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Algebraic Design Methods

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21.1 Introduction

One of the features of modern control theory is the growing presence of algebra. Algebraic formalism offers several useful tools for control system design, including the so-called "factorization" approach.

This approach is based on the input–output properties of linear systems. The central idea is that of "factoring" the transfer matrix of a (not necessarily stable) system as the "ratio" of two *stable* transfer matrices. This is a natural step for the linear systems whose transfer matrices are rational, that is, for the lumped-parameter systems. Under certain conditions, however, this approach is productive also for the distributed-parameter systems.

The starting point of the factorization approach is to obtain a simple parameterization of *all* controllers that stabilize a given plant. One could then, in principle, choose the best controller for various applications. The key point here is that the parameter appears in the closed-loop system transfer matrix in a linear manner, thus making it easier to meet additional design specifications.

The actual design of control systems is an engineering task that cannot be reduced to algebra. Design contains many additional aspects that have to be taken into account: sensor placement, computational constraints, actuator constraints, redundancy, performance robustness, among many others. There is a need for an understanding of the control process, a feeling for what kinds of performance objectives are unrealistic, or even dangerous, to ask for. The algebraic approach to be presented, nevertheless, is an elegant and useful tool for the mathematical part of the controller design.

21.2 Systems and Signals

The fundamentals of the factorization approach will be explained for linear systems with *rational* transfer functions whose input u and output y are scalar quantities. We suppose that u and y live in a space of functions mapping a time set into a value set. The time set is a subset of real numbers bounded on the left, say R_+ (the nonnegative reals) in the case of continuous-time systems and Z_+ (the nonnegative integers) for discrete-time systems. The value set is taken to be the set of real numbers R.

Let the input and output spaces of a continuous-time system be the spaces of locally (Lebesgue) integrable functions f from R_+ into R, and define a p-norm

$$||f||_{L_p} = \left[\int_0^\infty |f(t)|^p \, dt \right]^{1/p} \quad \text{if } i = 1 \le p < \infty,$$

$$||f||_{L_\infty} = \operatorname{ess\,sup}_{t \ge 0} |f(t)| \quad \text{if } p = \infty.$$

The corresponding normed space is denoted by L_p.

The systems having the desirable property of preserving these functional spaces are called *stable*. More precisely, a system is said to be L_p stable if any input $u \in L_p$ gives rise to an output $y \in L_p$. The systems that are L_∞ stable are also termed to be bounded-input bounded-output (BIBO) stable.

The transfer function of a continuous-time system is the Laplace transform of its impulse response $g(t)$,

$$G(s) = \int_0^\infty g(t)e^{-st} \, dt.$$

It is well known that a system with a rational transfer function $G(s)$ is BIBO stable if and only if $G(s)$ is proper and Hurwitz stable, that is, bounded at infinity with all poles having negative real parts.

In the study of discrete-time systems, we let the input and output spaces be the spaces of infinite sequences $f = (f_0, f_1, \ldots)$ mapping Z_+ into R and define a p-norm as follows:

$$\|f\|_{l_p} = \left[\sum_{i=0}^\infty |f_i|^p \right]^{1/p} \quad \text{if } 1 \le p < \infty,$$

$$||f||_{l_\infty} = \sup_{i \ge 0} |f_i| \quad \text{if } p = \infty.$$

A discrete-time system is said to be l_p stable if it transforms any input $u \in l_p$ to an output $y \in l_p$. The systems that are l_∞ stable are also known as BIBO stable systems.

The transfer function of a discrete-time system is defined as the z-transform of its unit pulse response $(h_0, h_1, \ldots)$,

$$H(z) = \sum_{i=0}^\infty h_i z^{-i},$$

and it is always proper. A system with a proper rational transfer function $H(z)$ is BIBO stable if and only if $H(z)$ is Schur stable, that is, its all poles have modulus less than one.

Of particular interest are discrete-time systems that are finite-input finite-output (FIFO) stable. Such a system transforms finite-input sequences into finite-output sequences, its unit pulse response is finite, and its transfer function $H(z)$ has no poles outside the origin $z = 0$, that is, $H(z)$ is a polynomial in z^{-1}.

21.3 Fractional Descriptions

Consider a rational function $G(s)$. By definition, it can be expressed as the ratio

$$G(s) = \frac{B(s)}{A(s)}$$

of two qualified rational functions A and B.

A well-known example is the *polynomial* description, in which case A and B are coprime polynomials, that is, polynomials having no roots in common.

Another example is to take for A and B two coprime, *proper* and *Hurwitz-stable* rational functions. When $G(s)$ is, say,

$$G(s) = \frac{s+1}{s^2+1},$$

then one can take

$$A(s) = \frac{s^2+1}{(s+\lambda)^2}, \quad B(s) = \frac{s+1}{(s+\lambda)^2},$$

where $\lambda > 0$ is a real number. We recall that two proper and Hurwitz-stable rational functions are coprime if they have no infinite nor unstable zeros in common. Therefore, in the example above, the denominator of A and B can be any strictly Hurwitz polynomial of degree exactly 2; if its degree is lower, then A would not be proper and if it is higher, then A and B would have a common zero at infinity. The set of proper and Hurwitz-stable rational functions is denoted by $R_H(s)$.

The proper rational functions $H(z)$ arising in discrete-time systems can be treated in a similar manner. One can write

$$H(z) = \frac{B(z)}{A(z)},$$

where A and B are coprime, *Schur-stable* (hence proper) rational functions. Coprimeness means having no unstable zeros (i.e., in the closed disc $|z| \geq 1$) in common. For example, if

$$H(z) = \frac{1}{z-1},$$

then one can take

$$A(z) = \frac{z-1}{z-\lambda}, \quad B(z) = \frac{1}{z-\lambda}$$

for any real number λ such that $|\lambda| < 1$. The set of Schur-stable rational functions is denoted by $R_S(z)$.

The particular choice of λ=0 in the example above leads to

$$A(z) = \frac{z-1}{z} = 1 - z^{-1}, \quad B(z) = \frac{1}{z} = z^{-1}.$$

In this case, A and B are in fact polynomials in z^{-1}.

21.4 Feedback Systems

To control a system means to alter its dynamics so that a desired behavior is obtained. This can be done by feedback. A typical feedback system consists of two subsystems, S_1 and S_2, connected, as shown in Figure 21.1.

