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MATHEMATICS

On submersivity assumption for nonlinear control systems on homogeneous time scales

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Abstract. The paper derives a condition that allows construction of the σ-differential fields for nonlinear control systems, described by the set of input–output (i/o) higher-order delta-differential equations, defined on a homogeneous time scale. This condition is related to the submersivity assumption of the extended system, associated with i/o equations, but is formulated directly in terms of i/o equations.

Key words: time scale, nonlinear system, submersion, submersive system.

1. INTRODUCTION

The submersivity property plays a crucial role in the study of discrete-time nonlinear control systems. The concept of submersivity was introduced into the nonlinear control theory by Grizzle [1] and since then this assumption has been made in the majority of papers on discrete-time nonlinear control systems. Especially, this assumption is vital in the algebraic approach based on differential forms as well as in the differential geometric approach based on vector fields. Under the submersivity assumption the backward shift, playing an important role in the above approaches, is a well-defined operator in the inversive closure of the difference field, associated with the control system. The submersivity condition is also necessary for system inversion [1] and in application of the Singh compensator to solve the dynamic input–output (i/o) linearization problem in the discrete-time case; see Example 4 in [2]. In [3] it was proved that the discrete-time nonlinear control system, described by the rational state transition function, is submersive if and only if the ideal, defined by the control system, is prime, proper, and reflexive. Finally, note that the submersivity assumption is not restrictive, since it is a necessary condition for the system to be accessible [1].

The submersivity condition also plays an important role in the study of nonlinear control systems on time scales [4]. The time scale is a framework that allows unifying the study of continuous-time and discrete-time systems into a single general formalism and also an extension to the cases when time may be partly continuous and partly discrete. The main concept of the time scale calculus is the so-called delta-derivative that is a generalization of both the time-derivative (in the continuous-time case) and the difference operator (in the discrete-time case) [5]. In a similar manner a related definition of a delta-differential equation

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describes both the differential and difference equations. Therefore, note that since the shift operator is not a delta-derivative, the time scale formalism accommodates the discrete-time system description in terms of the difference operator, in opposition to more conventional models based on the shift operator [1,6].

The submersivity assumption is given, in general, for systems described via state equations. If the system description is given in terms of i/o equations, then the condition may be formulated with the help of the so-called extended state equations, associated with i/o equations, as it has been done, for example, in [4] for a single-input single-output (SISO) system, defined on the time scale. As a result, every time one studies a system defined by i/o equations, one needs to introduce an additional description of the system. The only purpose of this description, called *the extended system*, is verifying the submersivity assumption. This approach, though possible in principle, would also complicate the proofs, because one has to go from one system description to the other every time the submersivity assumption shows up. It would be much more desirable to present the submersivity condition directly in terms of i/o equations under study. Finally, note that this task becomes especially urgent in addressing the multi-input multi-output (MIMO) systems defined on the time scale, since in such a case the relationship between two equivalent submersivity conditions is far from immediate unlike in the SISO case, or even in the MIMO case when the system is described in terms of the shift operator. The goal of this paper is to derive the submersivity condition that is presented directly in terms of the set of i/o delta-differential equations. We examine the structure of the extended system, associated with the set of i/o equations and its submersivity conditions.

2. TIME SCALES AND THE DELTA-DERIVATIVE

In this section the necessary facts are defined. For a more detailed presentation of this topic, see the references, e.g. [4,5,7]. Let us remark that the central problem is to define a generalization of the derivative operator (known from the real analysis) for functions defined on time scales.

A *time scale* $\mathbb{T}$ is defined as a nonempty closed subset of the set $\mathbb{R}$. We assume that $\mathbb{T}$ is a topological space with the topology induced by $\mathbb{R}$. Let $\mathbb{T}$ be a time scale and $t \in \mathbb{T}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined as $\sigma(t) = \inf\{s \in \mathbb{T},\ s > t\}$ if $t \neq \max\mathbb{T}$, $\sigma(\max\mathbb{T}) := \max\mathbb{T}$. Then the *graininess function* $\mu : \mathbb{T} \to \mathbb{T}$ is defined as $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. We are interested in homogeneous time scales. The time scale $\mathbb{T}$ is called *homogeneous* if $\mu = \text{const}$. Moreover, the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T},\ s < t\}$ if $t \neq \min\mathbb{T}$, $\rho(\min\mathbb{T}) := \min\mathbb{T}$.

Let $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max\mathbb{T}\}$ if $\rho(\max\mathbb{T}) < \max\mathbb{T}$, $\mathbb{T}^\kappa = \mathbb{T}$ in other cases. Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function defined on the time scale $\mathbb{T}$. The *delta-derivative* $f^\Delta(t),\ t \in \mathbb{T}^\kappa$ is defined as the real number (provided it exists) with the property that, given $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$,

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leqslant \varepsilon|\sigma(t) - s|.$$

Moreover, f is *delta-differentiable* (on $\mathbb{T}^\kappa$) provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The properties of the delta-derivative can be found in [5], therefore their description is omitted here. The *n*th delta-derivative of the function f, denoted by $f^{[n]}$, is defined recursively as $f^{[1]} := f^\Delta$ and $f^{[n]} := (f^{[n-1]})^\Delta$, $n \geqslant 2$.

Assume the function $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $g : \mathbb{T} \to \mathbb{R}$ is a delta-differentiable function. Then the function $f \circ g$ is delta-differentiable and the delta-derivative satisfies the relation

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^\Delta(t))dh \right\} \cdot g^\Delta(t).$$

At the end of this section we introduce some notation that will be useful in the following sections. Let f be a function admitting the delta-derivatives up to the qth order. Let q_1 and q_2 be integers such that $0 \leq q_1 < q_2 \leq q$. We set $f^{[0]} = f$. Let $f^{[q_1 \cdots q_2]}$ denote the set $\{f^{[q_1]}, \ldots, f^{[q_2]}\}$. Moreover, instead of $f(\sigma(t))$ the symbol $f^\sigma(t)$ is often used. We define also $f^{[q_1 \cdots q_2]\sigma}$ as $\{f^{[q_1]\sigma}, \ldots, f^{[q_2]\sigma}\}$.

