Chapter 2
On Support of Imperfect Bayesian Participants

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Abstract. Bayesian decision theory provides a strong theoretical basis for a single-participant decision making under uncertainty, that can be extended to multiple-participant decision making. However, this theory (similarly as others) assumes unlimited abilities of a participant to probabilistically model the participant's environment and to optimise its decision-making strategy. The proposed methodology solves knowledge and preference elicitation, as well as sharing of individual, possibly fragmental, knowledge and preferences among imperfect participants. The approach helps to overcome the non-realistic assumption on participants' unlimited abilities.

2.1 Introduction

Dynamic decision making (DM) under uncertainty concerns a dynamic interaction of a decision maker (participant) with its environment, a part of the World. During the interaction the participant selects among available actions while aiming to reach own DM goals expressing its DM preferences. The interacting participants may cooperate or compete to achieve their personal DM goals or may be engaged in a collaborative DM, i.e. may have an additional common DM goal. The solution of decentralised DM relies on the participant considering future behaviour of its neighbours [10]. This requires modelling knowledge and preferences of the neighbours that cannot be performed by participant's limited capabilities.

Unlike many other approaches to decision making, Bayesian decision theory with its solid axiomatic basis proposes a systematic treatment of the considered DM

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problem [23]. When the description of the environment and the participant's preferences are specified in probabilistic terms, the optimal strategy can be explicitly found [15, 30]. The assumption on availability of complete probabilistic description is, however, quite restrictive as the participant operates with its DM preferences expressed in domain-specific terms and (at most) with a part of domain-specific knowledge rising from its interaction with the environment. The limited cognitive and computational resources of the participant prevent it to make the proper inferences from this limited and uncertain knowledge and to transfer it into the relevant probabilistic descriptions. This needs an algorithmic solution of *knowledge elicitation* and *preference elicitation* problems.

Many of the proposed knowledge elicitation methods heavily depend on the quality of DM experts, see for instance [6]. Theoretical and algorithmic support of the preference elicitation still remains to be a fundamental problem and no efficient and feasible solution has been proposed. A promising exception [5] treats the preference elicitation as an independent DM problem optimising elicitation effort or time with respect to a gain in decision quality yielded by the elicited preferences.

Another hard problem within cooperative distributed DM is *sharing* individual fragmental[1] knowledge and preferences among other imperfect participants. The main challenge here is to improve DM quality of individual selfish participants, while respecting their limited cognitive, computational and interacting abilities.

Chapter proposes a unified approach to knowledge and preference elicitation as well as sharing. The approach recognises typical subtasks arisen within the mentioned problems, formulates them as the independent supporting DM tasks and solves them via a fully probabilistic design [15, 17]. The provided solutions do not force the selfish imperfect participants to increase complexity of their models of environment or preferences while allowing to handle partially incompatible, fragmental knowledge and preferences.

The methodology respects the participant's selfishness by preserving the participant's formulation of the DM task and allowing the participant to follow its DM preferences. The participant's imperfection is also unchallenged by restricting its interaction to a small number of neighbours directly influencing its environment, and by not requiring a detailed modelling of its neighbours.

The proposed solution considers a sort of passive cooperation even for a non-collaborative selfish DM, that can be implemented as follows: i) the interacting participants offer their probabilistic models or their parts to the neighbours; ii) a feasible and implementable algorithm merges these models and projects the resulted compromise back to domains of the respective participants; iii) the back-projected compromise[2] can be exploited by the participants in order to efficiently reach their selfish DM goals. The projected compromise represents an additional information source, which does not force a participant to increase its load.

[1] A group of participants may interact on an intersection of their individual behaviour. Thus each participant has only fragmental knowledge about the whole behaviour of the group and may not be aware of the complete behaviour of any its neighbour.

[2] The exploitation of the merged and back-projected information is out of the chapter's scope.

Chapter layout is as follows. Section 2.2 summarises a formalisation of DM under uncertainty. The adopted normative Bayesian DM is recalled in Section 2.3. Section 2.4 provides an essence of the *fully probabilistic design* of decision-making strategies that densely extends the standard DM design. Section 2.5 describes, formulates and solves the typical supporting DM tasks. Section 2.6 outlines the solutions of elicitation and merging tasks. Section 2.7 summarises open problems.

In description of the underlying theory, capital roman letters denote random variables as well as their realisations. The following common notation are used:

$t, \tau \in \mathbb{N}$ is a discrete time
$\boldsymbol{A}$ denotes a set of As
A_t is a random variable at time t
A^t is a finite sequence $(A_\tau)_{\tau=0}^{t}$, $A_\tau \in \boldsymbol{A}_\tau$ and $t \in \mathbb{N}$
A_n^m is a finite sequence $(A_\tau)_{\tau=n}^{m}$, $A_\tau \in \boldsymbol{A}_\tau$, $n, m \in \mathbb{N}$, $n \leq m$
mappings are distinguished by san serif font, for instance $\mathsf{S}(A)$.

Since Section 2.5 intertwined supporting and original DM problems are addressed. Capital letters denote variables and mappings related to the supporting DM and small letters are used for the original DM.

2.2 Dynamic Decision Making under Uncertainty

This section recalls a formalisation of dynamic DM. It exploits and unifies results presented in [7, 12, 17], introduces notation and provides a theoretical basis of the proposed support of imperfect participants described in Section 2.5 and exploited in Section 2.6.

Dynamic DM under uncertainty deals with a dynamic interaction of a *participant* (decision maker) with its *environment* (World part of the participant's interest) that aims to reach participant's DM goal. *DM goal* expresses the participant's preferences with respect to the behaviour $B \in \boldsymbol{B}$ of the closed decision-making loop formed of the participant and its environment.

DM considers a sequence of participant's actions with respect to the participant's environment. The actions are not independent and the state of the environment changes either due to always present development of the environment[3] or/and as a reaction on the participant's actions. The knowledge available to the participant for selecting an action includes: the knowledge gained from the environment (*observations*); the knowledge associated with the participant's *past actions* (generated by the participant's decision-making strategy) and the knowledge considered by the participant (*prior knowledge* of the environment). Always limited cognitive, computational and acting resources of the participant are considered as the *participant's imperfectness*.

[3] The environment's behaviour reflects some inherent laws of the environment that are unknown (or incompletely known) to the participant.

A closed-loop nature of interaction and information exchange between the participant and its environment has allowed to use the notion "closed-loop description", which models a coupling of the decision-making strategy and the environment forming a *closed (DM) loop*. Its *behaviour* $B \in \boldsymbol{B} \neq \emptyset$ is identified with observations Y, internals X, actions A and prior knowledge K_0 as follows

$$B = (Y, X, A, K_0) = (\text{observations}, \text{internals}, \text{actions}, \text{prior knowledge}) \in \boldsymbol{B}. \quad (2.1)$$

Here it is assumed that observations made on the environment are available to the participant. Besides, the closed-loop nature of information exchange considers both the case when an action affects the environment and the case when it does not, i.e. when actions applied have no influence on the future behaviour of the environment. The tasks of this type arise when the DM goal is to describe or predict the environment based on observations made[4].

Note, the "closed-loop description" should not be confused with the "closed-loop control system". The last assumes the controller (decision maker) observes the system (environment) and adjusts the control action (decision) to obtain the desired system's behaviour (environment's behaviour) while an opposite open loop notion considers no observations are available to design the action.