In most applications, it is desirable that the feedback system be BIBO stable in the sense that whenever the exegenous inputs u_1 and u_2 are bounded in magnitude, so too are the output signals y_1 and y_2.

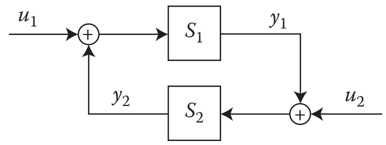

FIGURE 21.1 Feedback system.

In order to study this property, we express the transfer functions of S_1 and S_2 as ratios of proper stable rational functions and seek for conditions under which the transfer function of the feedback system is proper and stable.

To fix ideas, consider continuous-time systems and write

$$S_1 = \frac{B(s)}{A(s)}, \quad S_2 = -\frac{Y(s)}{X(s)},$$

where A, B and X, Y are two couples of coprime rational functions from $R_H(s)$. The transfer matrix of the feedback system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{S_1}{1 - S_1 S_2} & \frac{S_1 S_2}{1 - S_1 S_2} \\ \frac{S_1 S_2}{1 - S_1 S_2} & \frac{S_2}{1 - S_1 S_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is then given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{AX + BY} \begin{bmatrix} BX & -BY \\ -BY & -AY \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

We observe that the numerator matrix has all its elements in $R_H(s)$ and that no infinite or unstable zeros of the denominator can be absorbed in all these elements. We therefore conclude that the transfer functions belong to $R_H(s)$ if and only if the inverse of $AX + BY$ is in $R_H(s)$.

We illustrate with the example where S_1 is a differentiator and S_2 is an invertor such that

$$S_1(s) = s, \quad S_2(s) = -1.$$

We take

$$A(s) = \frac{1}{s + \lambda}, \quad B(s) = \frac{s}{s + \lambda}$$

for any real $\lambda > 0$ and

$$X(s) = 1, \quad Y(s) = 1.$$

Then

$$(AX + BY)^{-1}(s) = \frac{s + \lambda}{s + 1}$$

resides in $R_H(s)$ and hence the feedback system is BIBO stable.

The above analysis applies also to discrete-time systems; the set $R_H(s)$ is just replaced by $R_S(z)$. However, we note that any closed loop around a discrete-time system involves some information delay, no matter how small. Indeed, a control action applied to S_1 cannot affect the measurement from which it was calculated in S_2. Therefore, either $S_1(z)$ or $S_2(z)$ must be strictly proper; we shall assume that it is $S_1(z)$ that has this property.

To illustrate the analysis of discrete-time systems, consider a summator S_1 and an amplifier S_2,

$$S_1(z) = \frac{1}{z-1}, \quad S_2(z) = -k.$$

Taking

$$A(z) = \frac{z-1}{z-\lambda}, \quad B(z) = \frac{1}{z-\lambda}$$

for any real λ in magnitude less than 1 and

$$X(z) = 1, \quad Y(z) = k,$$

one obtains

$$(AX+BY)^{-1}(z) = \frac{z-\lambda}{z-(1-k)}.$$

Therefore, the closed-loop system is BIBO stable if and only if $|1-k| < 1$.

To summarize, the fractional representation used should be matched with the goal of the analysis. The denominators A, X and the numerators B, Y should be taken from the set of stable transfer functions, either $R_H(s)$ or $R_S(z)$, depending on the type of the stability studied. This choice makes the analysis more transparent and leads to a simple algebraic condition: the inverse of $AX+BY$ is stable. Any other type of stability can be handled in the same way, provided one can identify the set of the transfer functions that these stable systems will have.

21.5 Parameterization of Stabilizing Controllers

The design of feedback control systems consists of the following: given one subsystem, say S_1, we seek to determine the other subsystem, S_2, so that the resulting feedback system shown in Figure 21.1 meets the design specifications. We call S_1 the *plant* and S_2 the *controller*. Our focus is first on achieving BIBO stability. Any controller S_2 that BIBO stabilizes the plant S_1 is called a *stabilizing* controller for this plant.

Suppose S_1 is a continuous-time plant that gives rise to the transfer function

$$S_1(s) = \frac{B(s)}{A(s)}$$

for some coprime elements A and B of $R_H(s)$. It follows from the foregoing analysis that a stabilizing controller exists and that all controllers that stabilize the given plant are generated by all solution pairs X, Y with $X \neq 0$ of the Bézout equation

$$AX+BY = 1$$

over $R_H(s)$. There is no loss of generality in setting $AX+BY$ to the identity rather than to any rational function whose inverse is in $R_H(s)$: this inverse is absorbed by X and Y and therefore cancels in forming

$$S_2(s) = -\frac{Y(s)}{X(s)}.$$

The solution set of the equation $AX+BY=1$ with A and B coprimes in $R_H(s)$ can be parameterized as

$$X = X' + BW, \quad Y = Y' - AW,$$

where X', Y' represent a particular solution of the equation, and W is a free parameter, which is an arbitrary function in $R_H(s)$.

The parameterization of the family of all stabilizing controllers S_2 for the plant S_1 now falls out almost routinely:

$$S_2(s) = -\frac{Y'(s) - A(s)W(s)}{X'(s) + B(s)W(s)},$$

where the parameter W varies over $R_H(s)$ while satisfying $X' + BW \neq 0$.

In order to determine the set of all controllers S_2 that stabilize the plant S_1, one needs to do two things: (1) express $S_1(s)$ as a ratio of two coprime elements from $R_H(s)$ and (2) find a particular solution in $R_H(s)$ of a Bézout equation, which is equivalent to finding one stabilizing controller for S_1. Once these two steps are completed, the formula above provides a parameterization of the set of all stabilizing controllers for S_1. The condition $X' + BW \neq 0$ is not very restrictive, as $X' + BW$ can identically vanish for at most one choice of W.

As an example, we shall stabilize an integrator plant S_1. Its transfer function can be expressed as

$$S_1(s) = \frac{1/s+1}{s/s+1},$$

where $s+1$ is an arbitrarily chosen Hurwitz polynomial of degree one. Suppose that using some design procedure we have found a stabilizing controller for S_1, namely

$$S_2(s) = -1.$$

This corresponds to a particular solution $X' = 1$, $Y' = 1$ of the Bézout equation

$$\frac{s}{s+1}X + \frac{1}{s+1}Y = 1.$$

The solution set in $R_H(s)$ of this equation is

$$X(s) = 1 + \frac{1}{s+1}W(s), \quad Y(s) = 1 - \frac{s}{s+1}W(s).$$

Hence, all controllers S_2 that BIBO stabilize S_1 have the transfer function

$$S_2(s) = -\frac{1 - (s/s+1)W(s)}{1 + (1/s+1)W(s)},$$

where W is any function in $R_H(s)$.