3. THE CONTROL SYSTEM AND THE ASSOCIATED σ-DIFFERENTIAL FIELD

Consider a MIMO nonlinear control system (with m inputs and p outputs) described by a set of higher-order i/o delta-differential equations on the homogeneous time scale $\mathbb{T}$ relating the inputs $u_j,\ j=1,\ldots,m$, the outputs y_i, $i=1,\ldots,p$, and the finite number of their delta-derivatives:

$$\begin{array}{rcl} y_1^{[n_1]} & = & \Phi_1(y_1^{[0\ldots n_1-1]}, y_2^{[0\ldots n_{1,2}]},\ldots, y_p^{[0\ldots n_{1,p}]}, u_1^{[0\ldots s_{1,1}]},\ldots,u_m^{[0\ldots s_{1,m}]}) \\ \vdots & & \vdots \\ y_p^{[n_p]} & = & \Phi_p(y_1^{[0\ldots n_{p,1}]}, y_2^{[0\ldots n_{p,2}]},\ldots, y_p^{[0\ldots n_p-1]}, u_1^{[0\ldots s_{p,1}]},\ldots,u_m^{[0\ldots s_{p,m}]}), \end{array} \tag{1}$$

where the functions $\Phi_i,\ i=1,\ldots,p$ are analytic functions of their arguments and functions $y_i:\mathbb{T}\to\mathbb{R}$, $i=1,\ldots,p$ and $u_j:\mathbb{T}\to\mathbb{R},\ j=1,\ldots,m$ are delta-differentiable at least up to order n_i and $s_j:=\max_{1\leqslant i\leqslant p}(s_{i,j})$, respectively. It is assumed throughout this text that for each $i=1,\ldots,p$ the following inequalities hold: $n_i>n_{i,j}$ and $n_i>\max(n_{j,i}),\ j=1,\ldots,p,\ j\neq i$. In other words, the highest derivative of y_i appears in the ith equation. Moreover, one assumes that $s_k<n_i$ for $i=1,\ldots,p$, $k=1,\ldots,m$.

The σ-differential field to be defined in this section plays an important role in further investigations as all the functions defining the control systems are elements of this field.

Consider the set of (independent) variables

$$\mathscr{C}=\{y_1^{[0\ldots n_1-1]},\ldots,y_p^{[0\ldots n_p-1]},u_1^{[l_1]},\ldots,u_m^{[l_m]},\ \text{for all } l_1,\ldots,l_m\geq 0\}.$$

Denote by $\mathscr{R}$ the ring of analytic functions depending on a finite number of variables from the set $\mathscr{C}$. The operator σ can be extended onto this ring as follows. Let $\varphi\in\mathscr{R}$. Assume the function φ depends on the variables $y_1^{[0\ldots n_1-1]},\ldots,y_p^{[0\ldots n_p-1]},u_1^{[0\ldots s_1]},\ldots,u_m^{[0\ldots s_m]}$, $s_1,\ldots,s_m\geqslant 0$. Then

$$\begin{aligned}\sigma(\varphi)(y_1^{[0\ldots n_1-1]},\ldots,y_p^{[0\ldots n_p-1]},u_1^{[0\ldots s_1+1]},\ldots,u_m^{[0\ldots s_m+1]})&\\ :=\varphi(y_1^{[0\ldots n_1-1]\sigma},\ldots,y_p^{[0\ldots n_p-1]\sigma},u_1^{[0\ldots s_1]\sigma},\ldots,u_m^{[0\ldots s_m]\sigma}),&\end{aligned} \tag{2}$$

where

$$y_i^{[0\ldots n_i-1]\sigma}=y_i^{[0\ldots n_i-1]}+\mu\left[y_i^{[1\ldots n_i-1]},\Phi_i\big(y_1^{[0\ldots n_{i,1}]},\ldots,y_i^{[0\ldots n_i-1]},\ldots,y_p^{[0\ldots n_{i,p}]},u_1^{[0\ldots s_{p,s}]},\ldots,u_m^{[0\ldots s_{p,m}]}\big)\right],\ i=1,\ldots,p$$

and

$$u_j^{[0\ldots s_j]\sigma}=u_j^{[0\ldots s_j]}+\mu u_j^{[1\ldots s_j+1]},\ \ j=1,\ldots,m.$$

The operator σ defined on the ring $\mathscr{R}$ is an endomorphism. This means, it is a linear mapping satisfying the conditions

$$\sigma(\varphi\psi)=\sigma(\varphi)\sigma(\psi)$$

and

$$\sigma(1_{\mathscr{R}})=1_{\mathscr{R}},$$

where the symbol $1_{\mathscr{R}}$ stands for the unit in the ring $\mathscr{R}$.

Additionally, in the ring $\mathscr{R}$ one can define the operator $\Delta:\mathscr{R}\to\mathscr{R}$ as follows:

$$\begin{aligned}&\Delta(\varphi)\big(y_i,\ldots,y_i^{[n_i-1]},u_k,\ldots,u_k^{[l+1]}\big)\\ &:=\int_0^1\Bigg\{\mathrm{grad}\varphi\big(y_i+h\mu y_i^{\Delta},\ldots,y_i^{[n_i-1]}+h\mu\phi_i\big(y_1^{[0\ldots n_{i1}-1]},\ldots,y_p^{[0\ldots n_{ip}-1]},u_1^{[0\ldots s_{i1}]},\ldots,u_m^{[0\ldots s_{im}]}\big),\\ &\qquad u_k+h\mu u_k^{\Delta},\ldots,u_k^{[l]}+h\mu u_k^{[l+1]}\big)\\ &\qquad\times\begin{bmatrix}\big(y_i^{\Delta},\ldots,y_i^{[n_i-1]}\big)^{\mathrm{T}}\\ \phi_i\big(y_1^{[0\ldots n_{i1}-1]},\ldots,y_p^{[0\ldots n_{ip}-1]},u_1^{[0\ldots s_{i1}]},\ldots,u_m^{[0\ldots s_{im}]}\big)\\ \big(u_k^{\Delta},\ldots,u_k^{[l+1]}\big)^{\mathrm{T}}\end{bmatrix}\Bigg\}dh.\end{aligned} \tag{3}$$

We will use $\sigma(\varphi)$ and φ^{σ} to denote the action of σ on F. Similarly, both $\Delta(\varphi)$ and φ^{Δ} will be used interchangeably.

