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RESEARCH REPORT

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Mathematical Model of a Production Line Checkpoint

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1 Problem description

Almost everybody has already had the experience of purchasing a product which was defective in some way or does not function the way it was designed to. The challenge for producers is the eliminations of possible defects of final products. One of possible and very common ways to achieve this goal is to add checkpoints to the production process.

The main purpose of this report is to create a model of a production line checkpoint. The checkpoint is a place on a production line where the quality and functionality of products from the production line are tested. In all production processes, it is necessary to monitor whether the products meets the specifications and desired quality. Several statistical methods were developed and are used in real production lines for testing quality of products. These methods are usually based on the inspecting a random sample of the output from a process and deciding whether the process is producing products with characteristics that fall within a predetermined range. The checkpoint modeled in this paper is the checkpoint of the production line producing valves which are used for medical purposes. Since human lives depend on the functionality of these valves, it is not possible to test only random samples of valves but functionality of every produced valve must be tested.

Each produced valve must pass the testing procedure which consist of reading the unique product code, attaching the valve to the testing apparatus, setting the valve's pressure controller and safeguards to proper positions and the testing. The testing itself consists of the testing of airtightness, functionality of safeguards and proper pressure control. If the valve pass the test, it is send to the further processing. If the valve is defective, the code of defect is stored and the valve is repaired right at the checkpoint. The most common defect is caused by the defective gasket but sometimes also more complicated defects occur. After the repair, the valve is tested again and if it does not pass the test, it is again repaired. Therefore, a valve can be repaired several times till it is fully functional.

The large number of successive repairs causes large consumption of time for processing of one valve and the income from such a valve decreases. Since the checkpoint is a part of the production line and the performance of particular posts influence the overall performance of a production line, it is important to find the optimal number of successive repairs after which the valve is scrapped as a reject and which will not cause the blocking of a production line. Each repair costs except some time also some money as well as a scrapped valve means loss of money, therefore, it is important to optimize the number of possible successive repairs also with respect to the overall profit.

The flowchart representing the processing of a valve at the production line checkpoint can be seen in the figure 1. The processed valve can be seen in the figure 2 and the real production line checkpoint can be seen in the figure 3.

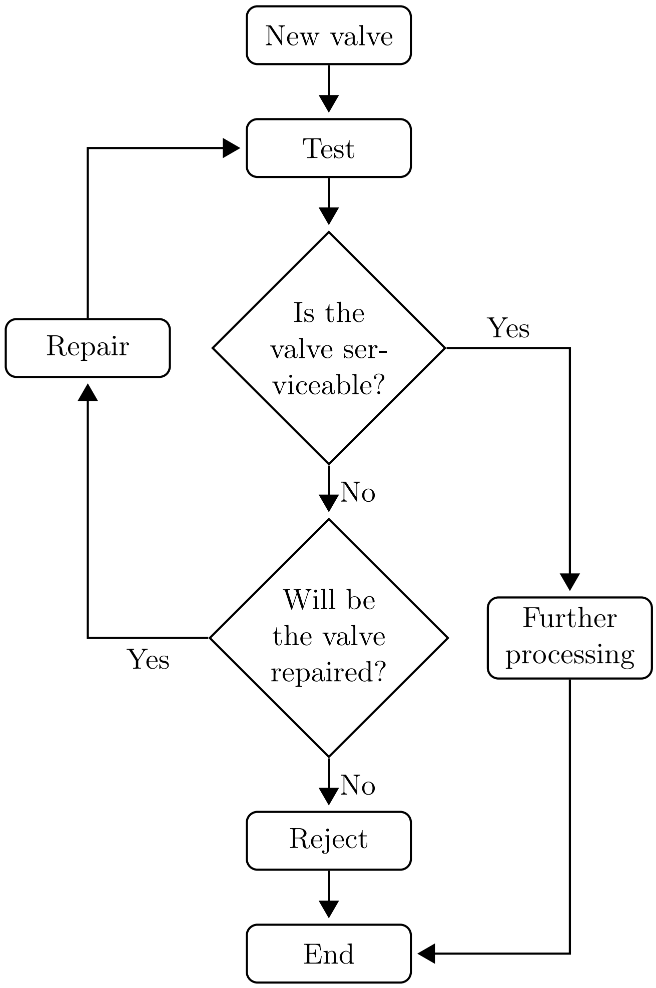

Figure 1: The flowchart representing the processing of a valve at the production line checkpoint

2 Modeling of the production line checkpoint

The processing of a valve at the production line checkpoint was described at the beginning of this paper and it is represented by the flowchart on the figure 1. If the valve passes the first test, it will be send to the further processing, which can consist for example from an additional assembling or packing. Important and interesting situation is the case of the defective valve, i.e., the valve which did not pass the functionality test and, therefore, it must be repaired. Each repair takes some time and the valve can be repaired several times in a row. Since the production line must not be blocked, the defective valve must be discarded after a certain number of repairs as the reject. Each repair also costs some money and the discarded valve means a loss of money, therefore, the important question is to find the optimal number of possible successive repairs.

The problem of finding the optimal number of possible successive repairs can be formulated as so called *optimal stopping problem* ([12]). In an optimal stopping problem, which is the special case of the Markov decision processes ([12], [4]), the system evolves as an uncon-

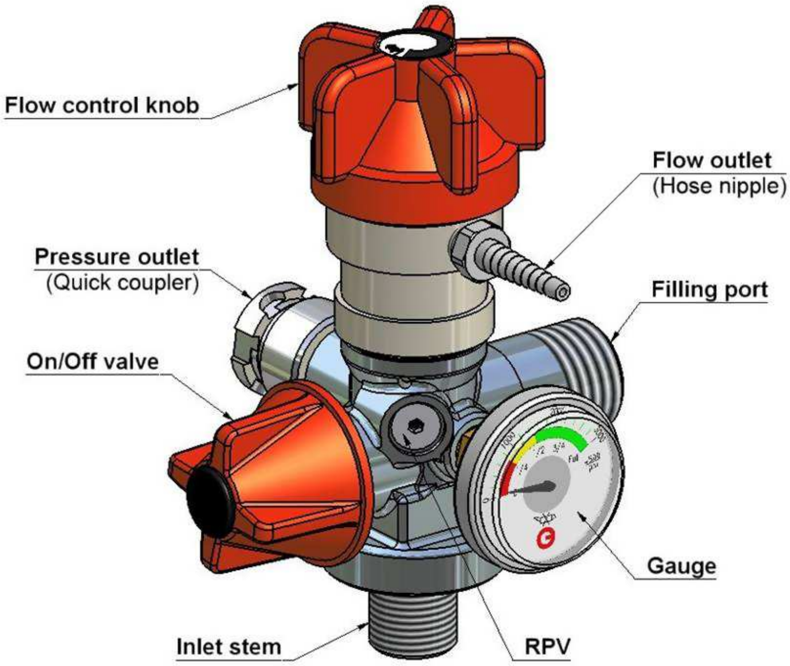

Figure 2: The processed valve.

Figure 3: The real production line checkpoint.

trolled Markov chain. At each decision epoch, the decision maker has two possible actions available in each state, to stop or to continue, and he receives a reward or pay a cost (which depends on the state and the chosen action). When stopped, it remains in stopped state with zero reward. The objective is to choose a policy to maximize the expected (total) reward.

First of all, the problem of finding the optimal number of possible successive repairs without considering the role of the time will be discussed.

Let us consider the valve which did not pass the first functionality test. Each valve is independent of the others and the number of possible successive repairs is limited, since the checkpoint is the part of the whole production line. This defines the maximal number of successive repairs, which will be denoted as K. If the valve is still defective after K repairs, it must be discarded as the reject. Therefore, the processing of a defective valve lasts maximally $K + 1$ periods.

If the valve is defective at a certain decision period, the decision maker must decide between the repair and the discarding of this valve. The overwhelming majority of defects is caused by the defective gasket and, therefore, most repairs consist from the replacing the defective gasket. The quality and functionality of the new gasket is not influenced by the previous repairs, therefore, it can be assumed that each repair is successful with probability $p \in (0, 1]$ (the new gasket is functional and the valve will pass the functionality test) and the repair fails with the probability $1 - p$ (and the valve will not pass the functionality test).

Each repair cost some amount of money (for the new gasket and the work related to its replacement), which will be denoted as c. The value received for the functional valve is denoted as C. This value will be lost if the valve is discarded. It is supposed that $0 < c < C$.

