# Blur Invariants Constructed From Arbitrary Moments

Jaroslav Kautsky and Jan Flusser, *Senior Member, IEEE*

***Abstract*—This paper deals with moment invariants with respect to image blurring. It is mainly a reaction to the works of Zhang *et al.* and Chen *et al.*, recently published in these Transactions. We present a general method on how to construct blur invariants from arbitrary moments and show that it is no longer necessary to separately derive the invariants for each polynomial basis. We show how to discard dependent terms in blur invariants definition and discuss a proper implementation of the invariants in orthogonal bases using recurrent relations. An example for Legendre moments is given.**

***Index Terms*—Blur invariants, image moments, moment invariants, orthogonal moments.**

## I. Introduction

In 1996, Flusser *et al.* [3] introduced a new class of moment-based image descriptors (features), which are invariant to convolution of an image with an arbitrary symmetric kernel. Their research has been motivated by the need for recognition of images degraded by an unknown blur (which might originate from wrong focus, media turbulence, object/camera motion, etc.) without the necessity of estimating this blur and restoring the image.

Assuming the image acquisition time is so short that the blurring factors do not change during the image formation and also assuming that the blurring is of the same kind for all pixels and all colors/gray levels, we can describe the observed blurred image $g(x,y)$ of a scene $f(x,y)$ as convolution, i.e.,

$$g(x,y) = (f * h)(x,y) \tag{1}$$

where kernel $h(x,y)$ stands for the point-spread function (PSF) of the imaging system. Model (1) is a frequently used compromise between universality and simplicity—it is general enough to describe many practical situations such as out-of-focus blur of a flat scene, motion blur of a flat scene in case of linear constant-velocity motion, and media turbulence blur. At the same time, its simplicity allows reasonable mathematical treatment.

In many cases, we do not need to know the whole original image of which the estimation may be ill posed, time consuming, or even impossible; we only need, for instance, to localize or recognize some objects on it (typical examples are matching of a blurred template against a database and recognition of blurred characters). In such situations, only the knowledge of a certain incomplete but robust representation of the image is sufficient. However, such a representation should be independent of the imaging system and should actually describe the original image, not the degraded one. We are looking for a functional $I$ that is invariant to degradation (1), i.e.,

$$I(f) = I(f * h) \tag{2}$$

must hold for any admissible $h(x,y)$. Descriptors satisfying condition (2) are called *blur invariants* or *convolution invariants*.

Although the PSF is supposed to be unknown, we still have to accept certain assumptions about it to find invariants. For an arbitrary PSF, no blur invariants exist; the more we assume about the PSF shape, the more invariants can be found. Since 1998, when a fundamental paper [4] was published, almost all authors have considered *centrosymmetric* blur for which $h(x,y) = h(-x,-y)$. This is a natural choice because many real imaging systems behave in this way, and we keep this assumption in this paper, too. Flusser and Suk [4] derived a system of blur invariants that were based on geometric moments of the image. Their first results initiated intensive research. These moment invariants (as well as their equivalent counterparts in Fourier domain) have become very popular image descriptors and have found numerous applications, namely, in image matching and registration in remote sensing [4]–[7], in medical imaging [8], [9], in face recognition on out-of-focus photographs [3], in normalizing blurred images into canonical forms [10], [11], in blurred digit and character recognition [12], in robot control [13], [14], in image forgeries detection [15], in aircraft silhouette recognition [16], in traffic sign recognition [17], and in animal shape-based classification [18] (interested readers can find a comprehensive review in [19]).

In the last few years, several authors attempted derivation of blur invariants, which are functions of orthogonal moments rather than of geometric moments. Legendre moments [1], [20] and Zernike moments [2], [21] were employed for this purpose. It should be noted that moment invariants in any two different polynomial bases are mutually dependent and theoretically equivalent in terms of discrimination power; therefore, there is no chance to derive "new" or "better" independent invariants just by changing the polynomial basis. It is, however, well known that numerical calculation of orthogonal moments is more robust to precision loss when properly implemented. That is a justifiable motivation for developing invariants using orthogonal polynomial bases, which, surprisingly, was not explicitly mentioned in the papers quoted above. The authors either skipped any deeper analysis and ended up with a small incomplete subset of the invariants (this is the case of [21] and

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J. Kautsky is with the Flinders University of South Australia, Adelaide, SA 5001, Australia (e-mail: jarka@infoeng.flinders.edu.au).