Denote by $\mathscr{K}$ the quotient field of $\mathscr{R}$. That means, $\mathscr{K}$ is the field of meromorphic functions in variables from $\mathscr{C}$. If the operator σ is injective, one can extend this operator onto the field $\mathscr{K}$ by

$$\sigma(\varphi/\psi) = \sigma(\varphi)/\sigma(\psi).$$

Note that if σ is not injective, there may exist a non-zero function $\varphi \in \mathscr{R}$ such that $\sigma(\varphi) = 0$. Then, taking $F = (\psi/\varphi) \in \mathscr{K}$, we get that the operator σ is not well defined on the field $\mathscr{K}$. For the operator σ to be injective the set of i/o equations has to satisfy an assumption equivalent to the submersivity of the extended system associated with the set of i/o equations.

3.1. Submersivity

In this section we derive a condition that is equivalent to the submersivity property of the extended system associated with the set of i/o equations. The condition is given in terms of i/o equations and it guarantees the injectivity of the operator σ.

First, associate with the set of i/o equations (1), the so-called *extended state-space system* with the state $(\mathbf{z}, \mathbf{w})$, where

$$\mathbf{z} := (z_{1,0}, \ldots, z_{1,n_1-1}, \ldots, z_{i,0}, \ldots, z_{i,n_i-1}, \ldots, z_{p,0}, \ldots, z_{p,n_p-1}) \in \mathbb{R}^{n_1+\cdots+n_p}$$

and

$$\mathbf{w} := (w_{1,0}, \ldots, w_{1,s}, \ldots, w_{i,0}, \ldots, w_{i,s}, \ldots, w_{m,0}, \ldots, w_{m,s}) \in \mathbb{R}^{ms},$$

whereas

$$\begin{aligned} z_{i,j} &= y_i^{[j]}, \quad i = 1, \ldots, p, \ j = 0, \ldots, n_i - 1, \\ w_{k,l} &= u_k^{[l]}, \quad k = 1, \ldots, m, \ l = 0, \ldots, s, \end{aligned}$$

and the inputs $v_k = u_k^{[s+1]}$, where $s := \max_{1 \leqslant k \leqslant m}(s_k)$. Then, with the set of equations (1) one can associate the following extended system Σ_e:

$$\begin{aligned} z_{i,j}^{\Delta} &= z_{i,j+1}, & i &= 1, \ldots, p, \ j = 0, \ldots, n_i - 2, \\ z_{i,n_i-1}^{\Delta} &= \Phi_i(\mathbf{z}, \mathbf{w}), & i &= 1, \ldots, p, \\ w_{k,l}^{\Delta} &= w_{k,l+1}, & k &= 1, \ldots, m, \ l = 0, \ldots, s-1, \\ w_{k,s}^{\Delta} &= v_k, & k &= 1, \ldots, m. \end{aligned} \tag{4}$$

The set of equations (4) governing the dynamics of the extended system Σ_e can be rewritten using the operator σ as follows:

$$\begin{aligned} z_{i,j}^{\sigma} &= z_{i,j} + \mu z_{i,j+1}, & i &= 1, \ldots, p, \ j = 0, \ldots, n_i - 2, \\ z_{i,n_i-1}^{\sigma} &= z_{i,n_i-1} + \mu \Phi_i(\mathbf{z}, \mathbf{w}), & i &= 1, \ldots, p, \\ w_{k,l}^{\sigma} &= w_{k,l} + \mu w_{k,l+1}, & k &= 1, \ldots, m, \ l = 0, \ldots, s-1, \\ w_{k,s}^{\sigma} &= w_{k,s} + \mu v_k, & k &= 1, \ldots, m. \end{aligned} \tag{5}$$

We are going to examine the structure of (5). Let $\mathbf{v} = (v_1, \ldots, v_m)$ be the vector of inputs and

$$(\mathbf{z}, \mathbf{w}, \mathbf{v}) \mapsto (\mathbf{z}, \mathbf{w}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, \mathbf{v}) \tag{6}$$

be the map corresponding to the right-hand side of (5), where $\mathbf{f}_{\mathrm{e}}(\mathbf{z},\mathbf{w},\mathbf{v}) := \left[z_{1,1},\ldots,z_{1,n_1-1},\Phi_1(\mathbf{z},\mathbf{w}),\ldots,\right.$ $\left.z_{p,1},\ldots,z_{p,n_p-1},\Phi_p(\mathbf{z},\mathbf{w}),w_{1,0},\ldots,w_{1,s},v_1,w_{m,0},\ldots,w_{m,s},v_m\right]^{\mathrm{T}}$ is the the right-hand side of (4). For the operator σ to be injective the map (6) has to define generically a *submersion*[1] [1,6], that is to satisfy (generically) the following condition:

$$\operatorname{rank}_{\mathscr{K}} \frac{\partial\left[(\mathbf{z},\mathbf{w})+\mu\mathbf{f}_{\mathrm{e}}(\mathbf{z},\mathbf{w},\mathbf{v})\right]}{\partial(\mathbf{z},\mathbf{w},\mathbf{v})} = n_1+\cdots+n_p+m\cdot(s+1)\,. \tag{7}$$

If the map (6) is a submersion, then systems (1) and (4) are called *submersive* [1,6]. The characterization of the submersive system can also be found in [3], where the authors have proved that submersivity of the system is equivalent to the ideal defined by the system to be prime, proper, and reflexive. By $\operatorname{rank}_{\mathscr{K}} A$ we mean the rank of the matrix A over the field $\mathscr{K}$.

Our goal is to formulate the condition (7) in terms of i/o equations. Before formulating the main theorem we define the following elementary *column transformations* which do not change the rank of the matrix:
1. Interchange of columns i and j.
2. Multiplication of column i by a nonzero scalar in $\mathscr{K}$.
3. Replacement of column i by itself plus any scalar in $\mathscr{K}$ multiplied by any other column j.