It is clear that the result is independent of the particular fraction taken to represent S_1. Indeed, if $s+1$ is replaced by another Hurwitz polynomial $s+\lambda$ in the above example, one obtains

$$S_2(s) = -\frac{\lambda - (s/s+\lambda)W'(s)}{1 + (1/s+\lambda)W'(s)},$$

which is the same set when

$$W'(s) = \left(\frac{s+\lambda}{s+1}\right)^2 W(s) + \frac{s+\lambda}{s+1}(\lambda - 1).$$

21.6 Parameterization of Closed-Loop Transfer Functions

The utility of the fractional approach derives not merely from the fact that it provides a parameterization of all controllers that stabilize a given plant in terms of a free parameter W, but also from the simple manner in which this parameter enters the resulting (stable) closed-loop transfer matrix.

In fact,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} B(X' + BW) & -B(Y' - AW) \\ -B(Y' - AW) & -A(Y' - AW) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

and we observe that all four transfer functions are *affine* in the free parameter W.

This result serves to parameterize the performance specifications, and it is the starting point for the selection of the best controller for the application at hand. The search for S_2 is thus replaced by a search for W. The crucial point is that the resulting selection/optimization problem is linear in W, whereas it is nonlinear in S_2.

21.7 Optimal Performance

The performance specifications often involve a norm minimization.

Let us consider the problem of *disturbance attenuation*. We are given, say, a continuous-time plant S_1 having two inputs: the control input u and an unmeasurable disturbance d (see Figure 21.2). The objective is to determine a BIBO stabilizing controller S_2 for the plant S_1 such that the effect of d on the plant output y is minimized in some sense.

We describe the plant by two transfer functions

$$S_{1u}(s) = \frac{B(s)}{A(s)}, \quad S_{1d}(s) = \frac{C(s)}{A(s)},$$

where A, B, and C is a triple of coprime functions from $R_H(s)$. The set of stabilizing controllers for S_1 is given by the transfer function

$$S_2(s) = -\frac{Y'(s) - A'(s)W(s)}{X'(s) + B'(s)W(s)},$$

where A', B' represent a *coprime* fraction over $R_H(s)$ for S_{1u},

$$\frac{B(s)}{A(s)} = \frac{B'(s)}{A'(s)}$$

and X', Y' represent a particular solution over $R_H(s)$ of the equation

$$A'X + B'Y = 1,$$

such that $X' + B'W \neq 0$.

The transfer function, $G(s)$, between d and y in a stable feedback system is

$$G = \frac{S_{1d}}{1 - S_{1u}S_2} = C(X' + B'W)$$

and it is affine in the proper and Hurwitz-stable rational parameter W.

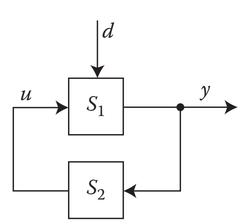

FIGURE 21.2 Disturbance attenuation.

Now suppose that the disturbance d is any function from L_∞, that is, any essentially bounded real function on R_+. Then

$$||y||_{L_\infty} \leq ||G||_1 ||d||_{L_\infty},$$

where

$$||G||_1 = \int_0^\infty |g(t)|\, dt$$

and $g(t)$ is the impulse response corresponding to $G(s)$. The parameter W can be used to minimize the norm $||G||_1$ and hence the maximum output amplitude.

If d is a stationary white noise, the steady-state output variance equals

$$Ey^2 = ||G||_2^2 Ed^2,$$

where

$$||G||_2^2 = \int_0^\infty |g(t)|^2\, dt = \frac{1}{2\pi j} \oint G(-s)G(s)\, ds.$$

The last integral is a contour integral up the imaginary axis and then around an infinite semicircle in the left half-plane. Again, W can be selected so as to minimize the norm $||G||_2$, thus minimizing the steady-state output variance.

Finally, suppose that d is any function from L_2, that is, any finite-energy real function on R_+. Then one obtains

$$||y||_{L_2} \leq ||G||_\infty ||d||_{L_2},$$

where

$$||G||_\infty = \sup_{Re\, s>0} |G(s)|.$$

Therefore, choosing W to make the norm $||G||_\infty$ minimal, one minimizes the maximum output energy.

The above system norms provide several examples showing how the effect of the disturbance on the plant output can be measured. The optimal attenuation is achieved by minimizing these norms.

Minimizing the 1-norm involves a linear program while minimizing the ∞-norm requires a search. The 2-norm minimization has a closed-form solution, which will be now described.

We recall that

$$G(s) = P(s) + Q(s)W(s),$$

where $P = CX'$ and $Q = CB'$. The norm $||G||_2$ is finite if and only if G is strictly proper and has no poles on the imaginary axis; hence we assume that Q has no zeros on the imaginary axis. We factorize

$$Q = Q_{ap}Q_{mp},$$

where Q_{ap} satisfies $Q_{ap}(-s)Q_{ap}(s) = 1$ (the so-called all-pass function) and Q_{mp} is such that Q_{mp}^{-1} is in $R_H(s)$ (the so-called minimum-phase function); this factorization is unique up to the sign. Let Q_{ap}^* denote the function $Q^*_{ap}(s) = Q_{ap}(-s)$. Then

$$\begin{aligned}\|G\|_2^2 &= ||P + QW||_2^2 \\ &= ||Q^*_{ap}P + Q_{mp}W||_2^2.\end{aligned}$$

Decompose $Q^*_{ap}P$ as

$$Q^*_{ap}P = (Q^*_{ap}P)_{st} + (Q^*_{ap}P)_{un}$$

where $(Q^*_{ap}P)_{st}$ is in $R_H(s)$ and $(Q^*_{ap}P)_{un}$ is unstable but strictly proper; this decomposition is unique. Then the cross-terms contribute nothing to the norm and

$$||G||_2^2 = ||(Q^*_{ap}P)_{un}||_2^2 + ||(Q^*_{ap}P)_{st} + Q_{mp}W||_2^2.$$

Since the first term is independent of W,

$$\min_W ||G||_2 = ||(Q^*_{ap}P)_{un}||_2$$

and this minimum is attained by

$$W = -\frac{(Q^*_{ap}P)_{st}}{Q_{mp}}.$$

Here is an illustrative example. The plant is given by

$$S_{1u}(s) = \frac{s-2}{s+1}, \quad S_{1d}(s) = 1$$

and we seek to find a stabilizing controller S_2 such that

$$G(s) = \frac{S_{1d}(s)}{1 - S_{1u}(s)S_2(s)}$$

has minimum 2-norm.