DM consists of the selection (also known as *DM design*) and of the application of a *DM strategy*, i.e. a sequence of mappings $\mathsf{S} \equiv (\mathsf{S}_t)_{t \in \boldsymbol{t}} \in \boldsymbol{\mathsf{S}}$ formed of *decision rules* S_t, $t \in \boldsymbol{t}$, where $t \in \boldsymbol{t} \equiv \{1, \ldots, h\}$, $h < \infty$ is a given *decision horizon*. Each strategy maps a *knowledge* sequence $K \equiv (K_{t-1})_{t \in \boldsymbol{t}}$ on an *action* sequence $A \equiv (A_t)_{t \in \boldsymbol{t}} \in \boldsymbol{A} \neq \emptyset$. Actions describe or influence the participant's environment. The processed knowledge sequence $K \in \boldsymbol{K}$ is assumed to be non-shrinking, i.e. the knowledge K^{t-1} is extended by *observations* $Y = (Y_t)_{t \in \boldsymbol{t}}$, $Y_t \in \boldsymbol{Y_t} \equiv \boldsymbol{K}^t \setminus (\boldsymbol{K}^{t-1} \cup \boldsymbol{A}_t)$. Thus, the knowledge K^{t-1} available for choosing the action A_t at time t is

$$K^{t-1} = (Y_1^{t-1}, A_1^{t-1}, K_0) = (\underbrace{(\text{observations, actions})}_{\text{from 1 until } t-1}, \text{prior knowledge}) \quad (2.2)$$

with K_0 denoting *prior knowledge*. Yet unmade observations and actions form a part of *ignorance* $I \equiv (I_t^h)_{t \in \boldsymbol{t}} \in \boldsymbol{I}$, which shrinks with time. Generally, ignorance also contains – considered but never observed –*internals* $X = (X_t)_{t=0}^h \in \boldsymbol{X}$, i.e.

$$I_t^h = (Y_t^h, X_0^h, A_{t+1}^h) = (\text{observations}, \text{internals}, \text{actions}). \quad (2.3)$$

Definition 2.1 (Admissible strategies). A set $\boldsymbol{\mathsf{S}}$ of *admissible strategies* is formed by sequences of *decision rules* $(\mathsf{S}_t(B))_{t \in \boldsymbol{t}}$ that are causal with respect to the available knowledge, i.e. $\mathsf{S}_t(B) = \mathsf{S}_t(K^{t-1}) \in \boldsymbol{A}_t$.

[4] This important class of DM tasks is widespread in many areas ranging from finance to medical applications, transportation, etc.

The strategy S is selected from a set of *compared strategies* $\mathbf{S}_\star \neq \emptyset$, which is usually a proper subset $\mathbf{S}_\star \subsetneq \mathbf{S}$ of the admissible strategies. The set of compared strategies is given by a detailed specification of the DM task solved. For instance, the strategies can have a prescribed form, complexity or can rely on a common description of environment.

2.3 Bayesian DM

Bayesian DM is based on the idea that the participant's choice among possible DM strategies indirectly reflects the participant's preference ordering of the closed-loop behaviour. The Savage's utility theory [23] allows a representation of this preference ordering by an expected utility. Thus, the optimal DM strategy ${}^{opt}\mathsf{S}$ can be found as follows

$$ {}^{opt}\mathsf{S} \in \operatorname{Arg} \min_{\mathsf{S} \in \mathbf{S}_\star} \int_{\boldsymbol{B}} \mathsf{U}_\mathsf{S}(B)\, \mu_\mathsf{S}(\mathrm{d}B), \tag{2.4} $$

where $\mathsf{U}_\mathsf{S}(B)$ is a real-valued utility function defined on $\boldsymbol{B}$ and $\mu_\mathsf{S}(B)$ is a countably additive probability measure on $\boldsymbol{B}$. Assuming that $\mu_\mathsf{S}(B)$, $\mathsf{S} \in \mathbf{S}_\star$, are absolutely continuous[5] with respect to a measure $\nu(\mathrm{d}B)$ operating on the same space $\boldsymbol{B}$, then each $\mu_\mathsf{S}(B)$ has a density, so-called *Radon-Nikodým derivative* (rnd) $\mathsf{F}_\mathsf{S}(B)$ with respect to $\mu_\mathsf{S}(B)$, i.e.

$$ \begin{aligned} &\mu_\mathsf{S}(\mathrm{d}B) = \mathsf{F}_\mathsf{S}(B)\nu(\mathrm{d}B) \quad \nu\textit{-almost everywhere} \\ &\mathsf{F}_\mathsf{S}(B) \geq 0, \quad \int_{\boldsymbol{B}} \mathsf{F}_\mathsf{S}(B)\,\nu(\mathrm{d}B) = 1. \end{aligned} \tag{2.5} $$

The rnd $\mathsf{F}_\mathsf{S}(B)$ defined by (2.5) can be interpreted as the *closed-loop model* describing an interaction of the participant's DM strategy S and the environment. The optimal DM strategy (2.4) then reads

$$ {}^{opt}\mathsf{S} \in \operatorname{Arg} \min_{\mathsf{S} \in \mathbf{S}_\star} \int_{\boldsymbol{B}} \mathsf{U}_\mathsf{S}(B)\, \mathsf{F}_\mathsf{S}(B)\nu(\mathrm{d}B). \tag{2.6} $$

Note that the participant actually defines a description of the optimal closed loop by selecting the optimal DM strategy.

2.4 Fully Probabilistic Design

Here an essence of Fully Probabilistic Design (FPD) of decision-making strategies is briefly outlined. For the detailed treatment, see, for instance [9, 11, 15, 14]. Its specification exploits the following notion.

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5 For any measurable subset $\boldsymbol{B}_\star \subset \boldsymbol{B}$ with $\nu(\boldsymbol{B}_\star) = 0$ also $\mu_\mathsf{S}(\boldsymbol{B}_\star) = 0$, $\forall \mathsf{S} \in \mathbf{S}_\star$.

Definition 2.2 (Ideal closed-loop model). *Ideal closed-loop model* ${}^{i}\mathsf{F}(B)$ is a closed-loop model (2.5) describing an interaction "participant-environment", when the participant's DM strategy is optimal ${}^{opt}\mathsf{S}$, (2.6), for the treated DM task.

The DM-design goal is thus to make a closed-loop behaviour close to the desired one, i.e. to the behaviour described by the ideal closed-loop model.

Let us consider the utility function in the form

$$\mathsf{U}_{\mathsf{S}}(B) = \ln\left(\frac{\mathsf{F}_{\mathsf{S}}(B)}{{}^{i}\mathsf{F}(B)}\right).$$

Then, by substituting it to (2.6), the optimised functional (2.6) becomes *Kullback-Leibler divergence* (KLD), [19], $\mathsf{D}\left(\mathsf{F}_{\mathsf{S}}||{}^{i}\mathsf{F}\right)$, of the closed-loop model $\mathsf{F}_{\mathsf{S}}(B)$ on the ideal closed-loop model ${}^{i}\mathsf{F}(B)$, i.e.

$$\int_{\boldsymbol{B}} \mathsf{F}_{\mathsf{S}}(B) \ln\left(\frac{\mathsf{F}_{\mathsf{S}}(B)}{{}^{i}\mathsf{F}(B)}\right) \nu(\mathrm{d}B) \equiv \mathsf{D}\left(\mathsf{F}_{\mathsf{S}}||{}^{i}\mathsf{F}\right). \tag{2.7}$$

KLD (2.7) of a pair of rnds H, F on $\boldsymbol{B}$ has the following properties, see [19],

$$\begin{aligned} &\mathsf{D}(\mathsf{H}||\mathsf{F}) \geq 0,\ \mathsf{D}(\mathsf{H}||\mathsf{F}) = 0 \quad \text{iff} \quad \mathsf{H} = \mathsf{F}\ \nu - \text{almost everywhere} \\ &\mathsf{D}(\mathsf{H}||\mathsf{F}) = \infty \text{ iff } \mathsf{H} \text{ is not absolutely continuous with respect to } \mathsf{F}. \end{aligned} \tag{2.8}$$

Definition 2.3 (Fully probabilistic design). The FPD searches a DM strategy via minimising the Kullback-Leibler divergence (2.7) of the closed-loop model F_{S} describing "participant-environment" behaviour to the ideal closed-loop model ${}^{i}\mathsf{F}$ determined by Definition 2.2. The optimal DM strategy ${}^{opt}\mathsf{S}$ reads

$${}^{opt}\mathsf{S} \in \operatorname{Arg}\min_{\mathsf{S}\in\boldsymbol{\mathsf{S}}_{\star}} \mathsf{D}\left(\mathsf{F}_{\mathsf{S}}||{}^{i}\mathsf{F}\right). \tag{2.9}$$

The formalised justification of the FPD can be found in [17].