Theorem 3.1. *The nonlinear control system, defined on a homogeneous time scale via the higher-order i/o equations* (1), *is submersive if and only if the following condition*

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1+\alpha_{11} & \ldots & \alpha_{1p} & \beta_{11} & \ldots & \beta_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{p1} & \ldots & 1+\alpha_{pp} & \beta_{p1} & \ldots & \beta_{pm} \end{pmatrix} = p \tag{8}$$

holds, where

$$\alpha_{ij} := \sum_{k=0}^{n_j-1} (-1)^{n_j-k-1}\mu^{n_j-k}\frac{\partial\Phi_i}{\partial y_j^{[k]}}\,, \tag{9}$$

$i,j=1,\ldots,p$, *and*

$$\beta_{lk} := \sum_{j=0}^{s} (-1)^{s-j+1}\mu^{s-j+2}\frac{\partial\Phi_l}{\partial u_k^{[j]}}\,, \tag{10}$$

$l=1,\ldots,p$, $k=1\ldots,m$.

Proof. Let us assume that the system (1) is submersive, i.e. the map (6) is a submersion (or, equivalently, the condition (7) holds). We are going to represent the Jacobi matrix $\frac{\partial[(\mathbf{z},\mathbf{w})+\mu\mathbf{f}_{\mathrm{e}}(\mathbf{z},\mathbf{w},\mathbf{v})]}{\partial(\mathbf{z},\mathbf{w},\mathbf{v})}$, using matrices that can be changed by elementary column transformations into the form for which the computation of the rank can be easily reduced to the computation of the ranks of its certain submatrices. Let us define the $n_i\times n_i$-matrices $A_i,\ i=1,\ldots,p$

$$A_i = \begin{pmatrix} 1 & \mu & 0 & \ldots & 0 \\ 0 & 1 & \mu & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \\ \mu\frac{\partial\Phi_i}{\partial z_{i,0}} & \mu\frac{\partial\Phi_i}{\partial z_{i,1}} & \mu\frac{\partial\Phi_i}{\partial z_{i,2}} & \ldots & 1+\mu\frac{\partial\Phi_i}{\partial z_{i,n_i-1}} \end{pmatrix}$$

[1] Recall that a map $\tilde{f}$ is called a *submersion at a point p* if its differential $D\tilde{f}_p$ is a surjective linear map. A differentiable map $\tilde{f}$ that is a submersion at each point is called a *submersion*. Equivalently, $\tilde{f}$ is a submersion if its differential has constant rank equal to the dimension of the codomain of $\tilde{f}$.

and also the matrices $B_{i,r}$, $i \neq r$, $i,r = 1,\dots,p$

$$B_{i,r} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \mu \frac{\partial \Phi_i}{\partial z_{r,0}} & \dots & \mu \frac{\partial \Phi_i}{\partial z_{r,n_r-1}} \end{pmatrix},$$

whose dimensions are $n_i \times n_r$. Moreover, matrices $C_{i,k}$ of dimension $n_i \times (s+2)$ are defined as follows:

$$C_{i,k} = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ \mu \frac{\partial \Phi_i}{\partial w_{k,0}} & \dots & \mu \frac{\partial \Phi_i}{\partial w_{k,s}} & 0 \end{pmatrix},$$

where $i = 1,\dots,p$, $k = 1,\dots,m$. Let now D be the following matrix:

$$D = \begin{pmatrix} 1 & \mu & 0 & \dots & 0 & 0 \\ 0 & 1 & \mu & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu & 0 \\ 0 & 0 & 0 & \dots & 1 & \mu \end{pmatrix}$$

with dimension $(s+1) \times (s+2)$.

The Jacobi matrix $\frac{\partial[(\mathbf{z},\mathbf{w})+\mu \mathbf{f}_e(\mathbf{z},\mathbf{w},\mathbf{v})]}{\partial(\mathbf{z},\mathbf{w},\mathbf{v})}$ in (7) is as follows:

$$A = \begin{pmatrix} A_1 & B_{1,2} & \dots & B_{1,p} & C_{1,1} & \dots & C_{1,m} \\ B_{2,1} & A_2 & & B_{2,p} & C_{2,1} & \dots & C_{2,m} \\ \vdots & & \ddots & \vdots & \vdots & & \vdots \\ B_{p,1} & B_{p,2} & \dots & A_p & C_{p,1} & \dots & C_{p,m} \\ 0 & 0 & \dots & 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & D \end{pmatrix}.$$

The submersivity condition (7) requires the following

$$\operatorname{rank}_{\mathscr{K}} A = n_1 + \dots + n_p + m \cdot (s+1). \tag{11}$$

The matrices A_i can be converted by using elementary column transformations of A (that do not change its rank) into the form where the only nonzero elements are placed on the last row and on the diagonal, the latter being units. This can be achieved by subtracting the first column multiplied by $-\mu$ from the second one, then subtracting the second column multiplied by $-\mu$ from the third column, and proceeding in this way up to the last column. This converts the matrix A_i into

$$\tilde{A}_i = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \mu \frac{\partial \Phi_i}{\partial z_{i,0}} & \mu \frac{\partial \Phi_i}{\partial z_{i,1}} - \mu^2 \frac{\partial \Phi_i}{\partial z_{i,0}} & \dots & 1 + \sum_{j=0}^{n_i-1} (-1)^{n_i-j-1} \mu^{n_i-j} \frac{\partial \Phi_i}{\partial z_{i,j}} \end{pmatrix}.$$

These transformations simultaneously change the matrices $B_{i,j}$ into $\tilde{B}_{i,j},\ i \neq j$:

$$\tilde{B}_{i,j} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \mu \frac{\partial \Phi_i}{\partial z_{j,0}} & \mu \frac{\partial \Phi_i}{\partial z_{j,1}} - \mu^2 \frac{\partial \Phi_i}{\partial z_{j,0}} & \dots & \sum\limits_{k=0}^{n_j-1} (-1)^{n_j-k-1} \mu^{n_j-k} \frac{\partial \Phi_i}{\partial z_{j,k}} \end{pmatrix}.$$