We write

$$A(s) = 1, \quad B(s) = \frac{s-2}{s+1}, \quad C(s) = 1$$

and find all stabilizing controllers first. Since the plant is already stable, these are given by

$$S_2(s) = -\frac{-W(s)}{1 + (s-2)/(s+1)W(s)},$$

where W is a free parameter in $R_H(s)$.

Then

$$G(s) = 1 + \frac{s-2}{s+1}W(s),$$

so that

$$P(s) = 1, \quad Q(s) = \frac{s-2}{s+1}.$$

Clearly,

$$Q_{ap}(s) = \frac{s-2}{s+2}, \quad Q_{mp}(s) = \frac{s+2}{s+1}$$

and

$$Q^*_{ap}(s)P(s) = \frac{s+2}{s-2} = 1 + \frac{4}{s-2}.$$

Therefore,

$$\|G\|_2^2 = \|\frac{4}{s-2}\|_2^2 + \|1 + \frac{s+2}{s+1}W\|_2^2,$$

so that the least norm

$$\min_W \|G\|_2 = \|\frac{4}{s-2}\|_2 = \|\frac{4}{s+2}\|_2 = 2$$

is attained by

$$W(s) = -\frac{s+1}{s+2}.$$

21.8 Robust Stabilization

The actual plant can differ from its nominal model. We suppose that a nominal plant description is available together with a description of the plant uncertainty. The objective is to design a controller that stabilizes all plants lying within the specified domain of uncertainty. Such a controller is said to *robustly* stabilize the family of plants.

The plant uncertainty can be modeled conveniently in terms of its fractional description. To fix ideas, we shall consider discrete-time plants factorized over $R_S(z)$ and endow $R_S(z)$ with the ∞-norm: for any function $H(z)$ from $R_S(z)$,

$$\|H\|_\infty = \sup_{|z|>1} |H(z)|.$$

For any two such functions, $H_1(z)$ and $H_2(z)$, we define

$$\begin{aligned}\|[H_1 H_2]\|_\infty &= \|\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}\|_\infty \\ &= \sup_{|z|>1} (|H_1(z)|^2 + |H_2(z)|^2)^{1/2}.\end{aligned}$$

Let S_{10} be a nominal plant giving rise to a strictly proper transfer function

$$S_{10}(z) = \frac{B(z)}{A(z)},$$

where A and B are coprime functions from $R_S(z)$. We denote $S_1(A, B, \mu)$ the family of plants having strictly proper transfer functions

$$S_1(z) = \frac{B(z) + \Delta B(z)}{A(z) + \Delta A(z)},$$

where ΔA and ΔB are functions from $R_S(z)$ such that

$$\|[\Delta A\ \Delta B]\|_\infty < \mu$$

for some nonnegative real number μ.

Now, let S_2 be a BIBO stabilizing controller for S_{10}. Therefore,

$$S_2 = -\frac{Y' - AW}{X' + BW},$$

where $AX' + BY' = 1$ and W is an element of $R_S(z)$. Then S_2 will BIBO stabilize all plants from $S_1(A, B, \mu)$ if and only if the inverse of

$$(A + \Delta A)(X' + BW) + (B + \Delta B)(Y' - AW) = 1 + [\Delta A \quad \Delta B] \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix}$$

is in $R_S(z)$. This is the case whenever

$$\|[\Delta A \ \Delta B] \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix} \|_\infty < 1;$$

thus, we have the following condition of robust stability:

$$\mu \| \begin{bmatrix} X' + BW \\ Y' - AW \end{bmatrix} \|_\infty \leq 1.$$

The best controller that robustly stabilizes the plant corresponds to the parameter W that minimizes the ∞-norm above. This requires a search; closed-form solutions exist only in special cases. One such case is presented next.

Suppose the nominal model

$$S_{10}(z) = \frac{1}{z-1}$$

has resulted from

$$S_1(z) = \frac{z + \delta}{(z-1)(z-\varepsilon)}$$

by neglecting the second-order dynamics, where $\delta \geq 0$ and $0 \leq \varepsilon < 1$. Rearranging,

$$S_1(z) = \frac{(1/z) + (1/z)\,(\delta + \varepsilon/z - \varepsilon)}{(z - 1/z)}$$

and one identifies

$$\Delta A = 0, \quad \Delta B = \frac{1}{z}\frac{\delta + \varepsilon}{z - \varepsilon}.$$

Hence,

$$\|[\Delta A \ \Delta B]\|_\infty = \frac{\delta + \varepsilon}{1 - \varepsilon}$$

and the true plant belongs to the family

$$S_1\left(\frac{z-1}{z}, \quad \frac{1}{z}, \quad \frac{\delta + \varepsilon}{1 - \varepsilon}\right).$$

All controllers that BIBO stabilize the nominal plant S_{10} are given by

$$S_2(z) = -\frac{1 - (z - 1/z)W(z)}{1 + (1/z)W(z)},$$

where W is a free parameter in $R_S(z)$. Which controller yields the best stability margin against δ and ε? The one that minimizes the ∞-norm in

$$\frac{\delta + \varepsilon}{1 - \varepsilon} \| \begin{bmatrix} 1 + \dfrac{1}{z} W \\ 1 - \dfrac{z-1}{z} W \end{bmatrix} \|_\infty < 1.$$

Suppose we wish to obtain a controller of McMillan degree zero, $S_2(z) = -K$. Then

$$W(z) = (1-K)\frac{z}{z-(1-K)}$$

and $|1-K| < 1$. The norm

$$\left\| \begin{bmatrix} 1+\frac{1}{z}W \\ 1-\frac{z-1}{z}W \end{bmatrix} \right\|_\infty = \sqrt{(1+K^2)}\left\|\frac{z}{z-(1-K)}\right\|_\infty$$

attains the least value of $\sqrt{2}$ by $K = 1$, which corresponds to $W(z) = 0$. It follows that the controller

$$S_2(z) = -1$$

stabilizes all plants $S_1(z)$ for which

$$\frac{\delta+\varepsilon}{1-\varepsilon} < \frac{1}{\sqrt{2}}.$$

21.9 Robust Performance

The performance specifications often result in divisibility conditions. A typical example is the problem of *reference tracking*.

