1_{\mathrm{comp}}(\mathbb{R})$. The result now follows since $A_p \cdot Z_p = 0$, $p \in M$. □

Appendix A. Some Useful Facts about Riemannian Geometry

In what follows we assume that M is a compact d-dimensional Riemannian manifold embedded isometrically in $\mathbb{R}^n$.

Lemma A.1. *There exist a C_0^∞-class function $P : \mathbb{R}^n \to \mathbb{R}^n$ and a neighbourhood V of M such that $P(V) = M$ and, for $p \in V$, it holds that $P(p) = p$ iff $p \in M$.*

Proof. We will use a suitable smooth projection of the ambient space $\mathbb{R}^n$ on the manifold. By [29, Proposition 7.26, p. 200], there exists an open neighbourhood $\widetilde{V}$ of the set $\{(p, 0) : p \in M\}$ in the normal bundle NM and an open set $O \subseteq \mathbb{R}^n$ such that the function $\mathscr{E} : \widetilde{V} \ni (p, \xi) \mapsto p + \xi$ is a diffeomorphism. Let us define a smooth map $\widetilde{P} : O \to M$ as the composition of the natural projection map $NM \to M : (p, \xi) \mapsto p \in M$ and $\mathscr{E}^{-1}$. Employing a partition of unity we can find a C_0^∞-class function $P : \mathbb{R}^n \to \mathbb{R}^n$ and a neighbourhood $V \subseteq O$ of M such that P has the claimed properties. □

Lemma A.2. *There exists $C_M > 0$ depending on M and Borel measurable mappings $\Theta_k : H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM) \to H^2_{\mathrm{loc}} \times H^1_{\mathrm{loc}}(TM)$ such that $\lim_{k\to\infty} |\Theta_k(z) - z|_{H^1(-R,R)\times L^2(-R,R)} = 0$ for every $z \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ and, for every $R > 0$ and $k \in \mathbb{N}$,*

$$|\Theta_k(z)|_{H^1(-R,R)\times L^2(-R,R)} \le C_M(R^{\frac{1}{2}} + |z|_{H^1(-R-1,R+1)\times L^2(-R-1,R+1)}), z \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM).$$

Proof. Let $(b_k)_{k=1}^\infty$ be an approximation of identity. The maps Θ_k, $k \in \mathbb{N}$ are constructed as follows. Let $z = (u, v) \in H^1_{\mathrm{loc}} \times L^2_{\mathrm{loc}}(TM)$ and set $\bar{u}_k(x) = u(\mathrm{sgn}(x)\min\{k, |x|\})$, $x \in \mathbb{R}$. Obviously, $\bar{u}_k \in H^1_{loc}(M)$. Let V be the neighbourhood of M introduced Lemma A.1. Then we can find $\delta > 0$ such that $p + z \in V$ provided $p \in M$ and $|z| \le \delta$. Set $m_k = \min\{j \ge k : j \ge \delta^{-2}|u_x|^2_{L^2(-k,k)}\}$ and $w_k = \bar{u}_k * b_{m_k}$. Since we have $\partial_x \bar{u}_k = \mathbf{1}_{(-k,k)}\partial_x u$, whence $\sup_{x\neq y} |x - y|^{-\frac{1}{2}}|\bar{u}_k(x) - \bar{u}_k(y)| \le |u_x|_{L^2(-k,k)}$ and thus we infer that $\sup_{x\in\mathbb{R}} |w_k(x) - \bar{u}_k(x)| \le [m_k]^{-\frac{1}{2}}|u_x|_{L^2(-k,k)} \le \delta$. Thus we infer that $w_{k,u}$ is a C_0^∞-class function taking values in a compact set $M + \overline{B_\delta} \subseteq V$. We set $u_k = P \circ w_{k,u}$, $v_k(x) = \pi_{u_k(x)}\big((b_k * v)(x)\big)$, $x \in \mathbb{R}^n$ and $\Theta_k(u) = (u_k, v_k)$, where $\pi_p : \mathbb{R}^n \to T_pM$ is the orthogonal projection at $p \in M$. It is easy to show that the map Θ_k is measurable in the sense as requested and that the first part of the claim holds true. □

The next proposition follows easily from the C^∞ partition of unity on M.