These assumptions lead to the definitions of possible states of the valve and actions in these states. The valve can be either defective and the decision maker has to decide between the repair or the discarding of this valve or the valve is already repaired or discarded and the optimization process is over.

The important part of the optimization is the choice of the decision criterion. The decision criterion should be the appropriate representation of the real process of obtaining rewards and it should also represent the decision maker's preferences. From the definition of the problem, it follows that the expected total reward criterion represents the best the real reward process ([12]).

The optimization model introduced above, which is also represented on the figure 4 can be summarized and formally described by:

- Decision periods ($K \geq 0$):
$$T = \{1, 2, \ldots, K, K + 1\}.$$
- States:
$$S = \{\text{Defective valve}, \text{Processed valve}\} = \{1, 2\}.$$
- Actions available at states:
$$\begin{aligned} A(\text{Defective valve}) &= A(1) = \{\text{repair, discard}\}, \\ A(\text{Processed valve}) &= A(2) = \{\text{end}\}. \end{aligned}$$
- Transition probabilities ($p \in (0, 1]$):
$$\begin{array}{lll} P_{11}(\text{repair}) = 1 - p, & P_{11}(\text{discard}) = 0, & P_{21}(\text{end}) = 0, \\ P_{12}(\text{repair}) = p, & P_{12}(\text{discard}) = 1, & P_{22}(\text{end}) = 1, \end{array}$$

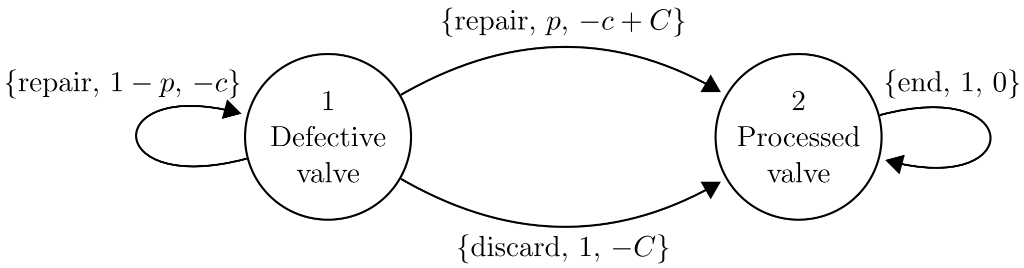

Figure 4: Transition graph representing the model for optimization. The values in the braces denote action, transition probability corresponding to this action and reward corresponding to this action respectively.

where P_{11}(repair) denotes the transition probability from the state 1 (defective valve) to the same state 1 (defective valve) if action "repair" is chosen and the other symbols have analogous meaning.

- Rewards ($C > c > 0$):
  For decision periods $1, 2 \ldots, K$:

$$r(1, \text{repair}, 1) = -c, \qquad r(1, \text{discard}, 1) = 0, \qquad r(2, \text{end}, 1) = 0,$$
$$r(1, \text{repair}, 2) = -c + C, \qquad r(1, \text{discard}, 2) = -C, \qquad r(2, \text{end}, 2) = 0,$$

  where $r(1, \text{repair}, 1)$ denotes the reward received when the process is in state 1 (defective valve), action "repair" is chosen and the next state is again state 1 (i.e., the repair was not successful). The other symbols have analogous meaning.

  The expected reward corresponding to the certain state i and chosen action a is (according to the equation $r(i, a) = \sum_{j \in S} r(i, a, j) P_{ij}(a)$.),

$$r(1, \text{repair}) = -c + pC, \qquad r(1, \text{discard}) = -C, \qquad r(2, \text{end}) = 0.$$

  Rewards received during the last decision epoch $(K + 1)$ in each of the states of process:

$$r(1) = -C, \qquad r(2) = 0.$$

- Decision criterion:
  The expected total reward criterion (the initial state is the state 1, $\psi \in \Psi$ is the policy, $X(n)$ represents the state of the valve at the nth decision epoch and $a(n)$ represents the action chosen at the nth decision epoch, $n = 1, 2, \ldots, K + 1$)

$$V_{K+1}(1, \psi) = \mathsf{E}_{i_0}^{\psi} \left(\sum_{n=1}^{K} r\left(X(n), a(n)\right) + r(X(K+1)) \right).$$

The optimal strategy can be found by using the backward induction algorithm ([12]). Since the both the state space and the action space are finite, it is not necessary to evaluate the algorithm for all possible histories of the process but only for all possible current states of the process, which significantly reduces the computational effort ([12]).

Therefore, the backward induction can be computed from the set of equations given by (for $n = 1, 2, \ldots, K$ and for all $i_n \in S$)

$$V_n(i_n) = \max_{a \in A(i_n)} \left\{ r(i_n, a) + \sum_{j \in S} P_{i_n j}(a) V_{n+1}(i_n, a, j) \right\}$$

and the recursion starts with (for all $i_{K+1} \in S$)

$$V_{K+1}(i_{K+1}) = r(i_{K+1}).$$

Application of the backward recursion ([12]) to the optimization model introduced above gives:

- $n = K + 1$:

$$\begin{aligned} V_{K+1}(1) &= r(1) = -C, \\ V_{K+1}(2) &= r(2) = 0. \end{aligned}$$

- $n = K$:

$$\begin{aligned} V_K(1) &= \max_{\{\text{repair, discard}\}} \{C(2p-1) - c, -C\}, \\ V_K(2) &= 0. \end{aligned}$$

  If the process is in state 1 (the valve is still defective) at decision period K, it is optimal to repair the valve if and only if

$$C(2p-1) - c > -C \Longleftrightarrow p > \frac{c}{2C},$$

  if $p < \frac{c}{2C}$, it is optimal to discard this valve and if $p = \frac{c}{2C}$ either action is optimal. If $p > \frac{c}{2C}$, $V_K(1) = C(2p-1) - c$, otherwise $V_K(1) = -C$.

- $n = K - 1$:

$$\begin{aligned} V_{K-1}(1) &= \max_{\{\text{repair, discard}\}} \left\{C(-2p^2 + 4p - 1) - c(2-p), -C\right\}, \\ V_{K-1}(2) &= 0. \end{aligned}$$

  If the process is in state 1 (the valve is still defective) at decision period $K - 1$, it is optimal to repair the valve if and only if

$$C(-2p^2 + 4p - 1) - c(2-p) > -C \Longleftrightarrow p > \frac{c}{2C},$$

  if $p < \frac{c}{2C}$, it is optimal to discard this valve and if $p = \frac{c}{2C}$ either action is optimal. If $p > \frac{c}{2C}$, $V_{K-1}(1) = C(-2p^2 + 4p - 1) - c(2-p)$, otherwise $V_{K-1}(1) = -C$.

$$\vdots$$

- $n = K - k$:

$$
\begin{aligned}
V_{K-k}(1) &= \max_{\{\text{repair, discard}\}} \left\{ C(1 - 2(1-p)^{k+1}) - c\frac{1-(1-p)^{k+1}}{p}, -C \right\}, \\
V_{K-k}(2) &= 0.
\end{aligned}
$$

If the process is in state 1 (the valve is still defective) at decision period $K - k$, it is optimal to repair the valve if and only if

$$
C(1 - 2(1-p)^{k+1}) - c\frac{1-(1-p)^{k+1}}{p} > -C \Longleftrightarrow p > \frac{c}{2C},
$$

if $p < \frac{c}{2C}$, it is optimal to discard this valve and if $p = \frac{c}{2C}$ either action is optimal.

If $p > \frac{c}{2C}$, $V_{K-k}(1) = C(1 - 2(1-p)^{k+1}) - c\frac{1-(1-p)^{k+1}}{p}$, otherwise $V_{K-k}(1) = -C$.

$\vdots$

- $n = 1$:

$$
\begin{aligned}
V_1(1) &= \max_{\{\text{repair, discard}\}} \left\{ C(1 - 2(1-p)^{K}) - c\frac{1-(1-p)^{K}}{p}, -C \right\}, \\
V_1(2) &= 0.
\end{aligned}
$$

It is optimal to repair the defective valve (which did not pass the first test) if and only if

$$
C(1 - 2(1-p)^{K}) - c\frac{1-(1-p)^{K}}{p} > -C \Longleftrightarrow p > \frac{c}{2C},
$$

if $p < \frac{c}{2C}$, it is optimal to discard this valve and if $p = \frac{c}{2C}$ either action is optimal.