Suppose we are given a discrete-time plant S_1, with transfer function

$$S_1(z) = \frac{B(z)}{A(z)}$$

in coprime fractional form over $R_S(z)$, together with a reference r whose z-transform is of the form

$$r = \frac{E(z)}{D(z)},$$

where only D is specified. We recall that $S_1(z)$ is strictly proper. The objective is to design a BIBO stabilizing controller S_2 such that the plant output y asymptotically tracks the reference r (see Figure 21.3). The controller can operate on both r (feedforward) and y (feedback), so it is described by two transfer functions

$$S_{2y}(z) = -\frac{Y(z)}{X(z)}, \quad S_{2r}(z) = \frac{Z(z)}{X(z)},$$

where X, Y, and Z are from $R_S(z)$.

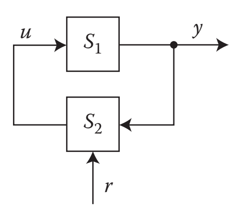

FIGURE 21.3 Reference tracking.

The requirement of tracking imposes that the tracking error

$$e = r - y = \left(1 - \frac{BZ}{AX + BY}\right)\frac{E}{D}$$

belong to $R_S(z)$. Since $AX + BY$ has inverse in $R_S(z)$ for every stabilizing controller and E is unspecified, D must divide $1 - BZ$ in $R_S(z)$. Hence, there must exist a function V in $R_S(z)$ such that $1 - BZ = DV$. Therefore, S_2 exists if and only if B and D are coprime in $R_S(z)$, and the two controller transfer functions evolve from solving the two Bézout equations

$$AX + BY = 1,$$
$$DV + BZ = 1,$$

where the function V serves to express the tracking error as

$$e = VE.$$

The reference tracking is said to be *robust* if the specifications are met even as the plant is slightly perturbed. We call S_{10} the nominal plant and $S_1(A, B, \mu)$ the neighborhood of S_{10} defined by

$$S_1 = \frac{B + \Delta B}{A + \Delta A},$$

where ΔA and ΔB are functions of $R_S(z)$, such that

$$\|[\Delta A\ \Delta B]\|_\infty < \mu$$

for some nonnegative real number μ. We recall that all $S_1(z)$ are strictly proper.

Now $A + \Delta A$ and $B + \Delta B$ are not specified, but $(A + \Delta A)X + (B + \Delta B)Y$ still has inverse in $R_S(z)$; call it U. We have

$$e = \left(\frac{A + \Delta A}{U}X + \frac{B + \Delta B}{U}(Y - Z)\right)\frac{E}{D}.$$

Hence, for robust tracking, D must divide both X and $Y - Z$ in $R_S(z)$. But, it is sufficient that D divides X; this condition already implies the other one as can be seen on subtracting the two Bézout equations above.

We illustrate on a discrete-time plant S_1 given by

$$S_1(z) = \frac{1}{z - 2}$$

whose output is to track every sinusoidal sequence of the form

$$r = \frac{az + b}{z^2 - z + 1},$$

where a, b are unspecified real numbers. Taking

$$A(z) = \frac{z - 2}{z}, \quad B(z) = \frac{1}{z}, \quad D(z) = \frac{z^2 - z + 1}{z^2}$$

and solving the pair of Bézout equations

$$\frac{z - 2}{z}X(z) + \frac{1}{z}Y(z) = 1,$$
$$\frac{z - z + 1}{z^2}V(z) + \frac{1}{z}Z(z) = 1$$

yields the tracking controllers in parametric form

$$S_{2y}(z) = -\frac{2-(z-2/z)W_1(z)}{1+(1/z)W_1(z)},$$
$$S_{2r}(z) = \frac{(z-1/z)-(z^2-z+1/z^2)W_2(z)}{1+(1/z)W_1(z)}$$

for any elements W_1, W_2 of $R_S(z)$. The resulting error is

$$e = \left[1+\frac{1}{z}W_2(z)\right]\frac{az+b}{z^2}.$$

Not all of these controllers, however, achieve a robust tracking of the reference. The divisibility condition is fulfilled if and only if W_1 is restricted to

$$W_1(z) = -\frac{z-1}{z} + \frac{z^2-z+1}{z^2}W(z),$$

where W is free in $R_S(z)$.

It is to be noted that the requirement of asymptotic tracking leaves enough degrees of freedom to meet additional design specifications.

21.10 Finite Impulse Response

Transients in discrete-time systems can settle in finite time. Systems having the property that any input sequence with a finite number of nonzero elements produces an output sequence with a finite number of nonzero elements have been called FIFO stable. We recall that a system with proper rational transfer function $H(z)$ is FIFO stable if and only if $H(z)$ is a polynomial in z^{-1}.

Let us consider the feedback system shown in Figure 21.1 and focus on achieving FIFO stability. To this end, we write the transfer function of the plant as

$$S_1(z) = \frac{B(z)}{A(z)},$$

where this time A and B are coprime polynomials in z^{-1}. We recall that the plant incorporates the necessary delay so that $S_1(z)$ is strictly proper. Repeating the arguments used to design a BIBO stable system, we conclude that all controllers S_2 that FIFO stabilize the plant S_1 have the transfer function

$$S_2(z) = -\frac{Y(z)}{X(z)},$$

where X, Y represent the solution class of the polynomial Bézout equation

$$AX + BY = 1.$$

In particular, if X' and Y' define any FIFO stabilizing controller for S_1, the set of all such controllers can be parameterized as

$$S_2(z) = -\frac{Y'-AW}{X'+BW},$$

where $W(z)$ is a free polynomial in z^{-1}.