J. Flusser is with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 182 08 Prague 8, Czech Republic (e-mail: flusser@utia.cas.cz).

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[22] where only Zernike moments of equal indexes are considered) or followed the original derivation presented in [4] "from scratch" and repeated the whole process for a chosen orthogonal basis [1], [2], [20]. Since the original derivation is long and difficult, even for the simplest polynomial basis $\{x^j y^k\}$, they ended up with extremely complicated and nontransparent formulas for Legendre and Zernike moments. Obviously, this nontransparency has led to several errors, incorrect conclusions, and misunderstandings.

In [20], the general formula for blur invariants (the authors used slightly different definition of blur symmetry) from Legendre moments (see [20, eq. (33)]) is incorrect. It can be easily proved that most invariants listed in the Appendix of [20] are not invariant at all (for instance $B_1, B_3, B_8$, etc.). A better attempt to derive Legendre invariants for centrosymmetric blur was published in [1] where the resulting invariants (see [1, eqs. (25) and (26)]) are "almost" correct. They are actually invariant, but due to the summation over redundant indexes, they become correlated since higher order invariants contain useless terms comprising lower order invariants, which were already used before (see [1, Appendix A]). Such terms should be discarded. Apparently, because of extremely complicated formulas (sextuple sums, complex recurrences, etc.) the authors were not able to analyze and correct this phenomenon. In the next paper of the same group of authors [23], the lack of deeper analysis led to a serious mistake in geometric normalization of blurred images (the normalization parameters with respect to rotation and stretching were calculated from the second-order Legendre moments, but these quantities depend on the particular blur).

This paper is mainly a reaction to the aforementioned papers [1], [2] published recently in these Transactions. Our primary motivation is to provide the readers (and prospective authors) with an insight into the subject and, consequently, to prevent mistakes both in theory and numerical computation. We demonstrate that it is useless to derive blur moment invariants with respect to each polynomial basis separately. We show that there exists a simple relation between blur invariants in different bases. As soon as we have the blur invariants in standard basis $\{x^j\}$, we can easily generate invariants in *any* polynomial basis $\{p_j(x)\}$, and in that way, we avoid error-prune individual derivations. For the sake of simplicity, we show that for a 1-D case. The generalization to the 2-D (or even $N$-dimensional) images presents no significant complications while being too lengthy for this paper. (More precisely, it is straightforward for separable polynomial bases of type $\{p_j(x)q_k(y)\}$ where $p(x)$ and $q(y)$ are any 1-D polynomial bases. Legendre and Chebyshev polynomials are typical examples. Extension to nonseparable 2-D orthogonal bases is more demanding because the construction and proper implementation of such bases may be complicated.)

The second goal of this paper is to develop an algorithm for discarding dependent terms in the definition formulas in blur invariants, which reduces correlation and simplifies the computations. Such method has never been proposed; in fact, this problem has never been identified and formulated.

Finally, the third goal is to suggest how to properly implement the blur invariants in orthogonal bases using recurrent relations.

In Section II, we recall the "traditional" 1-D blur invariants from geometric moments and present their definition in a new matrix notation. In Section III, we derive a recurrence for blur invariants for any polynomial basis, i.e., for arbitrary moments. In Section IV, we discuss blur invariants for any symmetric orthogonal polynomials, demonstrate how to systematically discard useless terms, and also present a special case for Legendre moments.