If $p > \frac{c}{2C}$, $V_1(1) = C(1 - 2(1-p)^{K}) - c\frac{1-(1-p)^{K}}{p}$, otherwise $V_1(1) = -C$.

It clearly follows from the backward induction performed above that if $p > \frac{c}{2C}$, it is optimal to use the maximal number of possible repairs K to achieve the maximal expected total reward, otherwise it is better to discard the defective valve without any repair. Let us suppose, that $p > \frac{c}{2C}$ and let us further consider this stationary deterministic policy of using maximal number of possible successive repairs.

The value of the expected total reward under this policy represented by $V_1(1)$ can be rewritten to the form

$$
V_1(1) = (1-p)^K \left(\frac{c - 2pC}{p} \right) + C - \frac{c}{p}.
$$

$V_1(1)$ can be viewed as the function of variable K with parameters c, C and p. Since $p > \frac{c}{2C}$ and $1 > 1 - \frac{c}{2C} > 1 - p \geq 0$, it is obvious, that $V_1(1)$ is in the variable K increasing function.

$V_1(1)$ represents the expected total reward from the defective valve, which can be repaired maximally K times. By the same reasoning as in the definition of the optimization model, the valve can pass the first initial test with the probability p and it can be defective with the complementary probability $1-p$. Let us denote $R(K)$ the expected reward from a processed valve (which can pass the initial test), if the maximal number of possible successive repairs is K (c, C and p are parameters of this function). It follows, that

$$R(K) = pC + (1-p)V_1(1) = (1-p)^{K+1}\left(\frac{c-2pC}{p}\right) + C - c\left(\frac{1-p}{p}\right). \tag{1}$$

Since $p > \frac{c}{2C}$ and $1 > 1 - \frac{c}{2C} > 1 - p \geq 0$, it is obvious, that $R(K)$ is also in the variable K increasing function, i.e., the expected reward from a valve increases with the increasing number of possible successive repairs. The limit of the expected reward from one processed valve (considering as a function of K) is

$$\lim_{K\to\infty} R(K) = C - c\left(\frac{1-p}{p}\right). \tag{2}$$

If no repair is allowed, i.e., $K = 0$, the expected reward is $R(0) = C(2p-1)$. This value is nonnegative for $p \geq \frac{1}{2}$, negative otherwise.

If $\frac{1}{2} > p > \frac{c}{2C}$, the important question is how many repairs is necessary to obtain the positive expected reward, i.e., for which K is $R(K) > 0$. This inequality has solution for $p > \frac{c}{C+c}$ (note that $\frac{1}{2} > \frac{c}{C+c} > \frac{c}{2C}$ because $C > c$) given by

$$K > \frac{\ln\left(\frac{p(C+c)-c}{2pC-c}\right)}{\ln(1-p)} - 1,$$

where $\ln(\cdot)$ denotes the natural logarithm.

That is, for $\frac{1}{2} > p > \frac{c}{C+c}$, the number of repairs necessary to obtain positive expected reward is given by the smallest integer bigger than $\frac{\ln\left(\frac{p(C+c)-c}{2pC-c}\right)}{\ln(1-p)} - 1$.

For $\frac{c}{C+c} \geq p > \frac{c}{2C}$, the expected reward from one processed valve is nonpositive for all $K \geq 0$ and approaches the value given by the limit (2). Since the expected reward from one processed valve is increasing function in variable K, the increasing number of possible successive repairs represents reducing the expected loss.

The process of testing and repairing a valve can be viewed as the process consisting of independent Bernoulli trials, therefore, for infinite number of possible successive repairs ($K = \infty$), the number of repairs before repairing the valve has geometric distribution with mean $\frac{1-p}{p}$ (see [7]) and the expected reward from one processed valve with this number of repairs is $C - c\left(\frac{1-p}{p}\right)$, which agrees with the value of the limit (2).

The variance of the reward, if the maximal number of possible successive repairs is K, can be computed as

$$\sum_{j=0}^{K}(C-jc)^2p(1-p)^j + (-C-Kc)^2(1-p)^{K+1} - (R(K))^2 = \tag{3}$$

$$= \frac{(1-p)^{K+1}}{p^2}\left(K(4Ccp^2 - 2c^2p) - c^2p - 4Ccp + 4Ccp^2 + 4C^2p^2\right) + \tag{4}$$

$$+ c^2\frac{1-p}{p^2} - \frac{(1-p)^{2(K+1)}}{p^2}(c-2Cp)^2. \tag{5}$$

Note, that the function $R(K)$ defined by (1) represents the expected reward from a processed valve not only for $p \in (\frac{c}{2C}, 1]$ but for all $p \in (0, 1]$, which can be easily proved by induction (since the repairing process consist of independent Bernoulli trials). If $p \in (0, \frac{c}{2C}]$, this function is decreasing and, therefore, as was already computed from the backward induction, the optimal strategy is not to repair the defective valve and the expected reward from a processed valve under this strategy is $R(0) = C(2p-1)$, which is negative number for $p \in (0, \frac{c}{2C}]$ and $C > c > 0$.

The most important results from the analysis performed above are concluded in the following proposition.

Proposition 2.1
If $p > \frac{c}{2C}$, then the optimal strategy is to use the maximal number of possible successive repairs K. The expected reward from one processed valve $R(K)$ is given by

$$R(K) = (1-p)^{K+1}\left(\frac{c-2pC}{p}\right) + C - c\left(\frac{1-p}{p}\right),$$

which is (under the assumption that $p > \frac{c}{2C}$) increasing function of the maximal number of possible successive repairs K.

If $p \leq \frac{c}{2C}$, then the optimal strategy is not to repair a defective valve and the expected reward from one valve is $C(2p-1)$.

Proof. See analysis performed above. □

On the basis of the results from the optimization model and from the analysis of the behavior of the expected reward, the Markov chain model ([13] and [14]) describing the processing of valves at the checkpoint can be proposed.

Since in the case of $0 < p \leq \frac{c}{2C}$, the optimal strategy is not to repair and the processing of valves is straightforward, therefore, it is supposed from this point further that $1 \geq p > \frac{c}{2C}$ and one valve can be repaired maximally K times.

The states of the proposed Markov chain and transition probabilities between these states are:

- 0 – Preprocessing:
  This state represents that the valve is taken from the production line and the initial functionality test is performed. If the valve passes the initial test, which happens with the probability p, the next state of the valve will be the state "Functional valve", otherwise the next state will be the state "1st repair" (in which the transition occurs with the complementary probability $1-p$).

- 1 – 1st repair:
  The valve in this state is repaired for the first time. The functionality test is performed after the repair and in case of successful repair, which happens with the probability p, the next state of the valve will be the state "Functional valve", otherwise the next state will be the state "2nd repair" (in which the transition occurs with the complementary probability $1-p$).

- 2 – 2nd repair:
  The valve in this state is repaired for the second time. The functionality test is performed after the repair and in case of successful repair, which happens with the probability p, the next state of the valve will be the state "Functional valve", otherwise the next state

will be the state "3rd repair" (in which the transition occurs with the complementary probability $1-p$).

$$\vdots$$

- K – Kth repair:
  The valve in this state is repaired for the Kth time. The functionality test is performed after the repair and in case of successful repair, which happens with the probability p, the next state of the valve will be the state "Functional valve", otherwise the valve will be discarded and the next state is the state "Reject" (in which the transition occurs with the complementary probability $1-p$).

- $K+1$ – Reject:
  This state represents the discarding of the defective valve (which can consist, for example, from disassembling or simple from scrapping), that was not repaired during the K possible repairs. After the discarding of the defective valve, a new valve enters to the checkpoint and, therefore, the next state is with probability 1 the state "Preprocessing".

- $K+2$ – Functional valve:
  This state represents the functional valve, which is send to the further processing on the production line (for example additional assembling or packing). After that, a new valve enters the production line checkpoint and, therefore, the next state is with probability 1 the state "Preprocessing".

The transition matrix (with dimensions $(K+3)\times(K+3)$) for this Markov chain is given by (the states are numbered by nonnegative integers as indicated above)

$$P = \begin{bmatrix} 0 & 1-p & 0 & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 1-p & 0 & 0 & \cdots & 0 & p \\ 0 & 0 & 0 & 1-p & 0 & \cdots & 0 & p \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1-p & p \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{6}$$

and the transition diagram of this Markov chain can be seen in the figure 5.

This model is a generalization of the model proposed in ([10]), which is at the same time adapted according to the optimization results.