The key features of the FPD approach are: i) a closed-loop behaviour and the preferred behaviour are probabilistically described; ii) the existence of the explicit minimiser in the stochastic dynamic programming significantly simplifies the optimisation; iii) FPD can approximate any standard Bayesian DM arbitrarily well; iv) some FPD formulations have no standard Bayesian counterpart.

The FPD approach also provides methodology for a feasible treatment of multiple-aim DM, as well as allows to efficiently solve an unsupervised cooperation of multiple participants including sharing non-probabilistic and probabilistic knowledge and preferences among participants, see Section 2.6.

2.4.1 DM Elements in FPD

This section introduces so-called DM elements, which are rnds processed by the FPD. The DM elements serve for the specification of the closed-loop model $\mathsf{F}_{\mathsf{S}}(B)$

and its ideal counterpart ${}^{i}\mathsf{F}(B)$, see Definition 2.2. These rnds act on a non-empty behaviour set $\boldsymbol{B}$. A behaviour $B \in \boldsymbol{B}$ describes an interaction of the participant and its environment and is structured as in (2.1). Let us stress that the internals $X \in \boldsymbol{X}$ can be influenced by the applied actions.

The chain rule applied to $\mathsf{F}_{\mathsf{S}}(B)$ yields the following factorisation of the closed-loop model

$$\mathsf{F}_{\mathsf{S}}(B) = \mathsf{F}_{\mathsf{S}}(X_0, K_0) \prod_{t \in \boldsymbol{t}} \mathsf{F}_{\mathsf{S}}(Y_t | A_t, X^t, K^{t-1}) \mathsf{F}_{\mathsf{S}}(X_t | A_t, X^{t-1}, K^{t-1}) \prod_{t \in \boldsymbol{t}} \mathsf{F}_{\mathsf{S}}(A_t | X^{t-1}, K^{t-1}). \tag{2.10}$$

It can be shown that a suitable definition of $X_t \in \boldsymbol{X}$ allows the simplification

$$\begin{aligned} \mathsf{F}_{\mathsf{S}}(Y_t | A_t, X^t, K^{t-1}) &= \mathsf{F}_{\mathsf{S}}(Y_t | A_t, X_t, K^{t-1}) \\ \mathsf{F}_{\mathsf{S}}(X_t | A_t, X^{t-1}, K^{t-1}) &= \mathsf{F}_{\mathsf{S}}(X_t | A_t, X_{t-1}, K^{t-1}) \end{aligned} \tag{2.11}$$

The following result is easy to check.

Lemma 2.1 (On strategy-independent models). *Let all compared strategies use a common factor* $\mathsf{F}_{\mathsf{S}}(X_0, K_0) = \mathsf{F}(X_0, K_0)$ *and common rnds (2.11). Let us consider the compared strategies* $S \in \mathsf{S}_\star \neq \emptyset$ *described by the identical model* $\mathsf{F}_{\mathsf{S}}(A_t | X^{t-1}, K^{t-1})$. *Then, all these strategies lead to the same closed-loop model (2.10) and need not be distinguished. This allows to introduce the simplified notation*

$$\mathsf{F}_{\mathsf{S}}(A_t | X^{t-1}, K^{t-1}) = \mathsf{S}(A_t | X^{t-1}, K^{t-1}),\ t \in \boldsymbol{t}. \tag{2.12}$$

Hereafter, the set of compared strategies meeting conditions of Lemma 2.1 are considered. Consequently, the subscript S of $\mathsf{F}_{\mathsf{S}}(X_0, K_0)$ and rnds (2.11) is dropped.
The following important conditions have been originally proposed within the control domain [22] and extended to decision making in [13]. These *natural conditions of DM* formalise that an admissible DM strategy cannot exploit the unknown realisations of internals, i.e.

$$\begin{aligned} \mathsf{S}(A_t | X^{t-1}, K^{t-1}) &= \mathsf{S}(A_t | K^{t-1}),\ \Leftrightarrow \\ \mathsf{F}(X_{t-1} | A_t, K^{t-1}) &= \mathsf{F}(X_{t-1} | K^{t-1}),\ t \in \boldsymbol{t}. \end{aligned} \tag{2.13}$$

Under (2.11), (2.12) and (2.13), the closed-loop model (2.10) gets the form

$$\mathsf{F}_{\mathsf{S}}(B) = \underbrace{\mathsf{F}(X_0 | K_0) \mathsf{F}(K_0)}_{\text{prior rnd}} \underbrace{\prod_{t \in \boldsymbol{t}} \mathsf{F}(Y_t | A_t, X_t, K^{t-1}) \mathsf{F}(X_t | A_t, X_{t-1}, K^{t-1})}_{\text{environment model}} \underbrace{\prod_{t \in \boldsymbol{t}} \mathsf{S}(A_t | K^{t-1})}_{\text{strategy } \mathsf{S}}.$$

The factors $\mathsf{S}(A_t | K^{t-1}),\ t \in \boldsymbol{t}$, model decision rules forming the strategy S. The closed decision loop given by the *ideal closed-loop model* ${}^{i}\mathsf{F}(B)$ (see Definition 2.2) can be factorised in a way similar to (2.14)

$$
{}^{i}\mathsf{F}_{\mathsf{S}}(B) = \overbrace{{}^{i}\mathsf{F}(X_0|K_0)\,{}^{i}\mathsf{F}(K_0)}^{\text{ideal prior rnd}} \tag{2.14}
$$
$$
\times \underbrace{\prod_{t\in\boldsymbol{t}} {}^{i}\mathsf{F}(Y_t|A_t,X_t,K^{t-1})\,{}^{i}\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1})}_{\text{ideal environment model}} \underbrace{\prod_{t\in\boldsymbol{t}} {}^{i}\mathsf{S}(A_t|X_{t-1},K^{t-1})}_{\text{ideal strategy } {}^{i}\mathsf{S}}.
$$

Note there is no reason to apply the natural conditions of DM (2.13) to the ideal DM strategy. On contrary, an explicit dependence on internals allow to respect incomplete knowledge of the participant's preferences regarding the behaviour B, see [14].

Definition 2.4 (DM elements). DM elements are rnds processed by the FPD and defined on the respective domains given by the decompositions (2.1) and (2.2). The DM elements consist of

observation model

$$
\mathsf{F}(Y_t|A_t,X_t,K^{t-1}), \tag{2.15}
$$

time evolution model of internals

$$
\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1}), \tag{2.16}
$$

prior rnd

$$
\mathsf{F}(X_0,K_0) = \mathsf{F}(X_0|K_0)\mathsf{F}(K_0), \text{ and} \tag{2.17}
$$

a set of the *compared strategies* $\mathsf{S}_\star \subset \mathsf{S}$, where S is a set of the admissible strategies,

ideal observation model

$$
{}^{i}\mathsf{F}(Y_t|A_t,X_t,K^{t-1}),
$$

ideal time evolution model of internals

$$
{}^{i}\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1}),
$$

ideal prior rnd

$$
{}^{i}\mathsf{F}(X_0,K_0) = {}^{i}\mathsf{F}(X_0|K_0)\,{}^{i}\mathsf{F}(K_0), \quad \text{and}
$$

ideal DM strategy

$$
{}^{i}\mathsf{S}(A_t|X_{t-1},K^{t-1}).
$$

The observation model (2.15) and the time evolution model of internals (2.16) determine *environment model* $\prod_{t\in\boldsymbol{t}} \mathsf{F}(Y_t|A_t,X_t,K^{t-1})\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1})$, see (2.14). The *ideal environmental model*, is defined in a similar way, see (2.14).