## II. Recalling Flusser–Suk Blur Invariants in 1-D

Let $f(x)$ be an arbitrary integrable image function and let

$$\mu_j = \int x^j f(x)dx, \quad j = 0, 1, \ldots$$

be its moments with respect to standard powers (commonly referred as *geometric moments*). We define the following for $j = 1, 2, \ldots$:

$$b_{2j-1} = \mu_{2j-1} - \frac{1}{\mu_0}\sum_{i=1}^{j-1}\binom{2j-1}{2i} b_{2(j-i)-1}\mu_{2i}. \tag{3}$$

Note that even-order invariants $b_2, b_4, b_6, \ldots$ do not exist (some authors formally define $b_{2j} = 0$ to get more compact formulas). Flusser and Suk proved that $b_1, b_3, b_5, \ldots$, defined by recurrence (3), form a complete and independent set of invariants with respect to arbitrary centrosymmetric blur (see [4] for the "full" proof in 2-D or [24] for a simplified 1-D version).

We want to express (3) in a matrix form. We introduce a vector notation, i.e.,

$$\boldsymbol{\mu}_a = \begin{pmatrix} \mu_1 \\ \mu_3 \\ \vdots \\ \mu_{2n-1} \end{pmatrix}, \quad \boldsymbol{\mu}_s = \begin{pmatrix} \mu_0 \\ \mu_2 \\ \vdots \\ \mu_{2n-2} \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_3 \\ \vdots \\ b_{2n-1} \end{pmatrix}.$$

The sum in (3) can be captured in two different ways. We have either

$$\boldsymbol{b} = \boldsymbol{\mu}_a - \frac{1}{\mu_0}B(\boldsymbol{b})\boldsymbol{\mu}_s \quad \text{or} \quad \boldsymbol{b} = \boldsymbol{\mu}_a - \frac{1}{\mu_0}C(\boldsymbol{\mu}_s)\boldsymbol{b} \tag{4}$$

where we indicate what matrices $B$ and $C$ depend on.

For example, for $n = 4$, matrices $B$ and $C$ are

$$B(\boldsymbol{b}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3b_1 & 0 & 0 \\ 0 & 10b_3 & 5b_1 & 0 \\ 0 & 21b_5 & 35b_3 & 7b_1 \end{pmatrix},$$

$$C(\boldsymbol{\mu}_s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3\mu_2 & 0 & 0 & 0 \\ 5\mu_4 & 10\mu_2 & 0 & 0 \\ 7\mu_6 & 35\mu_4 & 21\mu_2 & 0 \end{pmatrix}. \tag{5}$$

## III. Blur Invariants From Another Moment Set

In this section, we describe how to derive Flusser–Suk blur invariants (4) in terms of modified moments, that is, moments with respect to arbitrary polynomial basis, as long as this basis preserves the symmetric/antisymmetric property of the standard powers. This property has been essential in the original derivation of the Flusser–Suk blur invariants.

Let $p_j(x)$ be a sequence of polynomials of exact degree $j$, $j = 0, 1, \ldots, 2n-1$. To have the aforementioned symmetry property, odd-degree $p_{2j-1}$ (antisymmetric) must be combinations of odd powers, and even-degree $p_{2j}$ (symmetric) must be combinations of even powers only. Denote

$$\boldsymbol{x}_a = \begin{pmatrix} x \\ x^3 \\ \vdots \\ x^{2n-1} \end{pmatrix} \quad \boldsymbol{x}_s = \begin{pmatrix} 1 \\ x^2 \\ \vdots \\ x^{2n-2} \end{pmatrix},$$

$$\boldsymbol{p}_a = \begin{pmatrix} p_1 \\ p_3 \\ \vdots \\ p_{2n-1} \end{pmatrix}, \quad \boldsymbol{p}_s = \begin{pmatrix} p_0 \\ p_2 \\ \vdots \\ p_{2n-2} \end{pmatrix}.$$

Then, we have

$$\boldsymbol{p}_a = L_a \boldsymbol{x}_a \quad \text{and} \quad \boldsymbol{p}_s = L_s \boldsymbol{x}_s$$

for certain nonsingular lower triangular matrices $L_a$ and $L_s$. Similarly, if $K$ is any nonsingular lower triangular matrix, then

$$\tilde{\boldsymbol{b}} = (\tilde{b}_1 \quad \tilde{b}_3 \quad \ldots \quad \tilde{b}_{2n-1})^T = K\boldsymbol{b}$$

defines a complete and mutually independent set of blur invariants.