It is obvious, that the proposed Markov chain is irreducible and aperiodic and since the state space is finite ([14]), the limiting probabilities $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist ([13]). The limiting probabilities π_j's also represent the long-run proportion of time that the process will be in state j. They can be computed directly as the unique solution of the set of linear equations

$$\pi_j = \sum_{i\in S} \pi_i P_{ij}, \quad j \in S, \tag{7}$$

$$\sum_{i\in S} \pi_i = 1. \tag{8}$$

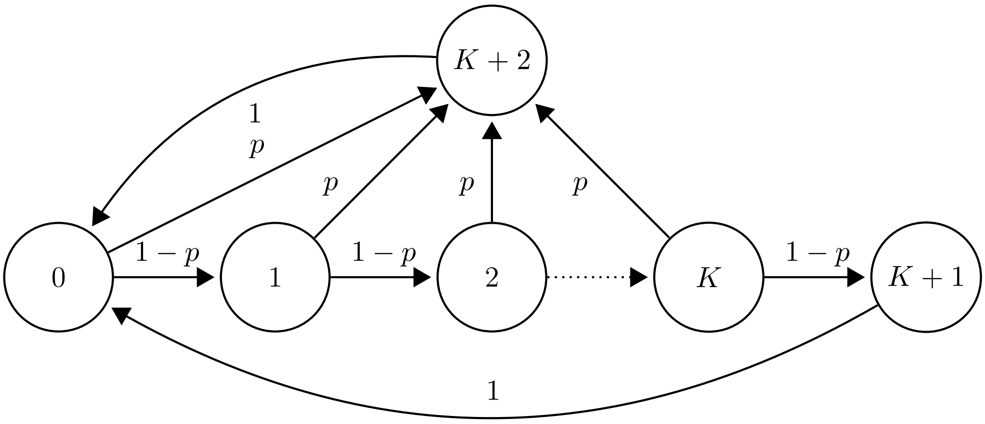

Figure 5: Transition graph representing the processing of one valve. The values at arrows represent the transition probabilities

Furthermore, it holds that

$$\pi_j = \frac{1}{\nu_{jj}} \tag{9}$$

where ν_{jj} represents the expected number of transitions until a Markov chain, starting in a state j, returns to that state.

The computed limiting probabilities are

$$\begin{aligned} \pi_i &= \frac{p(1-p)^i}{p+1-(1-p)^{K+1}}, \quad \text{for } i = 0, 1, \ldots, K+1 \\ \pi_{K+2} &= \frac{p(1-(1-p)^{K+1})}{p+1-(1-p)^{K+1}}. \end{aligned}$$

Since the mean number of transitions which a valve has to absolve is $\nu_{00} = \frac{1}{\pi_0}$ (according to the equation (9)), it is possible to compute the expected number of repairs as $\nu_{00} - 2$ (two transitions are subtracted since each valve has to be in the preprocessing state and each valve will be either discarded or rejected). Therefore, the expected number of repairs is

$$\frac{1-p}{p}\left(1-(1-p)^K\right). \tag{10}$$

The variance of the number of repairs can be computed as

$$\begin{aligned} &\sum_{j=0}^{K} j^2 p(1-p)^j + K^2(1-p)^{K+1} - \frac{(1-p)^2}{p^2}\left(1-(1-p)^K\right)^2 = \\ &= \frac{1}{p^2}\left(1-p-p(1-p)^{K+1}(2K+1)-(1-p)^{2K+2}\right). \end{aligned} \tag{11}$$

It is possible to redefine the states $K+1$ (Reject) and $K+2$ (Functional valve) as the absorbing states, i.e., the transitions occur only to the same state (if the process enter an absorbing state, it will never leave this state). Then the other states are transient states and the transition matrix can be partitioned to the submatrices which corresponds to the matrices representing transitions inside the transient classes or inside the recurrent classes and among these classes. The matrix E, which represents the transition matrix for the reccurent classes, is the identity matrix with the dimensions 2×2 and the matrix U, which represents the transition matrix among the transient classes, is the matrix with the dimensions $(K+1) \times (K+1)$ and with all entries equal to zero except the entries on the upper diagonal, which are equal to $1-p$. The matrix Y, which represents the transition matrix from transient states to the recurrent states, is the matrix with the dimensions $(K+1) \times 2$ with all rows equal to the vector $[0, p]$ except the last row, which is equal to the vector $[1-p, p]$. The following matrices can be used for the computation of the expected time spent in the transient states and in the probability of absorption in the recurrent states ([9], [5]):

$$[I-U]^{-1} = \begin{bmatrix} 1 & 1-p & (1-p)^2 & \cdots & (1-p)^K \\ 0 & 1 & 1-p & \cdots & (1-p)^{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$[I-U]^{-1} \cdot Y = \begin{bmatrix} (1-p)^{K+1} & (1-(1-p)^{K+1}) \\ (1-p)^K & (1-(1-p)^K) \\ \vdots & \vdots \\ 1-p & p \end{bmatrix}.$$

For the purposes of the modeling a production line checkpoint, the first row of these matrices is important. The values in the first row of the matrix $[I-U]^{-1}$ represent the expected number of time periods, that the processed valve spends in the corresponding transient state and since the value in the first column of this first row is equal to 1, the entries in the other columns of the first row also represent the probabilities, that the valve ever makes a transition into that transient states. The first value in the first row of the matrix $[I-U]^{-1} \cdot Y$ represents the probability, that the processed valve will be discarded and the other value in this row represent the probability, that the processed valve will be functional.

Now, the irreducible model will be again considered and the role of the transition times can be introduced. For further computations, it is necessary to specify the expected amount of time spent in some state during each visit given the next state of the process η_{ij}, and the concrete form of the time distribution need not to be considered ([13], [5]). It follows from the description of the process, that the expected time spent in a state does not depend on the next state of the process. Moreover, the expected time of repair does not depend on the number of previous repairs. Therefore, the matrix of η_{ij}'s is

$$\begin{bmatrix} 0 & \tau_P & 0 & 0 & 0 & \cdots & 0 & \tau_P \\ 0 & 0 & \tau_R & 0 & 0 & \cdots & 0 & \tau_R \\ 0 & 0 & 0 & \tau_R & 0 & \cdots & 0 & \tau_R \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \tau_R & \tau_R \\ \tau_D & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \tau_F & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where τ_P denotes the expected time of the preprocessing of a valve, τ_R denotes the expected time of one repair (the time of the functionality test is included), τ_D denotes the expected time of discarding defective valve and τ_F denotes the expected time of further processing of a functional valve at the production line checkpoint.

Let μ_i denote the expected amount of time spent in state i during each visit. From the nature of the process and from the equation $\mu_i = \sum_{j\in S} P_{ij}\eta_{ij}$ ([13]), it follows, that

$$\begin{aligned}\mu_0 &= \tau_P,\\ \mu_i &= \tau_R, \quad \text{for } i = 1, 2, \ldots, K,\\ \mu_{K+1} &= \tau_D,\\ \mu_{K+2} &= \tau_F.\end{aligned}$$

The limiting distribution is already computed and also other necessary conditions are fulfilled ([13]), therefore, the limiting probabilities P_j's, which also represent the (long-run) proportions of time, that the semi-Markov process is in state $j \in S$, and the mean recurrence times μ_{jj}'s, can be computed with use of the equations

$$\mu_{jj}\pi_j = \sum_{i\in S} \pi_i\mu_i, \tag{12}$$

$$P_j = \frac{\pi_j\mu_j}{\sum_{i\in S}\pi_i\mu_i} = \frac{\mu_j}{\mu_{jj}}. \tag{13}$$

The mean recurrence times are given by

$$\mu_{ii} = \frac{1}{(1-p)^i}\left((1-p)^{K+1}\left(\tau_D - \tau_F - \frac{\tau_R}{p}\right) + \tau_P + \tau_F + \frac{1-p}{p}\tau_R\right), \text{ for } i = 0, 1, \ldots, K+1,$$

$$\mu_{K+2,K+2} = \frac{1}{1-(1-p)^{K+1}}\left((1-p)^{K+1}\left(\tau_D - \tau_F - \frac{\tau_R}{p}\right) + \tau_P + \tau_F + \frac{1-p}{p}\tau_R\right).$$