Analogous elementary column transformations carried out with the matrix D convert this matrix into the following matrix:

$$\tilde{D} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Simultaneously, the matrices $C_{i,k}$ are transformed into the matrices

$$\tilde{C}_{i,k} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \mu \frac{\partial \Phi_i}{\partial w_{k,0}} & \mu \frac{\partial \Phi_i}{\partial w_{k,1}} - \mu^2 \frac{\partial \Phi_i}{\partial w_{k,0}} & \dots & \sum\limits_{j=0}^{s} (-1)^{s-j} \mu^{s-j+1} \frac{\partial \Phi_i}{\partial w_{k,j}} & \sum\limits_{j=0}^{s} (-1)^{s-j+1} \mu^{s-j+2} \frac{\partial \Phi_i}{\partial w_{k,j}} \end{pmatrix}.$$

To conclude, the matrix A is transformed into the matrix $\tilde{A}$ of the following form:

$$\tilde{A} = \begin{pmatrix} \tilde{A}_1 & \tilde{B}_{1,2} & \dots & \tilde{B}_{1,p} & \tilde{C}_{1,1} & \dots & \tilde{C}_{1,m} \\ \tilde{B}_{2,1} & \tilde{A}_2 & & \tilde{B}_{2,p} & \tilde{C}_{2,1} & \dots & \tilde{C}_{2,m} \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{p,1} & \tilde{B}_{p,2} & \dots & \tilde{A}_p & \tilde{C}_{p,1} & \dots & \tilde{C}_{p,m} \\ 0 & 0 & \dots & 0 & \tilde{D} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \tilde{D} \end{pmatrix}.$$

Note that $\frac{\partial \Phi_i}{\partial w_{k,j}} = 0$ for $j = s_k + 1, \dots, s$, so

$$\sum_{j=0}^{s} (-1)^{s-j+1} \mu^{s-j+2} \frac{\partial \Phi_i}{\partial w_{k,j}} = \sum_{j=0}^{s_k} (-1)^{s_k-j+1} \mu^{s_k-j+2} \frac{\partial \Phi_i}{\partial w_{k,j}}.$$

Moreover,

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \tilde{C}_{1,1} & \dots & \tilde{C}_{1,m} \\ \tilde{C}_{2,1} & \dots & \tilde{C}_{2,m} \\ \vdots & \ddots & \vdots \\ \tilde{C}_{p,1} & \dots & \tilde{C}_{p,m} \\ \tilde{D} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \tilde{D} \end{pmatrix} = m(s+1) + \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \Theta_{1,1} & \dots & \Theta_{1,m} \\ \Theta_{2,1} & \dots & \Theta_{2,m} \\ \vdots & \ddots & \vdots \\ \Theta_{p,1} & \dots & \Theta_{p,m} \end{pmatrix}, \tag{12}$$

where $\Theta_{i,k} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{j=0}^{s}(-1)^{s-j+1}\mu^{s-j+2}\frac{\partial\Phi_i}{\partial w_{k,j}} \end{pmatrix}$, $i = 1,\ldots,p$, $k = 1,\ldots,m$ are the matrices of dimensions $n_i \times 1$.

Since $\operatorname{rank}_{\mathscr{K}} A = \operatorname{rank}_{\mathscr{K}} \widetilde{A}$ and the condition (12) holds, we get that the condition (11) is equivalent to

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \tilde{A}_1 & \tilde{B}_{1,2} & \ldots & \tilde{B}_{1,p} & \Theta_{1,1} & \ldots & \Theta_{1,m} \\ \tilde{B}_{2,1} & \tilde{A}_2 & & \tilde{B}_{2,p} & \Theta_{2,1} & \ldots & \Theta_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{p,1} & \tilde{B}_{p,2} & \ldots & \tilde{A}_p & \Theta_{p,1} & \ldots & \Theta_{p,m} \end{pmatrix} = n_1 + n_2 + \cdots + n_p\,. \tag{13}$$

Note that for $i = 1,\ldots,p$

$$\begin{aligned} &\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \tilde{B}_{i,1} & \ldots & \tilde{B}_{i,i-1} & \tilde{A}_i & \tilde{B}_{i,i+1} & \ldots & \tilde{B}_{i,p} & \Theta_{i,1} & \ldots & \Theta_{i,m} \end{pmatrix} \\ &\qquad = n_i - 1 + \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \tilde{\alpha}_{i1} & \ldots & \tilde{\alpha}_{i,i-1} & 1+\tilde{\alpha}_{ii} & \tilde{\alpha}_{i,i+1} & \ldots & \tilde{\alpha}_{ip} & \tilde{\beta}_{i1} & \ldots & \tilde{\beta}_{im} \end{pmatrix}, \end{aligned}$$

where

$$\tilde{\alpha}_{ij} := \sum_{k=0}^{n_j-1}(-1)^{n_j-k-1}\mu^{n_j-k}\frac{\partial\Phi_i}{\partial z_{j,k}},$$

$i, j = 1,\ldots,p$,

$$\tilde{\beta}_{lk} := \sum_{j=0}^{s}(-1)^{s-j+1}\mu^{s-j+2}\frac{\partial\Phi_l}{\partial w_{k,j}},$$

$l = 1,\ldots,p$, $k = 1\ldots,m$, so

$$\begin{aligned} &\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \tilde{A}_1 & \tilde{B}_{1,2} & \ldots & \tilde{B}_{1,p} & \Theta_{1,1} & \ldots & \Theta_{1,m} \\ \tilde{B}_{2,1} & \tilde{A}_2 & & \tilde{B}_{2,p} & \Theta_{2,1} & \ldots & \Theta_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{B}_{p,1} & \tilde{B}_{p,2} & \ldots & \tilde{A}_p & \Theta_{p,1} & \ldots & \Theta_{p,m} \end{pmatrix} \\ &\qquad = n_1 + n_2 + \cdots + n_p - p + \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1+\tilde{\alpha}_{11} & \ldots & \tilde{\alpha}_{1p} & \tilde{\beta}_{11} & \ldots & \tilde{\beta}_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\alpha}_{p1} & \ldots & 1+\tilde{\alpha}_{pp} & \tilde{\beta}_{p1} & \ldots & \tilde{\beta}_{pm} \end{pmatrix}. \end{aligned}$$