It is a noteworthy fact that the parametric expressions for the sets of BIBO stable and FIFO stable controllers are the same; the only difference is that the free parameter of FIFO stabilizing controllers is permitted to range over only the smaller set of polynomials in z^{-1}, whereas in BIBO stabilizing controllers it is permitted to range over the larger set of Schur-stable rational functions in z. Indeed, FIFO stability is more restrictive than BIBO stability.

The design options offered by FIFO stability are remarkable. The parameter W can be selected so as to minimize the McMillan degree of S_2, or to achieve the shortest impulse response of the closed-loop system. Various norm minimizations can also be performed.

A well-known example is the deadbeat controller. We consider a double-summator plant with transfer function

$$S_1(z) = \frac{1}{(z-1)^2}$$

and interpret the exogenous inputs u_1 and u_2 as accounting for the effect of the initial conditions of S_1 and S_2. The requirement of FIFO stability is then equivalent to achieving finite responses y_1 and y_2 for all initial conditions. Since in this case

$$A(z) = (1-z^{-1})^2, \quad B(z) = z^{-2}$$

and the Bézout equation

$$(1-z^{-1})^2 X(z) + z^{-2} Y(z) = 1$$

has a particular solution

$$X'(z) = 1 + z^{-1}, \quad Y'(z) = 3 - 2z^{-1},$$

we obtain all deadbeat (or FIFO stabilizing) controllers as

$$S_2(z) = -\frac{3 - 2z^{-1} - (1-z^{-1})^2 W(z)}{1 + 2z^{-1} + z^{-2} W(z)}.$$

The deadbeat controller of least McMillan degree (=1) is obtained for $W(z) = 0$. The choice $W(z) = -3$ leads to a deadbeat controller that rejects step disturbances u_1 (hence, persistent) at the plant output y_1 in finite time. And when u_1 is a stationary white noise, then $W(z) = 2.5$ minimizes the steady-state variance of y_1 among all deadbeat controllers of McMillan degree 2.

21.11 Multivariable Systems

Up until now we have considered only single-input single-output (SISO) plants and controllers. In the case of multiple inputs and/or outputs, the input–output properties of linear systems are represented by a *matrix* of transfer functions. The additional intricacies introduced by these systems stem mainly from the fact that the matrix multiplication is not commutative.

Consider a rational transfer matrix $G(s)$ whose dimensions are, say, $m \times n$. Then it is always possible to factorize G as follows:

$$\begin{aligned} G(s) &= B_R(s) A_R^{-1}(s) \\ &= A_L^{-1}(s) B_L(s), \end{aligned}$$

where the factors B_R, A_R and A_L, B_L are, respectively, $m \times n$, $n \times n$ and $m \times m$, $m \times n$ matrices of qualified rational functions, say from $R_H(s)$, such that

$$\begin{aligned} &A_R, \quad B_R \text{ are right coprime,} \\ &A_L, \quad B_L \text{ are left coprime.} \end{aligned}$$

These “matrix fractions” are unique except for the possibility of multiplying the “numerator” and the “denominator” matrices by a matrix whose determinant has inverse in $R_H(s)$. That is, if $G(s)$ can also be expressed as

$$G(s) = B'_R(s)A'^{-1}_R(s) \\ = A'^{-1}_L(s)B'_L(s),$$

where the factors are matrices of functions from $R_H(s)$, such that

$$A'_R, B'_R \text{ are right coprime,} \\ A'_L, B'_L \text{ are left coprime,}$$

then

$$A'_R(s) = A_R(s)U_R(s), \quad B'_R(s) = B_R(s)U_R(s), \\ A'_L(s) = U_L(s)A_L(s), \quad B'_L(s) = U_L(s)B_L(s)$$

for some matrices U_R and U_L over $R_H(s)$, whose determinants have stable inverses in $R_H(s)$.

Analogous results hold for discrete-time systems. To illustrate, consider the transfer matrix

$$G(z) = \begin{bmatrix} \frac{1}{z-1} & \frac{2-z}{z^2-z} \\ \frac{1}{z-1} & \frac{1}{z-1} \end{bmatrix}$$

and determine its left and right coprime factorizations over $R_S(z)$. One obtains, for instance,

$$G(z) = \begin{bmatrix} \frac{1}{z} & \frac{2-z}{z^2-\lambda z} \\ 0 & \frac{1}{z-\lambda} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & \frac{z-1}{z-\lambda} \end{bmatrix}^{-1} \\ = \begin{bmatrix} 1 & -1 \\ 0 & \frac{z-1}{z-\mu} \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{2}{z} \\ \frac{1}{z-\mu} & \frac{1}{z-\mu} \end{bmatrix}$$

for any real λ and μ with modulus less than one.

Let us now consider the feedback system shown in Figure 21.1 where S_1 and S_2 are multivariable systems and analyze its BIBO stability. We therefore factorize the two transfer matrices over $R_H(s)$,

$$S_1(s) = B_R(s)A_R^{-1}(s) = A_L^{-1}(s)B_L(s), \\ S_2(s) = -X_L^{-1}(s)Y_L(s) = -Y_R(s)X_R^{-1}(s),$$

where the two pairs A_R, B_R and X_R, Y_R are right coprimes while the two pairs A_L, B_L and X_L, Y_L are left coprimes. The transfer matrix of the feedback system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} S_1(I-S_2S_1)^{-1} & S_1(I-S_2S_1)^{-1}S_2 \\ S_2(I-S_1S_2)^{-1}S_1 & S_2(I-S_1S_2)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

then reads

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} B_R(X_LA_R+Y_LB_R)^{-1}X_L & -B_R(X_LA_R+Y_LB_R)^{-1}Y_L \\ I-A_R(X_LA_R+Y_LB_R)^{-1}X_L & -A_R(X_LA_R+Y_LB_R)^{-1}Y_L \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

or alternatively

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_R(A_LX_R+B_LY_R)^{-1}B_L & X_R(A_LX_R+B_LY_R)^{-1}A_L - I \\ -Y_R(A_LX_R+B_LY_R)^{-1}B_L & -Y_R(A_LX_R+B_LY_R)^{-1}A_L \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The feedback system is BIBO stable if and only if this transfer matrix has entries in $R_H(s)$. We therefore conclude that the feedback system is BIBO stable if and only if the common denominator $X_LA_R + Y_LB_R$, or alternatively $A_LX_R + B_LY_R$, has inverse with entries in $R_H(s)$.