The very important quantity is the expected time of processing a valve, i.e., the expected length of the cycle at the production line checkpoint, which is given by the mean recurrence time to the state "Preprocessing" and its value is

$$\begin{aligned}\mu_{00} &= (1-p)^{K+1}\left(\tau_D - \tau_F - \frac{\tau_R}{p}\right) + \tau_P + \tau_F + \frac{1-p}{p}\tau_R \\ &= \tau_P + \frac{1-p}{p}\left(1-(1-p)^K\right)\tau_R + (1-p)^{K+1}\tau_D + \left(1-(1-p)^{K+1}\right)\tau_F.\end{aligned} \tag{14}$$

This result represents the fact, that each valve has to be in the preprocessing state, the expected number of repairs is $\frac{1-p}{p}\left(1-(1-p)^K\right)$ and the valve will be functional at the end of the process with the probability $1-(1-p)^{K+1}$ and it will be defective with the complementary probability $(1-p)^{K+1}$. The μ_{00} can be considered as the function of the number of possible repairs K with parameters $p, c, C, \tau_P, \tau_R, \tau_D$ and τ_F. Note, that μ_{00} is always positive and also note, that if $\tau_D - \tau_F - \frac{\tau_R}{p} > 0$, the expected length of the cycle is decreasing with the increasing number of possible successive repairs. This can correspond to the case, when the discarded valve is disassembled at the checkpoint, which can last much more time than the other operations. Therefore, each additional repair decreases the probability of this situation and the expected time of processing valve is decreasing function of the maximal number of possible successive repairs in this case.

Another important quantity is the expected time between two discarded valves which is given by the mean recurrence time to the state "Reject" and the result is

$$\mu_{K+1,K+1} = \tau_D - \tau_F - \frac{\tau_R}{p} + \frac{1}{(1-p)^{K+1}} \left(\tau_P + \tau_F + \frac{1-p}{p} \tau_R \right). \tag{15}$$

The time of processing of a valve which will not be functional at the end of process is $\tau_P + K\tau_R + \tau_D$ and it happens with the probability $(1-p)^{K+1}$. Therefore, the expected time of processing a valve, which will be functional at the end of process is given by the expression $\mu_{00} - (1-p)^{K+1}(\tau_P + K\tau_R + \tau_D)$.

The most important quantity is the expected average reward per time unit. Since the processing of valves at the production line checkpoint is the regenerative process ([13]) (the processing of one valve is the cycle of this regenerative process), the expected average reward per time unit can be computed as the ratio of the expected reward per one cycle and the expected length of one cycle (the conditions for this computation are obviously fulfilled ([13])). The expected average reward per time unit can be viewed as the function of number of possible successive repairs K with the parameters $p, c, C, \tau_P, \tau_R, \tau_D$ and τ_F. It will be denoted as $AR(K)$ and it is defined by

$$\begin{aligned} AR(K) &= \frac{R(K)}{\mu_{00}} \\ &= \frac{(1-p)^{K+1}\left(\frac{c-2pC}{p}\right) + C - c\left(\frac{1-p}{p}\right)}{(1-p)^{K+1}\left(\tau_D - \tau_F - \frac{\tau_R}{p}\right) + \tau_P + \tau_F + \frac{1-p}{p}\tau_R}. \end{aligned} \tag{16}$$

The behavior of the $AR(K)$ is very important for the overall performance of the production line checkpoint. The expected average reward per time unit can be written in the form

$$AR(K) = \frac{u^{K+1}v + w}{u^{K+1}x + y}$$

where

$$\begin{aligned} u &= 1 - p, \\ v &= \frac{c - 2pC}{p}, \\ w &= C - c\frac{1-p}{p}, \\ x &= \tau_D - \tau_F - \frac{\tau_R}{p}, \\ y &= \tau_P + \tau_F + \frac{1-p}{p}\tau_R. \end{aligned}$$

The important question is, whether the $AR(K)$ is monotone, increasing or decreasing in the variable K. This can be seen from the difference between two successive values of $AR(K)$:

$$AR(K+1) - AR(K) = \frac{u^{K+1}(u-1)(vy - wx)}{(u^{K+2}x + y)(u^{K+1}x + y)}. \tag{17}$$

Since there is the product of two positive values in the denominator of (17), $u-1<0$ and $u>0$, the behavior of the $AR(K)$ is determined by the value of $vy-wx$. The $AR(K)$ is increasing for $vy-wx<0$ and the best strategy is to use the maximal number of possible successive repairs. If $vy-wx>0$, the $AR(K)$ is decreasing with the increasing number of possible successive repairs and if $vy-wx=0$, the $AR(K)$ is constant. In both of these cases, the best strategy is to discard all defective valves without any repair (the advantage of this strategy for constant $AR(K)$ is that more valves is processed during the work shift).

The value of $vy-wx$ is given by

$$vy-wx=C(2\tau_R-2\tau_P-\tau_D-\tau_F)+c(\tau_F-\tau_D)+\frac{c(\tau_P+\tau_D)-C\tau_R}{p}$$

and for the concrete values of parameters, it can be determined, whether the $AR(K)$ is increasing or nonincreasing.

The positivity or the negativity of the expected average reward $AR(K)$ is given by the behavior of $R(K)$, which was already examined.

The most important results from this analysis are summarized in the following proposition.

Proposition 2.2
The expected time of processing a valve in the proposed model is given by

$$\mu_{00}=(1-p)^{K+1}\left(\tau_D-\tau_F-\frac{\tau_R}{p}\right)+\tau_P+\tau_F+\frac{1-p}{p}\tau_R.$$

The expected average reward per time unit (as a function of the maximal number of possible successive repairs) is given by

$$AR(K)=\frac{(1-p)^{K+1}\left(\frac{c-2pC}{p}\right)+C-c\left(\frac{1-p}{p}\right)}{(1-p)^{K+1}\left(\tau_D-\tau_F-\frac{\tau_R}{p}\right)+\tau_P+\tau_F+\frac{1-p}{p}\tau_R}.$$

The expected average reward per time unit is increasing for

$$C(2\tau_R-2\tau_P-\tau_D-\tau_F)+c(\tau_F-\tau_D)+\frac{c(\tau_P+\tau_D)-C\tau_R}{p}<0.$$

In that case, the optimal strategy is to use the maximal number of possible successive repairs, otherwise, it is optimal not to repair a defective valve.

Proof. See analysis performed above. □

The checkpoint is the part of the whole production line. The arrivals of valves can be modeled by a Poisson process with parameter λ_a and, therefore, the checkpoint can be viewed as the $M/G/1$ queueing system (see [1]). The first moment of the service distribution is given by μ_{00}, i.e., $\mathsf{E}(B)=\mu_{00}$. To avoid infinite queue, it is necessary to fulfill the condition $\lambda_a\mathsf{E}(B)<1$. From this inequality, it is possible to compute the maximal number of possible successive repairs. Also the optimization results must be adapted in some cases.

- If $\tau_D-\tau_F-\frac{\tau_R}{p}>0$, then it must hold that $1-\lambda_a\left(\tau_P+\tau_F+\frac{1-p}{p}\tau_R\right)>0$ (otherwise the production line will be blocked) and since the expected time of processing one valve is decreasing function in variable K, the minimal number of possible successive repairs which prevents from the blocking of production line is given by

$$K > \frac{\ln\left(\frac{1-\lambda_a\left(\tau_P+\tau_F+\frac{1-p}{p}\tau_R\right)}{\lambda_a\left(\tau_D-\tau_F-\frac{\tau_R}{p}\right)}\right)}{\ln(1-p)} - 1. \tag{18}$$

If $1 - \lambda_a\left(\tau_P + \tau_F + \frac{1-p}{p}\tau_R\right) > 0$ and if the expected average reward per time unit is increasing, the optimal strategy is to repair the valve till it is functional. If the expected average reward per time unit is nonincreasing function of the number of possible successive repairs, then it is necessary to make the minimal number of the possible successive repairs given by (18) to avoid the blocking of the production line checkpoint but the design of the process is obviously ineffective in this case.

- If $\tau_D - \tau_F - \frac{\tau_R}{p} < 0$, then for $1-\lambda_a\left(\tau_P + \tau_F + \frac{1-p}{p}\tau_R\right) \geq 0$, the production line will not be blocked for any number of possible successive repairs. For $1-\lambda_a\left(\tau_P + \tau_F + \frac{1-p}{p}\tau_R\right) < 0$, the maximal number of possible successive repairs is given by (since the expected time of processing one valve is increasing function in variable K)

$$K < \frac{\ln\left(\frac{1-\lambda_a\left(\tau_P+\tau_F+\frac{1-p}{p}\tau_R\right)}{\lambda_a\left(\tau_D-\tau_F-\frac{\tau_R}{p}\right)}\right)}{\ln(1-p)} - 1. \tag{19}$$

  If the expected average reward per time unit is increasing, the optimal strategy is to use the maximal number of possible successive repairs or to repair the valve till it is functional (for $1 - \lambda_a\left(\tau_P + \tau_F + \frac{1-p}{p}\tau_R\right) \geq 0$, but in this case the design of the process is obviously ineffective), otherwise, the optimal strategy is not to repair a defective valve.