Since $z_{i,j} = y_i^{[j]}$, $i = 1,\ldots,p$, $j = 0,\ldots,n_j - 1$ and $w_{k,l} = u_k^{[l]}$, $k = 1,\ldots,m$, $l = 0,\ldots,s$, we get $\tilde{\alpha}_{ij} = \alpha_{ij}$, $i, j = 1,\ldots,p$ and $\tilde{\beta}_{lk} = \beta_{lk}$, $l = 1,\ldots,p$, $k = 1\ldots,m$, where α_{ij} and β_{lk} are defined by (9) and (10), respectively, and therefore the condition (11) is equivalent to (13) and consequently to (8). □

Remark 3.2. Note that in the continuous-time case when $\mathbb{T} = \mathbb{R}$ we have $\mu \equiv 0$ and the condition (8) is satisfied automatically. Moreover, in this case $\sigma = \mathrm{id}$.

Example 4.1 in Section 4 illustrates the effects the nonsubmersive system may exhibit.

Assuming the submersivity property (8) to hold for the set of i/o delta-differential equations (1), we guarantee the injectivity of the operator σ. And, consequently, σ is well defined on the field of meromorphic functions depending on variables associated with the control system. The operator σ can be extended to $\mathscr{K}$, using the formula (2). Additionally, the operator Δ can be extended to $\mathscr{K}$, using the formula (3). The operator Δ, which is the extension of delta-derivative to the field $\mathscr{K}$, satisfies a generalization of the Leibniz rule

$$(FG)^{\Delta} = F^{\Delta}G + F^{\sigma}G^{\Delta}, \tag{14}$$

for $F, G \in \mathscr{K}$. The derivation satisfying the rule (14) is called a '*σ-derivation*' (for example, see [8]). Therefore $\mathscr{K}$ is a field equipped with a σ-derivation Δ such that σ is an injective endomorphism of $\mathscr{K}$. The field $\mathscr{K}$ with σ-derivation Δ is a *σ-differential field*. Since σ is injective, there exists a σ-differential overfield $\mathscr{K}^*$, called the *inversive closure* of $\mathscr{K}$, such that σ can be extended to $\mathscr{K}^*$ and this extension is an automorphism of $\mathscr{K}^*$ (see [8]). The inversive closure for a MIMO dynamical nonlinear system can be found in a similar way as that for the SISO case in [7], using the extended state-space model associated with the set of i/o equations.

4. EXAMPLES

The results of the previous section are illustrated by simple examples here.

Example 4.1. Consider the system described by the set of i/o delta-differential equations on a homogeneous time scale with the graininess function $\mu \neq 0$

$$\begin{aligned} y_1^\Delta &= -\tfrac{1}{\mu}(y_1 - y_2) + uy_2y_1, \\ y_2^\Delta &= uy_2y_1. \end{aligned} \tag{15}$$

The function $1/(y_1 - y_2)$ is meromorphic and belongs to the field $\mathscr{K}$, but $\sigma(1/(y_1 - y_2))$ is not defined because $\sigma(y_1 - y_2) = 0$. Note that the considered system is not submersive according to Theorem 3.1, since $\alpha_{11} = -1 + \mu uy_2$, $\alpha_{12} = 1 + \mu uy_1$, $\alpha_{21} = \mu uy_2$, $\alpha_{22} = \mu uy_1$, $\beta_{11} = -\mu^2 y_1y_2$, $\beta_{21} = -\mu^2 y_1y_2$, and the condition (8) is not satisfied, i.e.

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1+\alpha_{11} & \alpha_{12} & \beta_{11} \\ \alpha_{21} & 1+\alpha_{22} & \beta_{21} \end{pmatrix} = \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} \mu uy_2 & 1+\mu uy_1 & -\mu^2 y_1y_2 \\ \mu uy_2 & 1+\mu uy_1 & -\mu^2 y_1y_2 \end{pmatrix} = 1 \neq 2.$$

It is easy to see that for the considered system the operator σ is not injective and $\sigma(y_1) = \sigma(y_2)$. The noninjective operator is not well defined on the quotient field. Hence we would like the operator σ to be injective.

Example 4.2. Consider the system

$$\begin{aligned} y_1^\Delta &= y_2 + u_1u_2, \\ y_2^{[2]} &= y_1 + u_2^\Delta. \end{aligned} \tag{16}$$

Then the extended state-space system Σ_e with the state

$$(\mathbf{z}, \mathbf{w}) = (z_{10}, z_{20}, z_{21}, w_{10}, w_{11}, w_{20}, w_{21}) = (y_1, y_2, y_2^\Delta, u_1, u_1^\Delta, u_2, u_2^\Delta)$$

and the inputs $\mathbf{v} = (v_1, v_2) = (u_1^{[2]}, u_2^{[2]})$ has the following form:

$$\begin{aligned} z_{10}^\Delta &= z_{20} + w_{10}w_{20}, \\ z_{20}^\Delta &= z_{21}, \\ z_{21}^\Delta &= z_{10} + w_{21}, \\ w_{10}^\Delta &= w_{11}, \\ w_{11}^\Delta &= v_1, \\ w_{20}^\Delta &= w_{21}, \\ w_{21}^\Delta &= v_2. \end{aligned} \tag{17}$$

The set of equations governing the dynamics of the extended system Σ_e can be rewritten, using the operator σ as follows:

$$\begin{aligned} z_{10}^{\sigma} &= z_{10} + \mu(z_{20} + w_{10}w_{20}), \\ z_{20}^{\sigma} &= z_{20} + \mu z_{21}, \\ z_{21}^{\sigma} &= z_{21} + \mu(z_{10} + w_{21}), \\ w_{10}^{\sigma} &= w_{10} + \mu w_{11}, \\ w_{11}^{\sigma} &= w_{11} + \mu v_1, \\ w_{20}^{\sigma} &= w_{20} + \mu w_{21}, \\ w_{21}^{\sigma} &= w_{21} + \mu v_2. \end{aligned} \tag{18}$$