Can we derive a recurrence defining $\tilde{\boldsymbol{b}}$ in terms of moments

$$\pi_j = \int p_j(x) f(x) dx$$

with respect to the new polynomial basis? We note that

$$\boldsymbol{\pi}_a = L_a \boldsymbol{\mu}_a \quad \text{and} \quad \boldsymbol{\pi}_s = L_s \boldsymbol{\mu}_s$$

where, again, we denoted $\boldsymbol{\pi}_a = (\pi_1 \quad \cdots \quad \pi_{2n-1})^T$ and $\boldsymbol{\pi}_s = (\pi_0 \quad \cdots \quad \pi_{2n-2})^T$. Substituting into (4), we obtain

$$\tilde{\boldsymbol{b}} = \tilde{K}\boldsymbol{\pi}_a - \frac{1}{\mu_0}\tilde{B}(\tilde{\boldsymbol{b}})\boldsymbol{\pi}_s \tag{6}$$

and also

$$\tilde{\boldsymbol{b}} = \tilde{K}\boldsymbol{\pi}_a - \frac{1}{\mu_0}\tilde{C}(\boldsymbol{\pi}_s)\tilde{\boldsymbol{b}} \tag{7}$$

where $\tilde{K} = KL_a^{-1}$, $\tilde{B}(\tilde{\boldsymbol{b}}) = KB(K^{-1}\tilde{\boldsymbol{b}})L_s^{-1}$, and $\tilde{C}(\boldsymbol{\pi}_s) = KC(L_s^{-1}\boldsymbol{\pi}_s)K^{-1}$.

Thus, to obtain $\tilde{B}(\tilde{\boldsymbol{b}})$ from $B(\boldsymbol{b})$, we, besides the premultiplication and postmultiplication by the transform matrices, replace each $b_{2j-1}$ in $B(\boldsymbol{b})$ by a linear combination of $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_{2j-1}$, as found in the $j$th row of $K^{-1}\tilde{\boldsymbol{b}}$. Similar procedure applies to matrix $\tilde{C}(\boldsymbol{\pi}_s)$.

Before turning to orthogonal moments we make some general observations.

1) The lower triangular shape of $B$ and $C$ is preserved in $\tilde{B}$ and $\tilde{C}$. In fact, $C$ and $\tilde{C}$ are strictly lower triangular.
2) Both (6) and (7) are thus recurrences defining the same invariants $\tilde{\boldsymbol{b}}$.
3) System (7) appears more transparent and useful as it provides the invariants as the solution, by forward substitution, of the linear system,
$$\left(I + \frac{1}{\mu_0}\tilde{C}(\boldsymbol{\pi}_s)\right)\tilde{\boldsymbol{b}} = \tilde{K}\boldsymbol{\pi}_a \tag{8}$$
with system matrix $I + (1/\mu_0)\tilde{C}(\boldsymbol{\pi}_s)$, the numerical condition of which can be assessed for a given set of modified moments.
4) For simplicity and without loss of generality, we will restrict ourselves to unit lower triangular matrices $L_a$, $L_s$, and $K$. The diagonal elements represent just a scaling of the new polynomials/invariants. Then, always $\pi_0 = \mu_0$ and we can interchange $\mu_0$ with $\pi_0$ in the equations. Special cases of such matrices $L_a$ and $L_s$ generating orthogonal polynomials are shown in (10).
5) The choice of matrix $K$ influences the form of the blur invariants $\tilde{\boldsymbol{b}}$. Any nonsingular lower triangular $K$ generates some invariants, but apparently, some choices are more convenient than others. The most natural choice is $K = L_a$ so that $\tilde{K} = I$ and the recurrences (6) and (7) for the new invariants $\tilde{\boldsymbol{b}}$ resemble those for the original invariants $\boldsymbol{b}$. However, as we will see later, some other choices of $K$ may produce simpler invariants.