- If $\tau_D - \tau_F - \frac{\tau_R}{p} = 0$, then the expected time of processing one valve does not depend on the number of possible repairs and it must hold that $1 - \lambda_a\left(\tau_P + \tau_F + \frac{1-p}{p}\tau_R\right) > 0$ to avoid blocking the production line.

  If this condition is fulfilled, then the expected average reward per time unit is increasing (since it is supposed that $1 \geq p > \frac{c}{2C}$) and the optimal strategy is to repair the valve till it is functional (but the design of the process is ineffective).

The second moment of the service time can be computed as

$$\begin{aligned} p\sum_{j=0}^{K}(\tau_P + j\tau_R + \tau_F)^2(1-p)^j + (\tau_P + K\tau_R + \tau_D)^2(1-p)^{K+1} = \\ = (1-p)^{K+1}(\tau_D - \tau_F)(2\tau_P + \tau_D + \tau_F) + (\tau_P + \tau_F)^2 + \\ + 2\tau_R\left(\frac{1-p}{p}(\tau_P + \tau_F)(1-(1-p)^K) + (1-p)^{K+1}K(\tau_D + \tau_F)\right) + \\ + \frac{\tau_R^2}{p^2}\left((1-p)^{K+1}(p - 2pK - 2) + p^2 - 3p + 2\right). \end{aligned} \tag{20}$$

With use of the first and second moment of the service time, it is possible to compute the variance of the service time and also the other quantities which characterize the queueing system ([1]). The average amount of time, that a valve spends waiting in queue W_Q is given by

$$W_Q = \frac{\lambda_a \mathsf{E}\left(B^2\right)}{2(1-\lambda_a \mathsf{E}\left(B\right))}, \tag{21}$$

the average number of valves waiting in queue L_Q is given by

$$L_Q = \lambda_a W_Q = \frac{\lambda_a^2 \mathsf{E}\left(B^2\right)}{2(1-\lambda_a \mathsf{E}\left(B\right))}, \tag{22}$$

the average amount of time that a valve spends in the queueing system W equals to

$$W = W_Q + \mathsf{E}\left(B\right) = \frac{\lambda_a \mathsf{E}\left(B^2\right)}{2(1-\lambda_a \mathsf{E}\left(B\right))} + \mathsf{E}\left(B\right), \tag{23}$$

the average number of valves in the queueing system L is given by

$$L = \lambda_a W = \frac{\lambda_a^2 \mathsf{E}\left(B^2\right)}{2(1-\lambda_a \mathsf{E}\left(B\right))} + \lambda_a \mathsf{E}\left(B\right). \tag{24}$$

The expected length of idle periods is $\frac{1}{\lambda_a}$, the expected length of busy periods $\frac{\mathsf{E}(B)}{1-\lambda_a \mathsf{E}(B)}$ and the expected number of customers served during a busy period $\frac{1}{1-\lambda_a \mathsf{E}(B)}$. The explicit formulas for the discussed model are not stated here due to the very complicated forms of the moments of the service time.

2.0.1 Remarks

Note, that the most of the computed quantities could be computed without use of Markov and semi-Markov processes theory and Markov decision processes theory, due to the nature of the proposed model. However, the most of these quantities can be computed in more elegant way with use of the (semi-)Markov processes theory and Markov decision processes theory, which would be revealed in more complicated model.

For a general model, in which the transition probabilities among the successive repairs are different, it is not possible to obtain nice compact general result as for the proposed model. The general results in suitable form could be obtained in some special cases, e.g., when the transition probabilities between successive repairs are values of a suitable function of the order number of repair and a parameter. Since the modeling of the production line checkpoint is the finite horizon problem with finite number of states, the results can be always obtained numerically for concrete values of parameters.

3 Appropriate distributions for modeling transition times

Important task for the proper modeling of the processes at a production line checkpoint is the choice of suitable distributions describing the transition times among the states of the process. The choice of the distributions should be based on the real data observed from the process. Unfortunately, the real data were not available, therefore, it was necessary to determine the appropriate distributions from the nature of the process and from the known results for similar processes.

Operations performed at each state last some average time but because of human factor, there can be some deviations from this average time which can prolong or shorten the processing of a valve in each state. Due to the nature of the process, there exists some minimal time for which the process stays in the same state and also unexpected or unusual events can occur which, on the contrary, can significantly prolong the processing time. Therefore, the distributions with the positive skewness should be expected. In similar processes, the Weibull and the log-normal distributions are mostly used, therefore, the most important properties of these distributions relevant for the modeling of a production line checkpoint are described in this section. This section is based on ([6]), where it is possible to find more detailed description of the mentioned distributions.

The basic estimate of the distribution of the data can be done with use of the histograms and probability plots, the appropriateness of the choice of a distribution for the real data can be verified by statistical tests, for example, by the Kolmogorov-Smirnov test or by the Anderson-Darling test. More information about the goodness of fit tests can be found in ([8]).

3.1 Weibull distribution

The Weibull distribution is one of the most commonly used distributions for modeling time to repair due its versatility. The Weibull distribution is very flexible and can, through an appropriate choice of parameters, be used for modeling of many types of processes.

The Weibull probability density function is given by

$$f(x) = \begin{cases} \frac{\beta}{\theta}\left(\frac{x-\gamma}{\theta}\right)^{\beta-1} \exp\left(-\left(\frac{x-\gamma}{\theta}\right)^{\beta}\right) & x > \gamma, \\ 0 & x \leq \gamma, \end{cases}$$

where $\beta > 0$ is the shape parameter, $\theta > 0$ is the scale parameter and $\gamma \in \mathbb{R}$ is the location parameter.

The corresponding distribution function is

$$F(x) = \begin{cases} 1 - \exp\left(-\left(\frac{x-\gamma}{\theta}\right)^{\beta}\right) & x > \gamma, \\ 0 & x \leq \gamma. \end{cases}$$

The shape parameter β gives the Weibull distribution its flexibility. The Weibull distribution can model a wide variety of data simply by changing the values of the shape parameter. For $\beta = 1$, the Weibull distribution is identical to the exponential distribution, if $\beta = 2$, the Weibull distribution is identical to the Rayleigh distribution. For β between 3 and 4, the Weibull distribution approximates the normal distribution. The Weibull distribution can also approximate the log-normal distribution. The flexibility of the Weibull distribution is illustrated by the figure 6 (the scale and the location parameters are fixed, $\theta = 1$ and $\gamma = 0$).

The scale parameter θ determines the range of the distribution. Increasing the value of θ while holding other parameters constant has the effect of stretching out the probability density function and the "peak" of the probability density function decreases (because of the normalization condition).

The location parameter γ locates the distribution along the x-axis. Changing of the value of this parameter will cause "sliding" of the distribution along the x-axis.