A parameterization of all controllers S_2 that BIBO stabilize the plant S_1 is now at hand. Given left and right coprime factorizations over $R_H(s)$ of the plant transfer matrix

$$S_1 = B_R A_R^{-1} = A_L^{-1} B_L,$$

we select matrices X_L', Y_L' and X_R', Y_R' with entries in $R_H(s)$, such that

$$X_L' A_R + Y_L' B_R = I, \quad A_L X'_R + B_L Y_R' = I.$$

Then the family of all stabilizing controllers has the transfer matrix

$$\begin{aligned} S_2 &= -(X_L' + W_L B_L)^{-1}(Y_L' - W_L A_L) \\ &= -(Y_R' - A_R W_R)(X_R' + B_R W_R)^{-1}, \end{aligned}$$

where W_L is a matrix parameter whose entries vary over $R_H(s)$ such that $X_L' + W_L B_L$ is nonsingular, and W_R is a matrix parameter whose entries vary over $R_H(s)$ such that $X_R' + B_R W_R$ is nonsingular.

As an example, determine all BIBO stabilizing controllers for the discrete-time plant considered earlier, with the transfer matrix

$$S_1(z) = \begin{bmatrix} \dfrac{1}{z-1} & \dfrac{2-z}{z^2-z} \\ \dfrac{1}{z-1} & \dfrac{1}{z-1} \end{bmatrix}.$$

The left and right coprime factors over $R_S(z)$ can be taken as

$$A_R(z) = \begin{bmatrix} 1 & 0 \\ -1 & \dfrac{z-1}{z} \end{bmatrix}, \quad B_R(z) = \begin{bmatrix} \dfrac{1}{z} & \dfrac{2-z}{z^2} \\ 0 & \dfrac{1}{z} \end{bmatrix}$$

and

$$A_L(z) = \begin{bmatrix} 1 & -1 \\ 0 & \dfrac{z-1}{z} \end{bmatrix}, \quad B_L(z) = \begin{bmatrix} 0 & -\dfrac{2}{z} \\ \dfrac{1}{z} & \dfrac{1}{z} \end{bmatrix}.$$

The Bézout equations

$$X_L' A_R + Y_L' B_R = I, \quad A_L X_R' + B_L Y_R' = I$$

have particular solutions

$$X_L'(s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad Y_L'(s) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$X_R'(s) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Y_R'(s) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The set of stabilizing controllers is given by

$$S_2(s) = -\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + W_L \begin{bmatrix} 0 & -2z^{-1} \\ z^{-1} & z^{-1} \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - W_L \begin{bmatrix} 1 & -1 \\ 0 & 1-z^{-1} \end{bmatrix}\right),$$

where W_L varies over $R_S(z)$, or by

$$S_2(s) = -\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\-1 & 1-z^{-1}\end{bmatrix} W_R\right) \left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}z^{-1} & -z^{-1}+2z^{-2}\\0 & z^{-1}\end{bmatrix} W_R\right)^{-1},$$

where W_R varies over $R_S(z)$ as well.

It is clear that the two parameterizations of S_2 are equivalent. To each controller S_2 there is a unique parameter W_L such that $S_2 = -(X'_L + W_L B_L)^{-1}(Y'_L - W_L A_L)$ as well as a unique parameter W_R such that $S_2 = -(Y'_R - A_R W_R)(X'_R + B_R W_R)^{-1}$, and these two are related by

$$W_R - W_L = X'_L Y'_R - Y'_L X'_R.$$

It is easy to see that the transfer matrix of the closed-loop system is *affine* in the free parameter W_L or W_R. Indeed,

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}B_R(X'_L + W_L B_L) & -B_R(Y'_L - W_L A_L)\\ I - A_R(X'_L + W_L B_L) & -A_R(Y'_L - W_L A_L)\end{bmatrix} \begin{bmatrix}u_1\\u_2\end{bmatrix}$$

or alternatively

$$\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}(X'_R + B_R W_R)B_L & (X'_R + B_R W_R)A_L - I\\ -(Y'_R - A_R W_R)B_L & -(Y'_R - A_R W_R)A_L\end{bmatrix} \begin{bmatrix}u_1\\u_2\end{bmatrix}.$$

Thus, control synthesis problems beyond stabilization can be handled by determining the parameters W_L or W_R as described for SISO systems.

Let us consider the disturbance attenuation problem for the discrete-time plant

$$S_{1u}(z) = \begin{bmatrix}\frac{1}{z-1} & \frac{2-z}{z^2-z}\\ \frac{1}{z-1} & \frac{1}{z-1}\end{bmatrix}, \quad S_{1d}(z) = \begin{bmatrix}\frac{1}{z-1}\\ \frac{1}{z-1}\end{bmatrix},$$

where the disturbance d is assumed to be an arbitrary l_∞ sequence. We seek to find a BIBO stabilizing controller that minimizes the maximum amplitude of the plant output y.

We write

$$S_{1u}(z) = A_L^{-1}(z)B_L(z) = B_R(z)A_R^{-1}(z),$$
$$S_{1d}(z) = A_L^{-1}(z)C_L(z),$$

where

$$A_L(z) = \begin{bmatrix}1 & -1\\0 & 1-z^{-1}\end{bmatrix}, \quad B_L(z) = \begin{bmatrix}1 & -2z^{-1}\\z^{-1} & z^{-1}\end{bmatrix}, \quad C_L(z) = \begin{bmatrix}0\\z^{-1}\end{bmatrix}$$
$$A_R(z) = \begin{bmatrix}1 & 0\\-1 & 1-z^{-1}\end{bmatrix}, \quad B_R(z) = \begin{bmatrix}z^{-1} & -z^{-1}+2z^{-1}\\0 & z^{-1}\end{bmatrix}.$$

The set of BIBO stabilizing controllers has been found to have the transfer matrix

$$S_2(z) = -\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\-1 & 1-z^{-1}\end{bmatrix} W_R\right) \left(\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \begin{bmatrix}z^{-1} & -z^{-1}+2z^{-2}\\0 & z^{-1}\end{bmatrix} W_R\right)^{-1},$$

where W_R varies over $R_S(z)$.