2.4.2 Solution of FPD

The presented solution of the general FPD shows how DM elements, specified by Definition 2.4, are used in this DM design. It also exemplifies that the minimisation in the FPD is done explicitly. The proof is skipped as only the simplest variant of the FPD, Lemma 2.2, is used in Section 2.5. It essentially coincides with the basic DM lemma of the standard Bayesian DM, which transforms a minimisation over randomised decision rules into a minimisation over actions generated by them, see for instance [13].

The solution of the FPD requires the solution of *stochastic filtering* problem, i.e. evaluation of the *posterior rnds* $\mathsf{F}(X_t|A_t,K^{t-1})$, $\mathsf{F}(X_t|K^t)$. Stochastic filtering is summarised in the following theorem implied by the elementary operations with rnds. The proof can be found, for instance, in [15].

Theorem 2.1 (Stochastic filtering). *Let the compared strategies use a common prior rnd (2.17) and a common environment model (2.14) and meet (2.13). Then, the following recursions, describing stochastic filtering, hold*

$$\text{Time updating}\quad \mathsf{F}(X_{t+1}|A_{t+1},K^t) = \int_{\boldsymbol{X}_t} \mathsf{F}(X_{t+1}|A_{t+1},X_t,K^t)\mathsf{F}(X_t|K^t)\nu(\mathrm{d}X_t)$$

$$\text{Data updating}\quad \mathsf{F}(X_t|K^t) = \frac{\mathsf{F}(Y_t|A_t,X_t,K^{t-1})\mathsf{F}(X_t|A_t,K^{t-1})}{\mathsf{F}(Y_t|A_t,K^{t-1})}.$$

The recursions are initiated by the prior rnd $\mathsf{F}(X_0|K_0)$ and depend on action realisations but not on the strategy generating them.

Theorem 2.2 (Solution of FPD). *Let there is a stabilising strategy $\underline{\mathsf{S}} \in \mathsf{S}_\star$ such that KLD $\mathsf{D}(\mathsf{F}_{\underline{\mathsf{S}}}||{}^i\mathsf{F}) < \infty$ and the compared strategies use a common prior rnd (2.17) and a common environment model (2.14) and meet (2.13). Then, the optimal strategy ${}^{opt}\mathsf{S}$ (2.9) minimising KLD $\mathsf{D}(\mathsf{F}_{\mathsf{S}}||{}^i\mathsf{F})$, (2.7), of the closed-loop model F_{S} (2.14) on its ideal counterpart ${}^i\mathsf{F}$ (2.14) consists of the following decision rules, $t \in \boldsymbol{t}$,*

$$\begin{aligned}
{}^{opt}\mathsf{S}(A_t|K^{t-1}) &= \frac{\exp[-\omega(A_t,K^{t-1})]}{\gamma(K^{t-1})}\\
\gamma(K^{t-1}) &= \int_{\boldsymbol{A}_t} \exp[-\omega(A_t,K^{t-1})]\nu(\mathrm{d}A_t)\\
\omega(A_t,K^{t-1}) &= \int_{\boldsymbol{Y}_t,\boldsymbol{X}_t,\boldsymbol{X}_{t-1}} \mathsf{F}(Y_t|A_t,X_t,K^{t-1})\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1})\mathsf{F}(X_{t-1}|K^{t-1})\times
\end{aligned}$$
$$\ln\left(\frac{\mathsf{F}(Y_t|A_t,X_t,K^{t-1})\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1})}{\gamma(K_t)\,{}^i\mathsf{F}(Y_t|A_t,X_t,K^{t-1})\,{}^i\mathsf{F}(X_t|A_t,X_{t-1},K^{t-1})\,{}^i\mathsf{F}(A_t|X_{t-1},K^{t-1})}\right)\nu(\mathrm{d}Y_t\,\mathrm{d}X_t\,\mathrm{d}X_{t-1}).$$

Starting with $\gamma(K_{h+1}) \equiv 1$, the functions $\omega(A_t,K^{t-1})$ are generated in the backward manner for $t = h, h-1, \ldots, 1$. The evaluations exploit the given DM elements, Definition 2.4, and rnds $\mathsf{F}(X_{t-1}|K^{t-1})$ resulting from stochastic filtering, Theorem 2.1.

Proof. The proof just unites the results [15] and [14] while explicitly adding the need for existence of the stabilising strategy $\underline{\mathsf{S}}$. □

The application of the FPD requires a complete specification of all factors forming the ideal closed-loop model. However in many cases, the participant does not care about some factors in (2.14) and leaves them to their fate.

Definition 2.5 (Leave-to-fate option). If there is no requirement on a factor in the decomposition (2.14), then it is *left to its fate*, i.e. the corresponding factor in the ideal closed-loop model (2.14) is taken to be equal to its non-ideal counterparts resulting from the DM design.

The factors left to their fate cancel in logarithm occurring in the definition of KLD (2.7). Consequently, the FPD reduces to the standard Bayesian design, if the strategy is left to its fate. The following lemma makes this property explicit in the simplest case of *static design*, which: i) selects the optimal action in one-shot without modelling time evolution of internals; ii) uses no observations (behaviour (2.1) includes no Y), and iii) selects a single decision rule forming the strategy. In this case, the ignorance I (2.3) coincides with internals X.

Lemma 2.2 (Static design: basic DM lemma). *Let the behaviour* $B = (X,A,K) =$ *(internals,action,knowledge) be modelled by* $\mathsf{F}_{\mathsf{S}}(B) = \mathsf{F}(X|A,K)\mathsf{S}(A|K)\mathsf{F}(K)$ *and let the strategy be left to its fate, Definition 2.5, i.e.*

$$^{i}\mathsf{F}(B) = {}^{i}\mathsf{F}(X|A,K)\mathsf{S}(A|K)\,{}^{i}\mathsf{F}(K).$$

Let within the set of compared strategies $\mathsf{S}_{\star}$ *there is stabilising strategy* $\underline{\mathsf{S}} \in \mathsf{S}_{\star}$ *for which KLD* $\mathsf{D}(\mathsf{F}_{\underline{\mathsf{S}}}||\,^{i}\mathsf{F}) < \infty$.

Then, the optimal strategy (2.9) minimising $\mathsf{D}(\mathsf{F}_{\mathsf{S}}||\,^{i}\mathsf{F})$ *is deterministic one and the* optimal action $^{opt}A = {}^{opt}A(K)$

$$^{opt}A \in \operatorname{Arg}\min_{A \in \boldsymbol{A}} \int_{\boldsymbol{X}} \mathsf{F}(X|A,K) \ln\left(\frac{\mathsf{F}(X|A,K)}{^{i}\mathsf{F}(X|A,K)}\right) \nu(\mathrm{d}X). \tag{2.18}$$

Thus, the optimal action is found as a minimiser of the conditional version (2.18) of KLD (2.7). It is conditioned on the optimised action A and knowledge K that is available for the action choice. The ideal prior rnd does not influence the design and can always be left to its fate, Definition 2.5, $^{i}\mathsf{F}(K) = \mathsf{F}(K)$.