Proposition A.3. *There exists a natural number $N \in \mathbb{N}$ and there exist smooth vector fields $Z^1, \cdots, Z^N$ on M such that*

$$\xi = \sum_{i=1}^N \langle \xi, Z^i(p)\rangle Z^i(p), \quad \xi \in T_pM, \quad p \in M. \tag{A.1}$$

Appendix B. Some Useful Facts about Function Spaces

For a natural number $k \in \mathbb{N}$ let denote by $\mathbb{L}^k$ the vector space $C_w(\mathbb{R}_+; H^k_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n))$ of all weakly continuous functions from $\mathbb{R}_+$ to $H^k_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$. Let us denote by $|h|_{m,\varphi}$, with $m \in \mathbb{N}$ and $\varphi \in \prod_{j=0}^k L^2(\mathbb{R}; \mathbb{R}^n)$, the following seminorm on the space $\mathbb{L}^k$,

$$|h|_{m,\varphi} = \sup_{t\in[0,m]} \sum_{j=0}^k \big|\langle \partial_x^j h(t), \varphi_j\rangle_{L^2(-m,m)}\big|, \quad h \in \mathbb{L}^k. \tag{B.1}$$

The space $\mathbb{L}^k$ becomes a locally convex topological vector space when equipped with the locally convex topology generated by the above family of seminorms. By $\mathbb{L}$ we denote the following locally convex topological vector space $\mathbb{L} = \mathbb{L}^1 \oplus \mathbb{L}^0$.

We will also use the following family of Sobolev spaces, where $R > 0$,

$$H_0^1(-R, R) = \{\varphi \in H^1(-R, R) : \varphi(-R) = \varphi(R) = 0\}, \quad H^{-1}(-R, R) = [H_0^1(-R, R)]^*.$$

Let us remark that each space from this family is a separable Hilbert space. A proof of the following result can be found in [33, Proposition B.2].

Proposition B.1. *Let $a = (a_m)$ be a sequence of positive real numbers, $\alpha \in (0, 1]$ and $k \geq 0$. Then the set*

$$\left\{h \in \mathbb{L}^k : \sup_{t\in[0,m]} |h(t)|_{H^k(-m,m)} + \sup_{0\leq s<t\leq m} \left[\frac{|h(t) - h(s)|_{H^{-1}(-m,m)}}{(t-s)^\alpha}\right] \leq a_m,\, m \in \mathbb{N}\right\} \tag{B.2}$$

is a compact, convex and metrizable subset of $\mathbb{L}^k$.

Appendix C. The Skorokhod-Jakubowski Representation Theorem

Let X be a topological space such that there exists a sequence (f_j) of real continuous functions on X that separate points of X. According to Jakubowski [20], every compact subset of X is metrizable and a Borel probability measure on X is Radon iff it is supported by a σ-compact set. Moreover, the following result was proved in [20].

Theorem C.1. *Let (ν_j) be a tight sequence of Borel probability measures on X. Then there exist a subsequence (j_k) and Borel measurable maps $\theta, \theta_k : [0, 1] \to X$, $k \geq 1$ such that the range of each of these maps is σ-compact, for each $k \geq 1$, ν_{j_k} is equal to the law of θ_k and, for every $t \in [0, 1]$, $\theta_k(t) \to \theta(t)$ in X.*

The following result implies that the Borel σ-algebra on a Polish space Z that is continuously embedded into X coincides with the trace σ-algebra of X on Z.

Proposition C.2. *If Z is a Polish space and $b : Z \to X$ is a continuous injection, then $b(B)$ is a Borel set whenever B is Borel in Z.*

Proof. Since the map $F := (f_k)_{k\in\mathbb{N}} : X \to \mathbb{R}^{\mathbb{N}}$ is a continuous injection, the function $F \circ b : Z \to \mathbb{R}^{\mathbb{N}}$ is also a continuous injection. Let us take a Borel set $B \subseteq Z$. Since both Z and $\mathbb{R}^{\mathbb{N}}$ are Polish spaces, we infer that $(F \circ b)(B)$ is a Borel set. Therefore $b(B) = F^{-1}[(F \circ b)(B)] \subseteq X$ is Borel set too. □

Appendix D. A Measurability Lemma

Poposition D.1. *There exists a sequence (φ_k) of $C_0^\infty(\mathbb{R})$ functions such that, for every $L \in \mathbb{N}$, one can find a subsequence (k_j) such that the support of φ_{k_j} is contained in $(-L, L)$ for every $j \in \mathbb{N}$, $\{\varphi_{k_j}\}$ is dense in $H^m(-L, L)$ and the mappings*

$$H^m_{\text{loc}}(\mathbb{R}) \ni h \mapsto \langle h, \varphi_k\rangle_{L^2} \in \mathbb{R}, \quad k \in \mathbb{N}$$

generate the Borel σ-algebra on $H^m_{\text{loc}}(\mathbb{R})$ whenever $m \geq 0$.

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