The disturbance-output transfer matrix equals

$$\begin{aligned} G(z) &= (I - S_{1u}S_2)^{-1}S_{1d} \\ &= (X'_R + B_R W_R)^{-1}C_L. \end{aligned}$$

When

$$W_R = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

one obtains the expression

$$G(z) = \begin{bmatrix} z^{-1} + z^{-2}W_{12} - (z^{-2} - 2z^{-3})W_{22} \\ z^{-1} + z^{-2}W_{22} \end{bmatrix}.$$

The 1-norm of an $m \times n$ matrix $G(z)$ with entries

$$G_{ij}(z) = \sum_{k=0}^{\infty} g_{ij,k}\, z^{-k}$$

is defined by

$$\|G\|_1 = \max_{i=1,..,m} \sum_{j=1}^{n} \|G_{ij}\|_1,$$

where

$$\|G_{ij}\|_1 = \sum_{k=0}^{n} |g_{ij,k}|.$$

In our case

$$\|G\|_1 = \max\left(\|z^{-1} + z^{-2}W_{12} - (z^{-2} - 2z^{-3})\; W_{22}\|_1, \quad \|z^{-1} + z^{-2}W_{22}\|_1\right),$$

and it is clear by inspection that $\|G\|_1$ attains its minimum for $W_{12}(z) = 0$, $W_{22}(z) = 0$ and

$$\min_{W_R} \|G\|_1 = \max(1, 1) = 1.$$

The corresponding optimal BIBO stabilizing controllers are

$$S_2(z) = -\begin{bmatrix} -W_{11} & 1 \\ W_{11} - (1 - z^{-1})W_{21} & 0 \end{bmatrix} \begin{bmatrix} 1 + z^{-1}W_{11} - (z^{-1} - 2z^{-2})W_{22} & 1 \\ z^{-1}W_{21} & 1 \end{bmatrix}^{-1}$$

for any functions W_{11} and W_{21} in $R_S(z)$. The one of least McMillan degree reads

$$S_2(z) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

21.12 Extensions

The factorization approach presented here for linear time-invariant systems with rational transfer matrices can be generalized to extend the scope of the theory to include distributed-parameter systems, time-varying systems, and even nonlinear systems.

The transfer matrices of distributed-parameter systems are no longer rational and coprime factorizations cannot be assumed *a priori* to exist. The coefficients of time-varying systems are functions of time, and the operations of multiplication and differentiation do not commute. In nonlinear systems, transfer matrices are replaced by input–output maps. Suitable factorizations of these maps may not exist and, if they do, they are not commutative in general.

For many systems of physical and engineering interest, these difficulties can be circumvented and the algebraic factorization approach carries over with suitable modifications.

Acknowledgment

The author acknowledges Project 1M0567, Ministry of Education of the Czech Republic.

Q1

Further Reading

Tutorial textbooks

1. Desoer, C. A. and Vidyasagar, M., *Feedback Systems: Input–Output Properties,* Academic Press, New York, 1975.
2. Kuèera, V., *Discrete Linear Control: The Polynomial Equation Approach,* Wiley & Sons, Chichester, UK, 1979.
3. Vidyasagar, M., *Control System Synthesis: A Factorization Approach,* MIT Press, Cambridge, MA, 1985.
4. Doyle, J. C., Francis, B. A., and Tannenbaum, A. R., *Feedback Control Theory,* Macmillan, New York, 1992.

Survey papers

5. Kuèera, V., Diophantine equations in control—A survey, *Automatica,* 29, 1361–1375, 1993.
6. Kuèera, V., Polynomial control: Past, present, and future, *Int. J. Robust Nonlinear Control*, 17, 682–705, 2007.

Original sources on the parameterization of all stabilizing controllers

7. Kuèera, V., Stability of discrete linear feedback systems, Paper 44.1, *Prepr. 6th IFAC World Congr.,* Vol. 1, Boston, 1975.
8. Youla, D. C., Jabr, H. A., and Bongiorno, J. J., Modern Wiener–Hopf design of optimal controllers, II. The multivariable case, *IEEE Trans. Autom. Control,* 21, 319–338, 1976.
9. Desoer, C. A., Liu, R. W., Murray, J., and Saeks, R., Feedback system design: The fractional representation approach to analysis and synthesis, *IEEE Trans. Autom. Control,* 25, 399–412, 1980.

Original sources on norm minimization

10. Francis, B. A., On the Wiener–Hopf approach to optimal feedback design, *System Control Lett.,* 2, 197–201, 1982.
11. Chang, B. C. and Pearson, J. B., Optimal disturbance reduction in linear multivariable systems, *IEEE Trans. Autom. Control,* 29, 880–888, 1984.
12. Dahleh, M. A. and Pearson, J. B., l^1-optimal feedback controllers for MIMO discrete-time systems, *IEEE Trans. Autom. Control,* 32, 314–322, 1987.

Original sources on robust stabilization

13. Vidyasagar, M. and Kimura, H., Robust controllers for uncertain linear multivariable systems, *Automatica,* 22, 85–94, 1986.
14. Dahleh, M. A., BIBO stability robustness in the presence of coprime factor perturbations, *IEEE Trans. Autom. Control,* 37, 352–355, 1992.

Original source on FIFO stability and related designs

15. Kuèera, V. and Kraus, F. J., FIFO stable control systems, *Automatica,* 31, 605–609, 1995.

Advanced use of the parameterization of all stabilizing controllers

16. Kuèera, V., Parametrization of stabilizing controllers with applications, in *Advances in Automatic Control*, M. Voicu (Ed.), pp. 173–192, Kluwer, Boston, 2003.
17. Henrion, D., Tarbouriech, S., and Kuèera V., Control of linear systems subject to input constraints: A polynomial aproach, *Automatica*, 37, 597–604, 2001.
18. Henrion, D., Šebek, M., and Kuèera, V., Positive polynomials and robust stabilization with fixed-order controllers. *IEEE Trans. Autom. Control*, AC-48, 178–186, 2003.
19. Henrion, D., Tarbouriech, S., and Kuèera, V., Control of linear systems subject to time-domain constraints with polynomial pole placement and LMIs. *IEEE Trans. Autom. Control*, 50, 1360–1364, 2005.
20. Henrion, D., Kuèera, V., and Molina-Cristobal A., Optimizing simultaneously over the numerator and denominator polynomials in the Youla–Kuèera parametrization, *IEEE Trans. Autom. Control*, 50, 1369–1374, 2005.

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