Proof. The described deterministic strategy is admissible as $^{opt}A = {}^{opt}A(K) \in \boldsymbol{A}$, see (2.18). The definition of the minimum and independence of the expression $\ln\left(\frac{\mathsf{F}(K)}{^{i}\mathsf{F}(K)}\right) = \int_{\boldsymbol{X}} \mathsf{F}(X|A,K) \ln\left(\frac{\mathsf{F}(K)}{^{i}\mathsf{F}(K)}\right) \nu(\mathrm{d}X)$ of the action A, resulting from (2.5), imply that for any $A \in \boldsymbol{A}$

$$\begin{aligned} &\int_{\boldsymbol{X}} \mathsf{F}(X|\,^{opt}A,K) \ln\left(\frac{\mathsf{F}(X|\,^{opt}A,K)\mathsf{F}(K)}{^{i}\mathsf{F}(X|\,^{opt}A,K)\,^{i}\mathsf{F}(K)}\right) \nu(\mathrm{d}X) \\ \leq &\int_{\boldsymbol{X}} \mathsf{F}(X|A,K) \ln\left(\frac{\mathsf{F}(X|A,K)\mathsf{F}(K)}{^{i}\mathsf{F}(X|A,K)\,^{i}\mathsf{F}(K)}\right) \nu(\mathrm{d}X). \end{aligned}$$

Multiplying this inequality by an arbitrary strategy $\mathsf{S}(A|K) \geq 0$, integrating over $\boldsymbol{A}$ and using *Dirac delta* δ [29] for describing the deterministic strategy, we get

$$\int_{(\boldsymbol{A},\boldsymbol{X})} \mathsf{F}(X|A,K)\delta(A - {}^{opt}A(K)) \ln\left(\frac{\mathsf{F}(X|A,K)\mathsf{F}(K)}{{}^{i}\mathsf{F}(X|A,K)\,{}^{i}\mathsf{F}(K)}\right) \nu(\,\mathrm{d}(A,X))$$
$$\leq \int_{(\boldsymbol{A},\boldsymbol{X})} \mathsf{F}(X|A,K)\mathsf{S}(A|K) \ln\left(\frac{\mathsf{F}(X|A,K)\mathsf{F}(K)}{{}^{i}\mathsf{F}(X|A,K)\,{}^{i}\mathsf{F}(K)}\right) \nu(\,\mathrm{d}(A,X)).$$

Multiplication of this inequality by the rnd $\mathsf{F}(K) \geq 0$ and integration over $\boldsymbol{K}$ preserves it, while the left-hand side remains finite due to the assumed existence of $\underline{\mathsf{S}}$ making the inspected KLD for ${}^{opt}\mathsf{S}(A|K) = \delta(A - {}^{opt}A(K))$ finite. This, with the leave-to-fate option, Definition 2.5, $\mathsf{S}(A|K) = {}^{i}\mathsf{F}(A|K)$ demonstrates the claimed optimality. □

2.5 DM Tasks Supporting Imperfect Bayesian Participants

Many DM problems cannot be solved by imperfect individual participants working in isolation as they do not possess the necessary experience, information or resources. Such DM problems are successfully addressed by distributed solutions [24, 28]. Despite the evident positive effect of the distributed solutions, the lack of systematic support of multiple *imperfect* DM participants allowing them to cooperate and interact in complex, dynamic and uncertain environments has significantly restricted an efficient use of these solutions. The interdependencies between participants domains, the necessity of meeting individual and global constraints, as well as the participants' limited cognitive and computational abilities have indicated a strong need for the efficient algorithmic support of computational aspects of DM.

The needed support is mostly developing in the following interconnected directions: i) extending the solvable special cases dealing with linear systems, quadratic performance indices and Gaussian distributions, e.g., [1, 13, 21], or controlled Markov chains, e.g., [4]; ii) using various versions of approximate filtering like [20], and approximate dynamic programming, e.g. [27].

Still there is a significant gap between the needs of multiple imperfect participants and the available systematic support. The problems requiring primarily the support are: *knowledge and preferences elicitation* and *sharing of knowledge and preferences* among imperfect selfish participants. The section consider the typical tasks arising within these problems, formulates them as DM tasks and provides their solution via FPD. The use of the FPD relies on the ability to properly construct the DM elements: a detailed guide how to do that for the most common tasks from the participants' knowledge and preferences is given together with the solution.

As the considered DM tasks support decision making of multiple participants, they are called *supporting* DM tasks to distinguish them from the *original* DM task solved by the supported participant. The variables and DM elements related to the *supporting* task are denoted by capital letters, for instance, $B \in \boldsymbol{B}$ is a closed-loop behaviour of the supporting DM task, and $\mathsf{F_S}(B)$ is its closed-loop model. Variables and DM elements of the *original* DM task are denoted by small letters, for instance, $b \in \boldsymbol{b}$ is a closed-loop behaviour of the original DM task and the rnd $\mathsf{f}(b)$ denotes its closed-loop model[6]

$$\mathsf{f} \in \mathbf{f} \subset \mathfrak{F} = \left\{ \mathsf{f}(b) : \mathsf{f}(b) \geq 0, \int_{b \in \boldsymbol{b}} \mathsf{f}(b) \nu(\mathrm{d}b) = 1 \right\}. \tag{2.19}$$

A *finite cardinality* $|\boldsymbol{b}| < \infty$ of the behaviour set $\boldsymbol{b} = \{b_1, \ldots, b_{|\boldsymbol{b}|}\}$ is assumed. In this case, $\nu(\mathrm{d}b)$ is a counting measure and the rnd $\mathsf{f}(b)$ is a finite-dimensional vector belonging to the *probabilistic simplex* $\mathfrak{F}$ (2.19). General validity of the obtained results is conjectured.

The following types of supporting DM tasks met within distributed solutions are recognised: an approximation of a known rnd (Section 2.5.1); an approximation of an unknown rnd (Section 2.5.2) and a description of an unknown rnd based on available knowledge (Section 2.5.3). Section 2.6 illustrates how the provided solutions can further be employed to support a cooperative decision making.

2.5.1 Approximation of Known Rnd

Let us consider the closed-loop model $\mathsf{f}(b)$, $b \in \boldsymbol{b}$, derived from the available knowledge and/or preferences description. In reality, the constructed rnd $\mathsf{f}(b)$ can be intractable by an imperfect participant and needs to be approximated by a rnd $\hat{\mathsf{f}}(b) \in \hat{\mathbf{f}}$, where $\hat{\mathbf{f}}$ is a set of feasible rnds on $\boldsymbol{b}$

$$\hat{\mathsf{f}}(b) \in \hat{\mathbf{f}} \subset \mathfrak{F}, \text{ see (2.19)}. \tag{2.20}$$

The proposed approach formalises the considered *approximation task* as a static supporting DM problem solved by Lemma 2.2. Recall that in the static case the ignorance I coincides with internals X.

Definition 2.6 (Approximation of known rnd as supporting DM task). Approximation (2.20), formulated as a static supporting DM task in FPD sense, is characterised by the behaviour B, (2.1), structured as follows

$$B = (X, A, K) = (b, \hat{\mathsf{f}}(b), \mathsf{f}(b)), \text{ where} \tag{2.21}$$

the internals X, coinciding with the ignorance I (2.3), consist of an unknown realisation of behaviour $b \in \boldsymbol{b}$ within the original DM task; the action A represents a searched approximation $\hat{\mathsf{f}}(b)$ and the knowledge K is the known rnd $\mathsf{f}(b)$ to be approximated.

[6] The subscript referring to the strategy in the original DM problem is dropped as it plays no role in the solved supporting DM tasks.