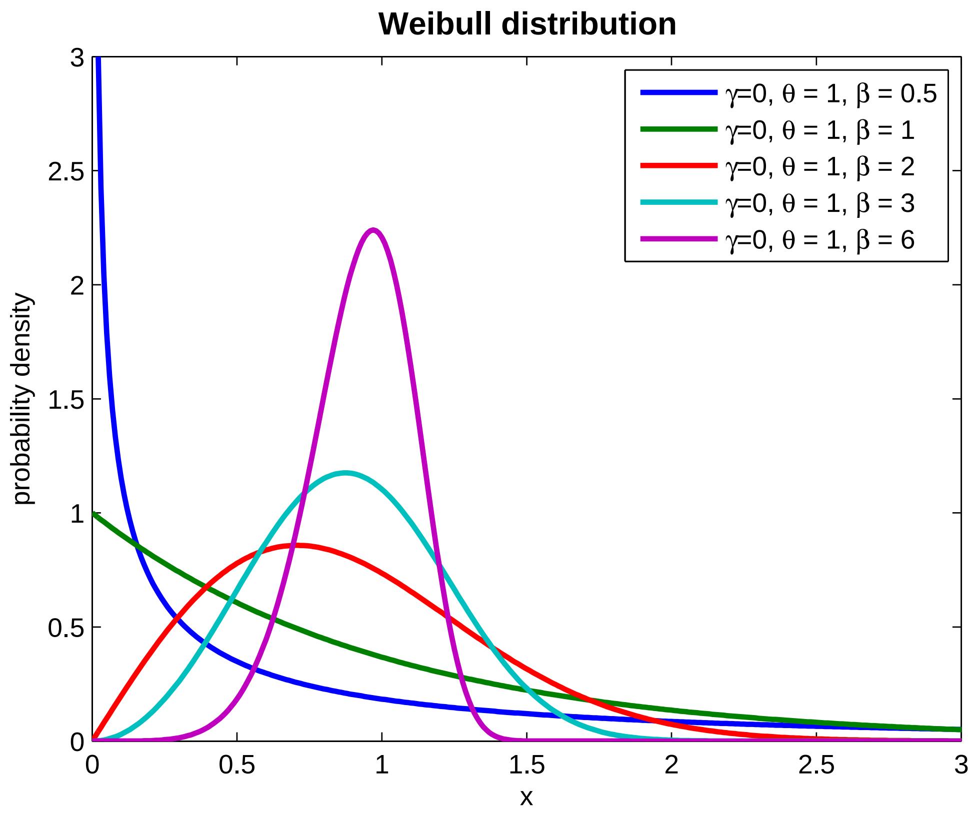

Figure 6: The influence of the shape parameter β of the Weibull distribution, other parameters remain constant

The mean and the variance of the Weibull distribution are given by

$$\mathsf{E}\,(x) = \theta\Gamma\left(1+\frac{1}{\beta}\right)+\gamma, \quad \mathsf{Var}\,(x) = \theta^2\left(\Gamma\left(1+\frac{2}{\beta}\right)-\left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right),$$

where $\Gamma(u) = \int\limits_0^\infty t^{u-1}\exp(-t)\mathrm{d}t$ denotes the gamma function.

Let us suppose, that n realizations $x_1,\ldots,x_n$ of the random variable with the Weibull distribution were observed. The most usual situation is γ known, which is also the case of modeling a production line checkpoint. Then the maximum likelihood estimators $\hat\beta$ and $\hat\theta$ of parameters β and θ, respectively, satisfy the equations

$$\hat\beta = \left(\frac{\sum\limits_{i=1}^{n}(x_i-\gamma)^{\hat\beta}\ln(x_i-\gamma)}{\sum\limits_{i=1}^{n}(x_i-\gamma)^{\hat\beta}} - \frac{1}{n}\sum_{i=1}^{n}(\ln(x_i-\gamma))\right)^{-1}, \tag{25}$$

$$\hat\theta = \left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\gamma)^{\hat\beta}\right)^{\frac{1}{\hat\beta}}. \tag{26}$$

The value of $\hat{\beta}$ needs to be obtained from the equation (25) and then used in the equation (26) to obtain $\hat{\alpha}$.

The maximum likelihood estimators for the case when γ is also unknown and, therefore, it must be also estimated, can be found in ([6]), where also the other properties of the Weibull distribution are summarized.

3.2 Log-normal distribution

Another commonly used distribution for modeling time to repair is the log-normal distribution.

The probability density function of log-normal distribution is given by

$$f(x) = \begin{cases} \frac{1}{(x-\kappa)\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x-\kappa)-\mu)^2}{2\sigma^2}\right) & x > \kappa, \\ 0 & x \leq \kappa, \end{cases}$$

where $\sigma > 0$ is the shape parameter, $\mu \in \mathbb{R}$ is the scale parameter and $\kappa \in \mathbb{R}$ is the location parameter (π denotes the Ludolph's Constant in this context). If a random variable X is log-normally distributed with parameters μ, σ and κ, then $\ln(X - \kappa)$ of this random variable is normally distributed with mean μ and variance σ^2.

The corresponding distribution function is

$$F(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} \int\limits_0^x \frac{1}{u-\kappa} \exp\left(-\frac{(\ln(u-\kappa)-\mu)^2}{2\sigma^2}\right) \mathrm{d}u & x > \kappa, \\ 0 & x \leq \kappa. \end{cases}$$

The shape of the log-normal distribution depends on the shape parameter σ. The log-normal distribution is skewed to the right and the skewness increases as the value of σ increases, as can be seen in the figure 7 (the other parameters remain constant, $\mu = 0$ and $\kappa = 0$).

Increasing values of the location parameter μ while holding the other parameters constant has the effect of stretching out the probability density function and the "peak" of the probability density function decreases (because of the normalization condition).

The location parameter κ represents the shift of the whole distribution along the x-axis.

The mean and variance of the log-normal distribution are given by

$$\mathsf{E}\,(x) = \exp\left(\mu + \frac{\sigma^2}{2}\right) + \kappa, \quad \mathsf{Var}\,(x) = \exp\left(2\mu + \sigma^2\right)\left(\exp(\sigma^2) - 1\right).$$

Let us suppose, that n realizations $x_1, \ldots, x_n$ of the random variable with the log-normal distribution were observed. The most usual situation is κ known, which is also the case of modeling a production line checkpoint. Then the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ of parameters μ and σ, respectively, are given by

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \ln(x_i - \kappa)$$

$$\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\ln(x_i - \kappa) - \hat{\mu})^2}.$$

The maximum likelihood estimators for the case when κ is also unknown and, therefore, it must be also estimated, can be found in ([6]), where also the other properties of the log-normal distribution are summarized.

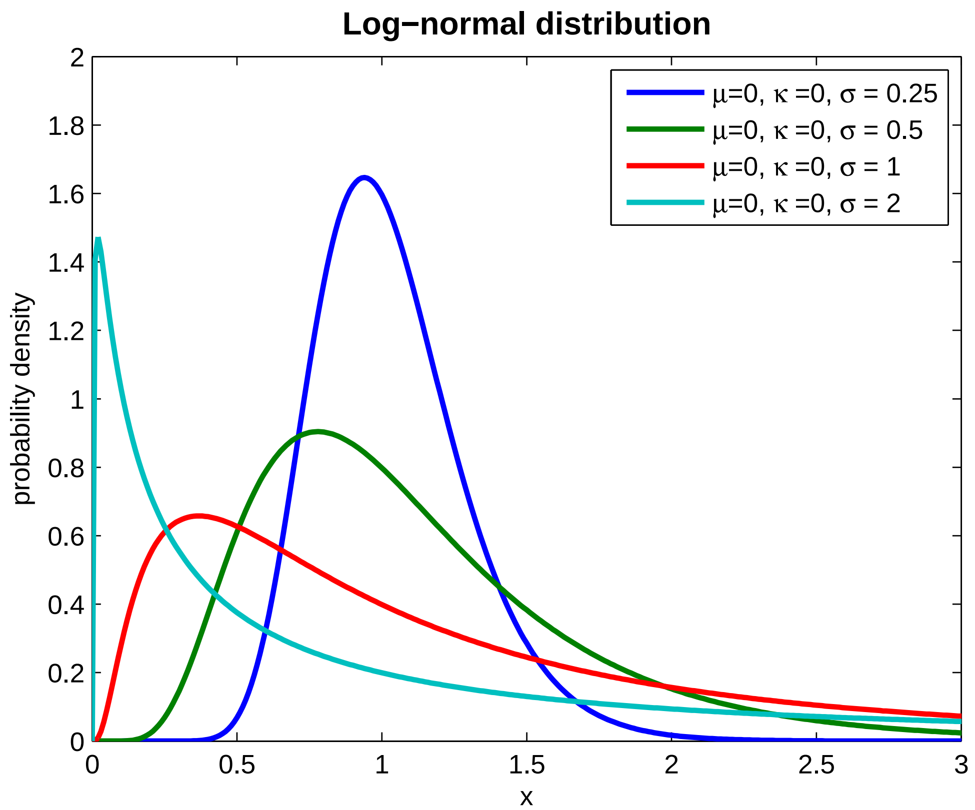

Figure 7: The influence of the shape parameter σ of the log-normal distribution, other parameters remain constant

4 Results for real data

The model which was proposed and analyzed in the previous sections was motivated by the real data from the real production line checkpoint. As was already mentioned, no data about the processing times were available, only the data about the number of repairs of one valve could be used.