Note that the map corresponding to the right-hand side of (18) has the following form:

$$(z_{10}, z_{20}, z_{21}, w_{10}, w_{11}, v_1, w_{20}, w_{21}, v_2) \mapsto (\mathbf{z}, \mathbf{w}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, \mathbf{v}), \tag{19}$$

where $\mathbf{f}_e(\mathbf{z}, \mathbf{w}, \mathbf{v}) := \big(z_{20} + w_{10}w_{20}, z_{21}, z_{10} + w_{21}, w_{11}, v_1, w_{21}, v_2\big)$ is the right-hand side of (17). For the operator σ to be injective the map (19) has to satisfy (generically) the condition

$$\operatorname{rank}_{\mathscr{K}} \frac{\partial\left[(\mathbf{z}, \mathbf{w}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, \mathbf{v})\right]}{\partial(\mathbf{z}, \mathbf{w}, \mathbf{v})} = 7. \tag{20}$$

Note that

$$A_1 := \frac{\partial\left[(\mathbf{z}, \mathbf{w}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, \mathbf{v})\right]}{\partial(\mathbf{z}, \mathbf{w}, \mathbf{v})} = \begin{pmatrix} 1 & \mu & 0 & \mu w_{20} & 0 & 0 & \mu w_{10} & 0 & 0 \\ 0 & 1 & \mu & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & 1 & 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 1 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \mu \end{pmatrix}.$$

Using the elementary column transformations specified in the proof of Theorem 3.1 and the fact that in the transformed matrix the 2nd, 4th, 5th, 6th, and 7th row vectors (or, equivalently, the 2nd, 4th, 5th, 7th, and 8th column vectors) are linearly independent, we get

$$\operatorname{rank}_{\mathscr{K}} A_1 = \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1 & \mu & -\mu^2 & \mu w_{20} & -\mu^2 w_{20} & \mu^3 w_{20} & \mu w_{10} & -\mu^2 w_{10} & \mu^3 w_{10} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu & 0 & 1 & 0 & 0 & 0 & 0 & \mu & -\mu^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= 5 + \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1 & -\mu^2 & \mu^3 w_{20} & \mu^3 w_{10} \\ \mu & 1 & 0 & -\mu^2 \end{pmatrix}.$$

Thus the condition (20) is equivalent to

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1 & -\mu^2 & \mu^3 w_{20} & \mu^3 w_{10} \\ \mu & 1 & 0 & -\mu^2 \end{pmatrix} = 2.$$

Since $(z_{10}, z_{20}, z_{21}, w_{10}, w_{11}, w_{20}, w_{21}) = (y_1, y_2, y_2^{\Delta}, u_1, u_1^{\Delta}, u_2, u_2^{\Delta})$, we get

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1 & -\mu^2 & \mu^3 u_2 & \mu^3 u_1 \\ \mu & 1 & 0 & -\mu^2 \end{pmatrix} = 2. \tag{21}$$

Therefore the condition which guarantees the injectivity of the operator $\sigma : \mathscr{K} \to \mathscr{K}$ is given in terms of variables $u_1, u_2 \in \mathscr{C}$. Moreover, using the elementary column transformations, namely by replacing the first column by itself plus $-\mu$ multiplied by the second column and replacing the fourth column by itself plus μ^2 multiplied by the second column, we get

$$\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1 & -\mu^2 & \mu^3 u_2 & \mu^3 u_1 \\ \mu & 1 & 0 & -\mu^2 \end{pmatrix} = \operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1+\mu^3 & -\mu^2 & \mu^2 u_2 & \mu^3 u_1 - \mu^4 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Therefore the condition (21) is equivalent to $\operatorname{rank}_{\mathscr{K}} \begin{pmatrix} 1+\mu^3 & \mu^2 u_2 & \mu^3 u_1 - \mu^4 \end{pmatrix} = 1$. Thus the system described by the set of equations (16) is submersive and the operator σ defined on the field of meromorphic functions in variables y_1, y_2, y_2^Δ, u_1, and u_2 is injective.

Example 4.3. Consider the system

$$\begin{aligned} y_1^{[2]} &= y_1 y_2^\Delta + u^2 y_1 y_2, \\ y_2^{[2]} &= u u^\Delta y_1 + y_1^\Delta y_2. \end{aligned} \tag{22}$$

Then the extended state-space system Σ_e with the state

$$(\mathbf{z}, \mathbf{w}) = (z_{10}, z_{11}, z_{20}, z_{21}, w_{10}, w_{11}) = (y_1, y_1^\Delta, y_2, y_2^\Delta, u, u^\Delta)$$

and the input $v_1 = u^{[2]}$ has the following form:

$$\begin{aligned} z_{10}^\Delta &= z_{11}, \\ z_{11}^\Delta &= z_{10} z_{21} + w_{10}^2 z_{10} z_{20}, \\ z_{20}^\Delta &= z_{21}, \\ z_{21}^\Delta &= w_{10} w_{11} z_{10} + z_{11} z_{20}, \\ w_{10}^\Delta &= w_{11}, \\ w_{11}^\Delta &= v_1. \end{aligned}$$

Using the operator σ, the system Σ_e can be rewritten as follows:

$$\begin{aligned} z_{10}^\sigma &= z_{10} + \mu z_{11}, \\ z_{11}^\sigma &= z_{10} + \mu(z_{10} z_{21} + w_{10}^2 z_{10} z_{20}), \\ z_{20}^\sigma &= z_{20} + \mu z_{21}, \\ z_{21}^\sigma &= z_{21} + \mu(w_{10} w_{11} z_{10} + z_{11} z_{20}), \\ w_{10}^\sigma &= w_{10} + \mu w_{11}, \\ w_{11}^\sigma &= w_{11} + \mu v_1. \end{aligned} \tag{23}$$