The next step is to specify the DM elements (see Definition 2.4) of the supporting DM task corresponding to the approximation considered. To simplify reading, the argument b is mostly omitted, i.e. $\mathsf{f}(b) = \mathsf{f}$ in the following expressions.

Definition 2.7 (Closed-loop model and its DM elements). For the supporting DM task summarised in Definition 2.6, the closed-loop model (2.14) reads

$$\begin{aligned} \mathsf{F_S}(B) = \mathsf{F_S}(X,A,K) = \mathsf{F}(X|A,K)\mathsf{F_S}(A|K)\mathsf{F}(K) &= \mathsf{F}(b|\hat{\mathsf{f}},\mathsf{f})\mathsf{F_S}(\hat{\mathsf{f}}|\mathsf{f})\mathsf{F}(\mathsf{f}) \\ &= \mathsf{f}(b)\mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})\mathsf{F}(\mathsf{f}). \end{aligned} \tag{2.22}$$

The motivation for the choice of DM elements (Definition 2.4) follows.

$\mathsf{F_S}(X,A,K) = \mathsf{F_S}(b,\hat{\mathsf{f}},\mathsf{f})$	is a model of the closed-loop behaviour.
$\mathsf{F}(X|A,K) = \mathsf{F}(b|\hat{\mathsf{f}},\mathsf{f}) = \mathsf{f}(b)$	is the environment model within the static supporting DM task. It equals the known description f of the behaviour $b \in \boldsymbol{b}$ of the original DM task.
$\mathsf{F_S}(A|K) = \mathsf{F_S}(\hat{\mathsf{f}}|\mathsf{f}) = \mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})$	is a model of the strategy within the supporting DM task. It is a single rule determining how to select an approximation $\hat{\mathsf{f}}$ based on the knowledge $K = \mathsf{f}$.
$\mathsf{F}(K) = \mathsf{F}(\mathsf{f})$	is a model of the knowledge K, which is the known approximated rnd f. Lemma 2.2 implies that its specific form is unimportant.

The following definition complements the DM elements by the ideal closed-loop model for the supporting approximation DM task and explains the choice made.

Definition 2.8 (Ideal closed-loop model and its elements). The considered ideal closed-loop model (2.14) for the supporting DM task, Definition 2.6, is

$$ {}^{i}\mathsf{F}(B) = {}^{i}\mathsf{F}(b,\hat{\mathsf{f}},\mathsf{f}) = {}^{i}\mathsf{F}(b|\hat{\mathsf{f}},\mathsf{f})\,{}^{i}\mathsf{F}(\hat{\mathsf{f}}|\mathsf{f})\,{}^{i}\mathsf{F}(\mathsf{f}) = \hat{\mathsf{f}}(b)\mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})\mathsf{F}(\mathsf{f}). \tag{2.23}$$

The motivation for the choice of DM elements (Definition 2.4) follows.

${}^{i}\mathsf{F}(X,A,K) = {}^{i}\mathsf{F}(b,\hat{\mathsf{f}},\mathsf{f})$	is an ideal model of the closed-loop behaviour.
${}^{i}\mathsf{F}(X|A,K) = {}^{i}\mathsf{F}(b|\hat{\mathsf{f}},\mathsf{f}) = \hat{\mathsf{f}}(b)$	is the ideal environment model, i.e. the ideal model of internals X in the static supporting approximation DM task. Its matching with the $\hat{\mathsf{f}}(b)$ expresses the wish to choose an approximating rnd $\hat{\mathsf{f}}(b)$, which well describes the original behaviour $b \in \boldsymbol{b}$, which is unknown when choosing the action $A = \hat{\mathsf{f}}(b)$.
${}^{i}\mathsf{F}(A|K) = {}^{i}\mathsf{F}(\hat{\mathsf{f}}|\mathsf{f}) = \mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})$	is the model of the ideal strategy. The strategy is left to its fate, Definition 2.5. This choice reflects a lack of common requirements on the way how to select an approximation $\hat{\mathsf{f}}(b)$ of the known approximated rnd $\mathsf{f}(b)$.

$^i\mathsf{F}(K) = \mathsf{F}(\mathsf{f})$ is an ideal model of the knowledge K, where K is the known approximated rnd f. Lemma 2.2 explains why it is left to its fate, Definition 2.5.

The static FPD, applied to the formalisation above, results in the following theorem.

Theorem 2.3 (Approximation of known rnd). *Let the static supporting DM task be given by Definitions 2.6, 2.7 and 2.8. Then, the strategy minimising KLD* $\mathsf{D}(\mathsf{F_S}||^i\mathsf{F})$ *is the optimal deterministic strategy, which generates the optimal approximation* ${}^{opt}\hat{\mathsf{f}} \in \hat{\mathbf{f}}$ *of the known rnd* f *describing the original closed-loop behaviour* $b \in \boldsymbol{b}$*, and*

$$ {}^{opt}\hat{\mathsf{f}} \in \operatorname{Arg}\min_{\hat{\mathsf{f}} \in \hat{\mathbf{f}}} \mathsf{D}(\mathsf{f}||\hat{\mathsf{f}}) = \int_{\boldsymbol{b}} \mathsf{f}(b) \ln\left(\frac{\mathsf{f}(b)}{\hat{\mathsf{f}}(b)}\right) \nu(\mathrm{d}b). \tag{2.24} $$

Proof. For the models considered in Definitions 2.7 and 2.8, the minimised KLD becomes linear in the optimised strategy. According to Lemma 2.2, the optimal strategy is deterministic with the optimal action being a minimising argument in (2.18). The minimised functionals (2.18) and (2.24) coincide. □

Note that a Bayesian formulation of the considered approximation task has been inspected in [2]. Under the widely acceptable conditions, the optimisation (2.24) has been found as the preferred approximation principle.

2.5.2 *Approximation of Unknown Rnd*

The approximation discussed in Section 2.5.1 assumes the knowledge of the approximated rnd for selecting the approximating rnd. This section considers an approximation of an unknown rnd $\mathsf{f}(b) \in \mathbf{f}$, $b \in \boldsymbol{b}$, describing the available knowledge and/or preferences of the original DM task[7]. The set $\mathbf{f}$ and the prior guess $\hat{\mathsf{f}}_0(b)$ about $\mathsf{f}(b)$, $b \in \boldsymbol{b}$, is the only available knowledge K of $\mathsf{f}(b)$

$$ K: \ \mathsf{f}(b) \in \mathbf{f} \subset \mathfrak{F} \text{ (2.19) and a rnd } \hat{\mathsf{f}}_0 \in \mathfrak{F} \text{ is a prior (flat) guess about } \mathsf{f}(b). \tag{2.25} $$

The corresponding static supporting DM task constructs the approximating rnd $\hat{\mathsf{f}}(b) \in \hat{\mathbf{f}} \equiv \mathbf{f}$ based on the available knowledge (2.25). The incompleteness of the knowledge implies the approximated unknown rnd $\mathsf{f}(b)$ is to be treated as internals (see Section 2.2) within the supporting DM task.

Definition 2.9 (Approximation of unknown rnd as supporting DM task). The static supporting DM task (in FPD sense) searching the approximation $\hat{\mathsf{f}}(b) \in \mathbf{f} \subset \mathfrak{F}$ of an unknown rnd $\mathsf{f}(b)$, $b \in \boldsymbol{b}$, with knowledge (2.25), is characterised by the behaviour B, (2.1), structured as follows

$$ B = (X, A, K) = \big((\mathsf{f}(b), b),\ \hat{\mathsf{f}}(b),\ (\mathbf{f}, \hat{\mathsf{f}}_0)\big), \text{ where} \tag{2.26} $$

---

[7] A content and goal of the original DM task is not important here.

the internals X, (2.3), consist of an unknown rnd $\mathsf{f}(b)$ to be approximated and unknown realisation of the original behaviour $b \in \boldsymbol{b}$; the action A is the searched approximation $\hat{\mathsf{f}}(b)$; and the knowledge K is represented by the set $\mathbf{f}$ and the prior guess $\hat{\mathsf{f}}_0$.