## IV. Orthogonal Polynomials

### A. Recurrence Relations for Orthogonal Polynomials

The main reason for replacing the invariants with respect to moments using standard powers by invariants with respect to modified moments using another polynomial basis is the numerical instability inherent in any calculation with powers of higher degree. A typical choice of a well-conditioned basis involves polynomials orthogonal on some domain with respect to certain weight function $w(x)$. To obtain polynomial basis satisfying the symmetry condition previously mentioned, it is sufficient and necessary to choose a weight function symmetric with respect to the origin (see [25, Theorem 4.3]).

Monic polynomials (i.e., those leading coefficients that are equal to one) orthogonal with respect to a symmetric weight function satisfy a three-term recurrence (see [25, Theorem 4.1]), i.e.,

$$p_{j+1}(x) = xp_j(x) - \beta_j p_{j-1}(x), \quad j = 1, 2, \ldots \tag{9}$$

with $p_0 = 1$ and $p_1 = x$. Here, constants $\beta_j$ must be positive, and they fully determine the orthogonal polynomials. Matrices $L_a$ and $L_s$ then depend only on constants $\beta_j$; here, we show them for size $3 \times 3$ as

$$L_a = \begin{bmatrix} 1 & 0 & 0 \\ -\beta_1 - \beta_2 & 1 & 0 \\ \beta_3\beta_1 + \beta_4(\beta_1 + \beta_2) & -\beta_1 - \beta_2 - \beta_3 - \beta_4 & 1 \end{bmatrix} \tag{10}$$

$$L_s = \begin{bmatrix} 1 & 0 & 0 \\ -\beta_1 & 1 & 0 \\ \beta_3\beta_1 & -\beta_1 - \beta_2 - \beta_3 & 1 \end{bmatrix}. \tag{11}$$

Using a formula-manipulating software (e.g., Maple), we can find matrix $\tilde{C}$ (or $\tilde{B}$) of any reasonable size and translate it into a form suitable for numerical evaluation. In the following is the case where $n = 3$ for any orthogonal polynomials and with $K = L_a$:

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 0 \\ 3\beta_1\pi_0 + 3\pi_2 & 0 & 0 \\ \tilde{C}_{3,1} & 10\beta_1\pi_0 + 10\pi_2 & 0 \end{bmatrix} \tag{12}$$

where

$$\tilde{C}_{3,1} = 3\left(4(\beta_1+\beta_2) - \beta_3 - \beta_4\right)\beta_1\pi_0 + \left(12(\beta_1+\beta_2) + 2\beta_3 - 3\beta_4\right)\pi_2 + 5\pi_4.$$

### B. Simplifying the New Invariants

When we choose a new polynomial basis (matrices $L_a$ and $L_s$, which also determine matrix $C(L_s^{-1}\boldsymbol{\pi}_s)$), then any choice of matrix $K$ will give a set of blur invariants as a solution of (7). We already mentioned that the natural choice is $K = L_a$. However, this choice (and most of other "random" choices of $K$) leads to invariants containing useless terms. This means that the invariants of higher orders include terms consisting of lower order invariants. Such terms are completely useless—they cannot contribute to discrimination power, and they increase not only computing complexity but also correlation between the invariants. Discarding such terms does not affect the invariance property and is highly desirable. (Note that the blur invariants published recently in [1] and [2] contain such useless terms.) An important question arises: What is the "optimal" (or at least "good") choice of $K$ with respect to the number of terms in invariants $\tilde{b}_1, \tilde{b}_3, \tilde{b}_5, \ldots$?

To illustrate what we are talking about, let us start with $K = L_a$. Then, the first invariant is simply $\tilde{b}_1 = \pi_1$. For the next one

$$\tilde{b}_3 = -\frac{3\pi_1\beta_1\pi_0 + 3\pi_1\pi_2 - \pi_3\pi_0}{\pi_0} = \pi_3 - 3\frac{\pi_1\pi_2}{\pi_0} - 3\beta_1\pi_1.$$

The last term is just a multiple of $\tilde{b}_1$ and should be omitted. The simplified form of this invariant (we denote the simplified invariants as $\hat{\boldsymbol{b}}$) is

$$\hat{b}_3 = \pi_3 - 3\frac{\pi_1\pi_2}{\pi_0}. \tag{13}$$

(Note that $\hat{b}_3$ does not depend on $\beta$; thus, it is the same for any set of orthogonal moments.)