The main task is to estimate the probability of passing the functionality test p. Data contained information about 500 processed valves. On the real production line, each valve is repaired to the full functionality and no valve is discarded. One valve was maximally repaired 8 times in the available data set. The number of repairs which one valve absolved till the full functionality is summarized in the following table:

Number of repairs	0	1	2	3	4	5	6	7	8
Number of valves	301	120	48	15	9	4	2	0	1

Based on these results and from the nature of the repairing process, the probability of passing the functionality test can be estimated from the relative frequencies of transitions from a certain state (“Preprocessing” or one of the successive repairs) to the state “Functional valve”. These relative frequencies are summarized in the following table (state “Preprocessing” is denoted as 0, the other states are labeled by the number of repairs which a valve has already absolved and the estimate of the probability of the passing the functionality test is denoted as $\hat{p}$):

State	0	1	2	3	4	5	6	7	8
$\hat{p}$	0.602	0.603	0.607	0.484	0.500	0.571	0.667	0.000	1.000

As can be seen from the previous table, the estimates of the probability of passing the functionality test $\hat{p}$ are almost the same from the states "Preprocessing", "1st repair" and "2nd repair", which confirms the assumption that the most of defects is caused by the defective gasket which is, therefore, replaced during repair and the new gasket may be defective with the same probability as the original gasket. The estimates of $\hat{p}$ for higher order of repairs are not very accurate because there was not enough data (the most of defective valves is already processed). The inaccuracy can be also caused by the more complicated defects and unexpected situations.

This processing of a valve can be viewed as the series of the Bernoulli trials, therefore, the probability of passing the functionality test $\hat{p}$ may be estimated with use of the maximum likelihood estimator (which eliminates the influence of lack of data for the higher orders of repairs) from the values summarized in the first table. The estimated value is

$$\hat{p} = 0.597.$$

As already mentioned, no data about the processing times nor about the rate of the incoming valves were available. For the purposes of illustration of the theoretical results, the mean processing times in particular states and the rate of the incoming valves were estimated by the expert on the basis of results from similar processes. These estimated values are (in minutes and for λ_a in number of incoming valves per one minute)

$$\tau_P = 0.75, \quad \tau_R = 1.5, \quad \tau_D = 0.5, \quad \tau_F = 0.5, \quad \lambda_a = 0.45. \tag{27}$$

The value of valve C and the cost of repair c were also unknown, therefore, for the purposes of illustration of the theoretical results were chosen values (the concrete currency is not important)

$$C = 80, \quad c = 30.$$

All necessary parameters for the analysis and modeling of the processes at the production line checkpoint are set up.

Since $C(2\tau_R - 2\tau_P - \tau_D - \tau_F) + c(\tau_F - \tau_D) + \frac{c(\tau_P+\tau_D)-C\tau_R}{\hat{p}} = -98.191$, the expected reward per time unit $AR(K)$ is increasing function in K and the optimal strategy is to use the maximal number of possible successive repairs.

Since $\tau_D - \tau_F - \frac{\tau_R}{\hat{p}} = -2.513$ and $1 - \lambda_a \left(\tau_P + \tau_F + \frac{1-\hat{p}}{\hat{p}}\tau_R\right) = -0.018$, the maximal number of possible successive repairs K which prevent the blocking of the production line checkpoint is defined by the inequality (19), which states that $K < 3.546$ and, therefore, the value of K is (after the rounding down)

$$K = 3.$$

This maximal number of possible successive repairs will be considered in further analysis.

The expected number of repairs defined by the equation (10) is 0.631 and the variance of the number of repairs defined by the equation (11) is 0.819. The probability of discarding a valve is 0.026.

The expected reward from one valve is according to the equation (1)

$$R(3) = 56.854$$

and the variance of the reward from one valve is 1994.884 (according to the equation (3)).

The expected time of processing one valve given by the equation (14) is

$$\mu_{00} = \mathsf{E}(B) = 2.196 \text{ minutes}$$

and the second moment of the time of processing a valve given by the equation (20) is 6.905, which determines the variance of the time of processing one valve as 2.081. This implies, that during one hour is processed $60 : 2.196 = 27.322$ valves. The expected time between two discarded valves given by the equation (15) is 83.266 minutes.

The proportions of time that the process is in particular states P_j's are

$$P_0 = 0.341, \quad P_1 = 0.275, \quad P_2 = 0.111, \quad P_3 = 0.045, \quad P_4 = 0.006, \quad P_5 = 0.222.$$

The expected reward per minute $AR(3)$ is (according to the equation (16))

$$AR(3) = 25.886.$$

The behavior of the expected reward per time unit for the various number of maximal possible successive repairs and with the values of other parameters defined above can be seen in the figure 8.

The production line checkpoint may be also analyzed with use of the results from the queueing theory. The fraction of time that the checkpoint is working (which is equal to the probability that there is a valve at the checkpoint) can be computed from the equation $\rho = \lambda_a \mathsf{E}(B)$ and the result is $\rho = 0.988$. The average number of valves in the queueing system (i.e., in the queue and at the checkpoint) given by the equation (24) is $L = 60.898$ and the average number of valves in the queue given by the equation (22) is

$$L_Q = 59.910.$$

The average number of valves in the queue L_Q can be used for the determining of the size of the container for unprocessed valves.

Each valve will spend (according to the equation (23)) on average 135.329 minutes in the queueing system and the average time which a valve spends in the queue is (according to the equation (21))

$$W_Q = 133.133 \text{ minutes}.$$

The expected length of busy periods of the production line checkpoint is 188.206 minutes during which 85.693 valves will be processed on average. The expected length of idle periods at the production line checkpoint is 2.222 minutes.

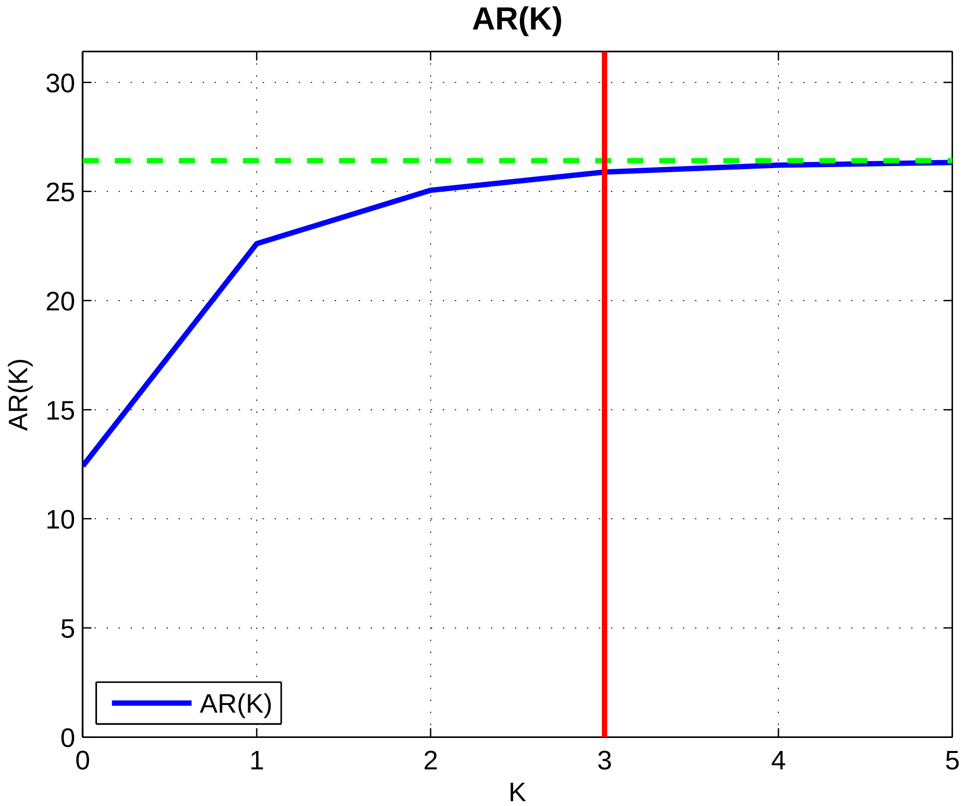

Figure 8: The expected reward per time unit $AR(K)$ as a function of maximal number of possible successive repairs K. The values of parameters are $p = 0.597$, $\tau_P = 0.75$, $\tau_R = 1.5$, $\tau_D = 0.5$, $\tau_F = 0.5$, $C = 80$ and $c = 30$. The maximal number of possible successive repairs which will not block the production line for $\lambda_a = 0.45$ is marked by the vertical red line, the limiting value is marked by the horizontal dashed green line.

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