The following definitions specify the DM elements (see Definition 2.4) of the supporting DM task considered. To simplify reading, the argument b is mostly omitted, i.e. $\mathsf{f}(b) = \mathsf{f}$ in the following expressions.

Definition 2.10 (Closed-loop model and its DM elements). For the supporting DM task summarised in Definition (2.9), the closed-loop model (2.14) reads

$$\begin{aligned} \mathsf{F}_{\mathsf{S}}(B) = \mathsf{F}_{\mathsf{S}}(X,A,K) &= \mathsf{F}_{\mathsf{S}}((\mathsf{f},b),\ \hat{\mathsf{f}},\ (\mathbf{f},\hat{\mathsf{f}}_0)) \\ &= \mathsf{F}(\mathsf{f}|b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\mathsf{F}(b|\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\mathsf{F}_{\mathsf{S}}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0)\mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0) \\ &= \mathsf{F}(\mathsf{f}|b,\mathbf{f},\hat{\mathsf{f}}_0)\hat{\mathsf{f}}(b)\mathsf{S}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0)\mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0). \end{aligned} \tag{2.27}$$

The motivation for the choice of DM elements (Definition 2.4) follows.

$\mathsf{F}_{\mathsf{S}}(X,A,K) = \mathsf{F}_{\mathsf{S}}(\mathsf{f},b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)$ is a model of the closed-loop behaviour.

$\mathsf{F}(X|A,K) = \mathsf{F}(\mathsf{f}|b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\mathsf{F}(b|\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0) = \mathsf{F}(\mathsf{f}|b,\mathbf{f},\hat{\mathsf{f}}_0)\hat{\mathsf{f}}(b)$
is the environment model within the static supporting DM task. The first factor is a model of unknown approximated rnd f for the given behaviour $b \in \boldsymbol{b}$ of the original DM task. The second factor is a description of behaviour $b \in \boldsymbol{b}$ for a fixed approximating rnd $\hat{\mathsf{f}}$. Easy to see that it equals $\hat{\mathsf{f}}(b)$. Note, the omitted condition in the first factor reflects the obvious assumption that the approximated rnd f is not influenced by its approximation $\hat{\mathsf{f}}$.

$\mathsf{F}_{\mathsf{S}}(A|K) = \mathsf{F}_{\mathsf{S}}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0) = \mathsf{S}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0)$ is a model of the strategy within the supporting DM task. It is a single rule determining how to select an approximation $\hat{\mathsf{f}}$ based on the knowledge $K = (\mathbf{f},\hat{\mathsf{f}}_0)$.

$\mathsf{F}(K) = \mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0)$ is a model of the knowledge, which is determined by the chosen $\mathbf{f}$ and $\hat{\mathsf{f}}_0$. Lemma 2.2 implies that its specific form is unimportant.

The following definition specifies the DM elements of the ideal closed-loop model for the discussed static supporting DM task and explains the choice made.

Definition 2.11 (Ideal closed-loop model and its elements). The considered ideal closed-loop model for the supporting DM task summarised by Definition 2.9 is

$$\begin{aligned} {}^{i}\mathsf{F}(B) = {}^{i}\mathsf{F}(X,A,K) &= {}^{i}\mathsf{F}((\mathsf{f},b),\ \hat{\mathsf{f}},\ (\mathbf{f},\hat{\mathsf{f}}_0)) \\ &= {}^{i}\mathsf{F}(\mathsf{f}|b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\,{}^{i}\mathsf{F}(b|\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\,{}^{i}\mathsf{F}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0)\,{}^{i}\mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0) \\ &= {}^{i}\mathsf{F}(\mathsf{f}|b,\mathbf{f},\hat{\mathsf{f}}_0)\ \hat{\mathsf{f}}_0(b)\ \mathsf{S}(\hat{\mathsf{f}}|\mathbf{f},\hat{\mathsf{f}}_0)\ \mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0). \end{aligned} \tag{2.28}$$

The motivation for the choice of DM elements (Definition 2.4) follows.

$^i\mathsf{F}(X,A,K) = {}^i\mathsf{F}(\mathsf{f},b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)$ is an ideal model of the closed-loop behaviour.

$^i\mathsf{F}(X|A,K) = {}^i\mathsf{F}(\mathsf{f}|b,\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0)\,{}^i\mathsf{F}(b|\hat{\mathsf{f}},\mathbf{f},\hat{\mathsf{f}}_0) = {}^i\mathsf{F}(\mathsf{f}|b,\mathbf{f},\hat{\mathsf{f}}_0)\;\hat{\mathsf{f}}_0(b)$
is an ideal model of the environment within the static supporting DM task. The unknown approximated rnd f and behaviour b play role of the task's internals X, hence two constituents contribute its ideal description. The first factor is an ideal model of the rnd f at given $b \in \boldsymbol{b}$, while the second one is an ideal model of behaviour $b \in \boldsymbol{b}$. Since there is no knowledge of f at disposal, the rnd $\hat{\mathsf{f}}_0(b)$ serves as the best a priori available description of $b \in \boldsymbol{b}$. Note $\hat{\mathsf{f}}$ is excluded from the condition in the first factor as the unknown approximated rnd f cannot be influenced by its approximation $\hat{\mathsf{f}}$.

$^i\mathsf{F}(A|K) = {}^i\mathsf{F}(\hat{\mathsf{f}}|\mathsf{f}) = \mathsf{S}(\hat{\mathsf{f}}|\mathsf{f})$ is the model of the ideal strategy. The strategy is left to its fate, Definition 2.5. This choice reflects a lack of common requirements on the way how to select an approximation $\hat{\mathsf{f}}(b)$ of the unknown approximated rnd $\mathsf{f}(b)$.

$^i\mathsf{F}(K) = \mathsf{F}(\mathbf{f},\hat{\mathsf{f}}_0)$ is an ideal model of the knowledge $\mathbf{f}$ and $\hat{\mathsf{f}}_0$. The model is left to its fate, Definition 2.5. This choice is implied by Lemma 2.2.

The static FPD, applied to the formalisation above, results in the following theorem.

Theorem 2.4 (Approximation of unknown rnd). *Let the DM task be given by Definitions 2.9, 2.10 and 2.11. Then, the strategy minimising KLD* $\mathsf{D}(\mathsf{F_S}||{}^i\mathsf{F})$ *is the optimal deterministic strategy, which generates the optimal approximation* $^{opt}\hat{\mathsf{f}} \in \mathbf{f}$ *of the unknown rnd* f *using the knowledge (2.25), and*

$$^{opt}\hat{\mathsf{f}} \in \operatorname{Arg}\min_{\hat{\mathsf{f}}\in\mathbf{f}} \mathsf{D}(\hat{\mathsf{f}}||\hat{\mathsf{f}}_0) = \int_{\boldsymbol{b}} \hat{\mathsf{f}}(b)\ln\left(\frac{\hat{\mathsf{f}}(b)}{\hat{\mathsf{f}}_0(b)}\right)\nu(\mathrm{d}b). \tag{2.29}$$