Simplification of $\tilde{b}_{2j-1}$ for $j > 2$ is not so obvious. Looking at recurrence (6), we observe that the first column of $\tilde{B}$ is always multiplied, and also divided, by $\pi_0$. Thus, this column generates terms with isolated lower order invariants not being multiplied by any moments. This makes it a candidate for bringing in useless terms. The first column of $\tilde{B}$ can be expressed as

$$\tilde{B}(\cdot,1) = S\tilde{\boldsymbol{b}}$$

where $S$ is a strictly lower triangular matrix (displayed here for size $3 \times 3$), i.e.,

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 3\beta_1 & 0 & 0 \\ 3(4\beta_1 + 4\beta_2 - \beta_3 - \beta_4)\beta_1 & 10\beta_1 & 0 \end{bmatrix}.$$

Note that the $S_{2,1}$ element exactly represents what we omitted from $\tilde{b}_3$.

Now, we can observe (using, for instance, formal calculations in Maple) that choosing $K = (I+S)L_a$ for this particular $S$ leads to invariants $\hat{\boldsymbol{b}}$ with significantly less number of terms. We tested this up to the order nine, and the saving against $\tilde{\boldsymbol{b}}$ was always about 30%. Hence, we consider this choice a very good one and worth recommending. (Although we did not formally prove that, with this particular $S$, $K = (I+S)L_a$ minimizes the number of terms, it is highly probable. We tested several other choices of $S$ and $K$, but they never yielded such a big saving.)

The recurrence analogous to (7) for these simpler invariants is

$$\hat{\boldsymbol{b}} = (I+S)\boldsymbol{\pi}_a - \frac{1}{\mu_0}\hat{C}(\boldsymbol{\pi}_s)\hat{\boldsymbol{b}}$$

where

$$\hat{C}(\boldsymbol{\pi}_s) = (I+S)\tilde{C}(\boldsymbol{\pi}_s)(I+S)^{-1}.$$

However, $S$ has the remarkable property that $S$ and $\tilde{C}(\boldsymbol{\pi}_s)$ commute. Then, $I + S$ and $\tilde{C}(\boldsymbol{\pi}_s)$ also commute and, as a consequence

$$\hat{C}(\boldsymbol{\pi}_s) = \tilde{C}(\boldsymbol{\pi}_s).$$

This implies that to obtain the simpler invariants, we use in (8) the same system matrix $I + (1/\pi_0)\tilde{C}(\boldsymbol{\pi}_s)$ and only change the right-hand side from $\boldsymbol{\pi}_a$ to $(I+S)\boldsymbol{\pi}_a$. Now, we can also see the relationship between the simplified invariants $\hat{\boldsymbol{b}}$ and the "natural" ones (obtained by choosing $K = L_a$): $\hat{\boldsymbol{b}} = \tilde{\boldsymbol{b}} + S\tilde{\boldsymbol{b}}$.

### C. Legendre Polynomials

Here, we present blur invariants for a particular case of Legendre polynomials, which were also used in [1], [20], and [23]. Legendre polynomials are orthogonal on interval $[-1, 1]$, with a constant weight function. The coefficients in their defining recurrence are

$$\beta_j = \frac{j^2}{4j^2 - 1}.$$

The matrix $\tilde{C}(\boldsymbol{\pi}_s)$ defining blur invariants in terms of Legendre moments is at the bottom of the page.

$$\tilde{C}(\boldsymbol{\pi}_s) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \pi_0 + 3\pi_2 & 0 & 0 & 0 \\ \frac{17}{9}\pi_0 + \frac{146}{21}\pi_2 + 5\pi_4 & \frac{10}{3}\pi_0 + 10\pi_2 & 0 & 0 \\ \frac{2923}{715}\pi_0 + \frac{2515}{143}\pi_2 + \frac{3213}{143}\pi_4 + 7\pi_6 & \frac{1099}{117}\pi_0 + \frac{1450}{39}\pi_2 + 35\pi_4 & 7\pi_0 + 21\pi_2 & 0 \end{bmatrix}$$