Then the map corresponding to the right-hand side of (23) is as follows:

$$(z_{10}, z_{11}, z_{20}, z_{21}, w_{10}, w_{11}, v_1) \mapsto (z_{10}, z_{11}, z_{20}, z_{21}, w_{10}, w_{11}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, v_1), \tag{24}$$

where $\mathbf{f}_e(\mathbf{z}, \mathbf{w}, v_1) := \left[z_{11}, z_{10} z_{21} + w_{10}^2 z_{10} z_{20}, z_{21}, w_{10} w_{11} z_{10} + z_{11} z_{20}, w_{11}, v_1\right]^T$ is the the right-hand side of Σ_e. For the operator σ to be injective the map (24) has to satisfy (generically) the condition

$$\operatorname{rank}_{\mathscr{K}} \frac{\partial\left[(\mathbf{z}, \mathbf{w}) + \mu \mathbf{f}_e(\mathbf{z}, \mathbf{w}, v_1)\right]}{\partial(\mathbf{z}, \mathbf{w}, v_1)} = 6. \tag{25}$$

Note that

$$A_2 := \frac{\partial\left[(\mathbf{z},\mathbf{w})+\mu\mathbf{f}_{\mathrm{e}}(\mathbf{z},\mathbf{w},v_1)\right]}{\partial(\mathbf{z},\mathbf{w},v_1)} = \begin{pmatrix} 1 & \mu & 0 & 0 & 0 & 0 & 0 \\ \mu(z_{21}+w_{10}^2 z_{20}) & 1 & \mu w_{10}^2 z_{10} & \mu z_{10} & 2\mu w_{10}z_{10}z_{20} & 0 & 0 \\ 0 & 0 & 1 & \mu & 0 & 0 & 0 \\ \mu w_{10}w_{11} & \mu z_{20} & \mu z_{11} & 1 & \mu w_{11}z_{10} & \mu w_{10}z_{10} & 0 \\ 0 & 0 & 0 & 0 & 1 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \mu \end{pmatrix}.$$

By using the elementary column transformations, the matrix A_2 can be transformed to $\tilde{A}_2$ in the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu(z_{21}+w_{10}^2 z_{20}) & 1-\mu^2(z_{21}+w_{10}^2 z_{20}) & \mu w_{10}^2 z_{10} & \mu z_{10}-\mu^2 w_{10}^2 z_{10} & 2\mu w_{10}z_{10}z_{20} & -2\mu^2 w_{10}z_{10}z_{20} & 2\mu^3 w_{10}z_{10}z_{20} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \mu w_{10}w_{11} & \mu(z_{20}-\mu w_{10}w_{11}) & \mu z_{11} & 1-\mu^2 z_{11} & \mu w_{11}z_{10} & \mu(w_{10}z_{10}-\mu w_{11}z_{10}) & \mu^2(\mu w_{11}z_{10}-w_{10}z_{10}) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that $\mathrm{rank}_{\mathscr{K}} A_2 = \mathrm{rank}_{\mathscr{K}} \tilde{A}_2$ and the 1st, 3rd, 5th, and 6th row vectors (or, equivalently, the 1st, 3rd, 5th, and 6th column vectors) are linearly independent over $\mathscr{K}$. Then

$$\mathrm{rank}_{\mathscr{K}} A_2 = 4 + \mathrm{rank}_{\mathscr{K}} \begin{pmatrix} 1-\mu^2(z_{21}+w_{10}^2 z_{20}) & \mu z_{10}-\mu^2 w_{10}^2 z_{10} & 2\mu^3 w_{10}z_{10}z_{20} \\ \mu(z_{20}-\mu w_{10}w_{11}) & 1-\mu^2 z_{11} & \mu^2(\mu w_{11}z_{10}-w_{10}z_{10}) \end{pmatrix}.$$

So the condition (25) is equivalent to

$$\mathrm{rank}_{\mathscr{K}} \begin{pmatrix} 1-\mu^2(z_{21}+w_{10}^2 z_{20}) & \mu z_{10}-\mu^2 w_{10}^2 z_{10} & 2\mu^3 w_{10}z_{10}z_{20} \\ \mu(z_{20}-\mu w_{10}w_{11}) & 1-\mu^2 z_{11} & \mu^2(\mu w_{11}z_{10}-w_{10}z_{10}) \end{pmatrix} = 2\,.$$

Since $(z_{10},z_{11},z_{20},z_{21},w_{10},w_{11}) = (y_1,y_1^\Delta,y_2,y_2^\Delta,u,u^\Delta)$, we get

$$\mathrm{rank}_{\mathscr{K}} \begin{pmatrix} 1-\mu^2(y_2^\Delta+u^2 y_2) & \mu y_1-\mu^2 u^2 y_1 & 2\mu^3 u y_1 y_2 \\ \mu(y_2-\mu u u^\Delta) & 1-\mu^2 y_1^\Delta & \mu^2(\mu u^\Delta y_1 - u y_1) \end{pmatrix} = 2\,. \tag{26}$$

Therefore the condition which guarantees the injectivity of the operator $\sigma : \mathscr{K} \to \mathscr{K}$ is given in terms of variables $y_1, y_1^\Delta, y_2, y_2^\Delta, u, u^\Delta \in \mathscr{C}$. Assuming the condition (26) holds generically, we get that the system described by the set of equations (22) is submersive and the operator σ defined on the field of meromorphic functions in variables y_1, y_1^Δ, y_2, y_2^Δ, u, and u^Δ is injective.

5. CONCLUSION

The submersivity condition for a MIMO nonlinear control system on a homogeneous time scale is derived directly in terms of i/o equations. The result is illustrated by several examples, including the demonstration of necessity of this assumption for the construction of the delta-differential field, associated with the control system.

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Submersiivsuse eeldusest mittelineaarsete juhtimissüsteemide jaoks homogeensetel ajaskaaladel

Ülle Kotta, Branislav Rehák ja Małgorzata Wyrwas

On tuletatud alternatiivne submersiivsuse tingimus mittelineaarsete juhtimissüsteemide jaoks, mis on kirjeldatud kõrgemat järku delta-diferentsiaalvõrranditega. Viimased on sisend-väljundvõrrandid homogeensetel ajaskaaladel, mis seovad juhtimissüsteemi sisendeid, väljundeid ja lõplikku arvu nende delta-tuletisi. Alternatiivne submersiivsuse tingimus on esitatud otseselt sisend-väljundvõrrandite kaudu ja on tuletatud süsteemi submersiivsuse tingimusest, mis on esitatud sisend-väljundvõrranditega seotud nn laiendatud olekuvõrrandite abil.