The matrix needed to premultiply the right-hand side to obtain the simpler form of the invariants is

$$I+S=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{17}{9} & \frac{10}{3} & 1 & 0 \\ \frac{2923}{715} & \frac{1099}{117} & 7 & 1 \end{bmatrix}.$$

The "natural" invariants (with $K=L_a$) are

$$\begin{aligned}
\tilde{b}_3 &= -\pi_1+\pi_3-3\frac{\pi_1\pi_2}{\pi_0}\\
\tilde{b}_5 &= \frac{13}{9}\pi_1+\pi_5-\frac{10}{3}\pi_3\\
&\quad+\left(\left(\frac{274}{21}\pi_2-5\pi_4\right)\pi_1-10\pi_2\pi_3\right){\pi_0}^{-1}\\
&\quad+30\frac{\pi_1{\pi_2}^2}{{\pi_0}^2},\\
\tilde{b}_7 &= \pi_7+\frac{1631}{117}\pi_3-7\pi_5-\frac{793}{165}\pi_1\\
&\quad+\left(\left(-7\pi_6+\frac{6797}{143}\pi_4-\frac{10567}{143}\pi_2\right)\pi_1\right.\\
&\qquad\left.+\left(\frac{4010}{39}\pi_3-21\pi_5\right)\pi_2-35\pi_4\pi_3\right){\pi_0}^{-1}\\
&\quad+\left(\left(210\pi_4\pi_2-\frac{4842}{13}{\pi_2}^2\right)\pi_1+210{\pi_2}^2\pi_3\right){\pi_0}^{-2}\\
&\quad-630\frac{\pi_1{\pi_2}^3}{{\pi_0}^3}
\end{aligned}$$

whereas the simplified invariants are

$$\begin{aligned}
\hat{b}_3 &= \pi_3-3\frac{\pi_1\pi_2}{\pi_0}\\
\hat{b}_5 &= \pi_5+\left(\left(\frac{64}{21}\pi_2-5\pi_4\right)\pi_1-10\pi_2\pi_3\right){\pi_0}^{-1}\\
&\quad+30\frac{\pi_1{\pi_2}^2}{{\pi_0}^2}\\
\hat{b}_7 &= \pi_7+\left(\left(\frac{1792}{143}\pi_4-7\pi_6-\frac{1536}{143}\pi_2\right)\pi_1\right.\\
&\qquad\left.+\left(-21\pi_5+\frac{1280}{39}\pi_3\right)\pi_2-35\pi_4\pi_3\right){\pi_0}^{-1}\\
&\quad+\left(\left(-\frac{2112}{13}{\pi_2}^2+210\pi_4\pi_2\right)\pi_1+210{\pi_2}^2\pi_3\right)\\
&\quad\times{\pi_0}^{-2}-630\frac{\pi_1{\pi_2}^3}{{\pi_0}^3}.
\end{aligned}$$

### *D. Practical Consequences of the Simplification*

Apart from a clear theoretical result—simplifying the explicit formulas—discarding the dependent useless terms, as described in the two previous sections, has also a practical impact. Although each system is algebraically independent, one may expect that removing unnecessary terms should decrease the correlation between individual invariants. We tried to verify this assumption experimentally. We took 40 audio signals of length 80 samples, calculated both "original" and "simplified" invariants, and then estimated correlation between individual invariants. The sample correlation between the invariants are summarized in Table I. The values for the simplified invariants are above the diagonal, whereas the values for the original invariants are below the diagonal. One can observe that the correlation between the simplified invariants is always less than or equal to the correlation between the original invariants and significantly less in some cases. Hence, simplification also decorelates the blur invariants, which is a desirable property in practice. However, the correlation values depend on the signals, and the previous observation cannot be absolutely generalized—one can create artificial examples where this correlation decrease is 1 and counterexamples where it is 0. We repeated this correlation measurement many times with various audio and other signals. The results were mostly similar to those presented in Table I—some invariants were decorrelated, whereas correlation between the others changed only slightly.

TABLE I
CORRELATION OF THE SIMPLIFIED INVARIANTS (ABOVE THE DIAGONAL) AND THE ORIGINAL INVARIANTS (BELOW THE DIAGONAL)

| | $b_1$ | $b_3$ | $b_5$ | $b_7$ |
|---|---|---|---|---|
| $b_1$ | 1 | 0.83 | -0.66 | 0.84 |
| $b_3$ | -0.96 | 1 | -0.16 | 0.43 |
| $b_5$ | 0.88 | -0.97 | 1 | -0.83 |
| $b_7$ | -0.85 | 0.94 | -0.99 | 1 |

We also tested robustness of both systems to noise because one might expect that, due to error accumulation, the original invariants are more vulnerable than the simplified ones. Although this effect is observable in some cases, no statistically significant differences were found.

## V. CONCLUSION

We have presented a general method on how to derive moment invariants to image blurring from an arbitrary kind of moments when knowing them in terms of one particular basis. We have proven that if we want to derive invariants from moments of a new type, there is no need to construct them "from scratch" as other authors did. We have shown that there exist simple one-to-one transformations between any two polynomial bases and, consequently, between any two systems of blur invariants. Due to this, the whole process is much simpler and correct. The invariants presented in [1], [2], [20], and [23] can be derived as particular examples of our general approach. Moreover, we showed how to avoid the useless dependent terms that provide us with computationally more efficient and less correlated invariants. This issue has been totally ignored so far.

We also wish to provide the readers with several general comments and recommendations on how to use orthogonal moments (and the respective invariants) in numerical applications. Most of them are not restricted just to blur invariants. Apparently, the researchers using orthogonal moments in practice are quite often not familiar with their proper implementation.

1) The choice of the domain is critical. While orthogonal polynomials are perfectly conditioned on the interval of orthogonality, they are useless (and even worse than the standard powers) outside this interval. Hence, the whole image domain must be mapped into the area of orthogonality.

2) Matrices $L_a$ and $L_s$, which express the orthogonal polynomials in terms of powers, are badly conditioned, getting worse with increasing size $n$. They are, therefore, only of theoretical interest, useful in deriving relations and final formulas but must be avoided in actual calculations.
3) Therefore, modified moments $\boldsymbol{\pi}$ must not be calculated from the geometric moments but directly using recurrent relations and other properties of orthogonal polynomials [25].

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**Jaroslav Kautsky** received the M.Sc. degree in mathematics from Charles University, Prague, Czech Republic, in 1957 and the Ph.D. degree in mathematics in 1961 from the Czechoslovak Academy of Sciences, Prague, where he held research positions in the Institute of Mathematics until 1968.

Since 1966, he has taught mathematics in Australia, first at the University of Adelaide, Adelaide, SA, Australia, from 1966 to 1969 and then at the Flinders University of South Australia, Adelaide, SA, where he has been an Honorary Research Fellow since retiring in 1987. For over 15 years, he has been a Regular Visiting Scientist in the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague, where he participated in research on invariants and wavelets. He is also a Principal Scientist at Prometheus Inc., Newport, RI. He has published about 60 papers in numerical analysis, particularly on integration, orthogonal polynomials, in control theory, numerical linear algebra, wavelets, and signal and image processing.



**Jan Flusser** (M'93–SM'03) received the M.Sc. degree in mathematical engineering from the Czech Technical University, Prague, Czech Republic, in 1985, the Ph.D. degree in computer science from the Czechoslovak Academy of Sciences, Prague, in 1990, and the D.Sc. degree in technical cybernetics in 2001.

Since 1985, he has been with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Prague. From 1995 to 2007, he was the Head the of the Department of Image Processing. Since 2007, he has been the Director of the Institute. He is a Full Professor of computer science with Czech Technical University, Prague, and with Charles University, Prague, where he gives undergraduate and graduate courses on digital image processing, pattern recognition, and moment invariants and wavelets. He has authored or coauthored more than 150 research publications in these areas, including the monograph *Moments and Moment Invariants in Pattern Recognition* (Wiley, 2009), tutorials, and invited/keynote talks at major international conferences. His research interest covers moments and moment invariants, image registration, image fusion, multichannel blind deconvolution, and superresolution imaging.