Proof. For the models considered by Definitions 2.10 and 2.11, the optimised KLD becomes linear in the optimised strategy. According to Lemma 2.2, a minimising argument of its version conditioned on $A = \hat{\mathsf{f}}$ and $K = (\mathbf{f},\hat{\mathsf{f}}_0)$ is the corresponding optimal action. Thus,

$$\begin{aligned} ^{opt}\hat{\mathsf{f}} &\in \operatorname{Arg}\min_{\hat{\mathsf{f}}\in\mathbf{f}} \int_{(\boldsymbol{b},\mathbf{f})} \mathsf{F}(\mathsf{f}|b,K)\hat{\mathsf{f}}(b)\ln\left(\frac{\mathsf{F}(\mathsf{f}|b,K)\hat{\mathsf{f}}(b)}{{}^i\mathsf{F}(\mathsf{f}|b,K)\hat{\mathsf{f}}_0(b)}\right)\nu(\mathrm{d}(b,\mathsf{f})) \\ &= \operatorname{Arg}\min_{\hat{\mathsf{f}}\in\mathbf{f}} \int_{\boldsymbol{b}} \hat{\mathsf{f}}(b)\ln\left(\frac{\hat{\mathsf{f}}(b)}{\hat{\mathsf{f}}_0(b)}\right)\nu(\mathrm{d}b) \end{aligned}$$

and the optimal $^{opt}\hat{\mathsf{f}}$ is given by (2.29). □

The result (2.29) coincides with the *minimum KLD principle* and reduces to the maximum entropy principle if $\hat{\mathsf{f}}_0$ is a uniform rnd. It has been axiomatically justified in [26] for the set $\mathbf{f}$ specified by the given values of linear functionals on $\mathfrak{F}$ (2.19).

2.5.3 Description of Unknown Rnd

The previous sections formulate the approximation problems as a supporting DM tasks and solve them. Section 2.5.1 searches an approximation of the known rnd by a rnd from the predefined set. Section 2.5.2 describes how to approximate an unknown random rnd, i.e. how to construct its point estimate based on the prior knowledge available.

The present section addresses the problem of how to find the complete probabilistic description of the unknown rnd $\mathsf{f}(b) \in \mathbf{f}$, $b \in \boldsymbol{b}$ using the knowledge

$$K:\ \mathsf{f}(b) \in \mathbf{f} \subset \mathfrak{F} \text{ and a rnd } \hat{\mathsf{F}}_0(\mathsf{f}) \text{ is a (flat) prior guess about } \mathsf{F}(\mathsf{f}) \text{ on } \mathfrak{F}. \tag{2.30}$$

The following definition specifies the static supporting DM task corresponding to the description problem. To simplify reading, the arguments are mostly omitted, i.e. $\mathsf{f}(b) = \mathsf{f}$, $\hat{\mathsf{F}}(\mathsf{f}) = \hat{\mathsf{F}}$ and $\hat{\mathsf{F}}_0(\mathsf{f}) = \hat{\mathsf{F}}_0$ in the following expressions.

Definition 2.12 (Description of unknown rnd as supporting DM task). The static supporting DM task (in FPD sense) searching for a probabilistic description $\hat{\mathsf{F}} \in \mathfrak{F}$ of an unknown rnd $\mathsf{f}(b) \in \mathbf{f}$, $b \in \boldsymbol{b}$, with the available knowledge (2.30), is characterised by the behaviour B, (2.1), structured as follows

$$B = (X,A,K) = \Big(\big(b,\mathsf{f}(b)\big),\ \hat{\mathsf{F}}\big(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0\big),\ \big(\mathbf{f},\hat{\mathsf{F}}_0\big)\Big), \text{ where} \tag{2.31}$$

the internals X, (2.3), consist of an unknown rnd $\mathsf{f}(b)$ to be described and realisation of the original closed-loop behaviour $b \in \boldsymbol{b}$. The action A is a searched rnd $\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$, where the set of admissible actions is a set of all rnds having the support in $\mathbf{f} \subset \mathfrak{F}$, (2.30). The knowledge K is represented by the set $\mathbf{f}$, defining the domain of $\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$, and by the prior guess $\hat{\mathsf{F}}_0$ about the targeted description.

The following definitions specify the DM elements (see Definition 2.4) of the static supporting DM task considered.

Definition 2.13 (Closed-loop model and its DM elements). For the supporting DM task with the behaviour (2.31), the closed-loop model (2.14) reads

$$\begin{aligned}
\mathsf{F}_{\mathsf{S}}(B) = \mathsf{F}_{\mathsf{S}}(X,A,K) &= \mathsf{F}_{\mathsf{S}}\left((b,\mathsf{f}),\ \hat{\mathsf{F}},\ (\mathbf{f},\hat{\mathsf{F}}_0)\right) \\
&= \mathsf{F}(b|\mathsf{f},\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ \mathsf{F}(\mathsf{f}|\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ \mathsf{F}_{\mathsf{S}}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)\ \mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0) \\
&= \mathsf{f}(b)\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)\mathsf{S}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)\ \mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0).
\end{aligned} \tag{2.32}$$

The motivation for the choice of DM elements (Definition 2.4) follows.

$\mathsf{F_S}(X,A,K) = \mathsf{F_S}\left(b,\mathsf{f},\ \hat{\mathsf{F}},\ \mathbf{f},\hat{\mathsf{F}}_0\right)$ is a model of the closed-loop behaviour.

$\mathsf{F}(X|A,K) = \mathsf{F}(b|\mathsf{f},\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ \mathsf{F}(\mathsf{f}|\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0) = \mathsf{f}(b)\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$
is the environment model within the static supporting DM task. The first factor describes an original behaviour $b \in \boldsymbol{b}$ that equals $\mathsf{f}(b)$ for a fixed f. The omitted condition on $\hat{\mathsf{F}}$ in the first factor reflects the assumption that the description of behaviour f is not influenced by selecting $\hat{\mathsf{F}}$.
The second factor is the opted description of rnds $\mathsf{f} \in \mathbf{f}$ based on (2.30) and given $\hat{\mathsf{F}} \in \mathfrak{F}$. Obviously, $\mathsf{F}(\mathsf{f}|\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0) = \hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$.

$\mathsf{F_S}(A|K) = \mathsf{F_S}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0) = \mathsf{S}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)$ is a model of the strategy within the supporting DM task. It is a single rule determining how to select the description $\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$ of an unknown $\mathsf{f} \in \mathbf{f}$ based on the available knowledge $K = (\mathbf{f},\hat{\mathsf{F}}_0)$.

$\mathsf{F}(K) = \mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0)$ is a model of knowledge K, which is determined by the chosen $\mathbf{f}$ and $\hat{\mathsf{F}}_0$. Lemma 2.2 implies that its specific form is unimportant.

The following definition specifies the DM elements of the ideal closed-loop model for the supporting DM task and explains the choice made.

Definition 2.14 (Ideal closed-loop model and its DM elements). The considered ideal closed-loop model (2.14) for the static supporting DM task described in Definition 2.12 is

$$
\begin{aligned}
{}^{i}\mathsf{F}(B) = {}^{i}\mathsf{F}(X,A,K) &= {}^{i}\mathsf{F}\left((b,\mathsf{f}),\ \hat{\mathsf{F}},\ (\mathbf{f},\hat{\mathsf{F}}_0)\right) \\
&= {}^{i}\mathsf{F}(b|\mathsf{f},\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ {}^{i}\mathsf{F}(\mathsf{f}|\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ {}^{i}\mathsf{F}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)\ {}^{i}\mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0) \\
&= \mathsf{f}(b)\hat{\mathsf{F}}_0(\mathsf{f})\ \mathsf{S}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)\mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0).
\end{aligned}
\tag{2.33}
$$

The motivation for the choice of DM elements (Definition 2.4) follows.

${}^{i}\mathsf{F}(X,A,K) = {}^{i}\mathsf{F}\left(b,\mathsf{f},\ \hat{\mathsf{F}},\ \mathbf{f},\hat{\mathsf{F}}_0\right)$ is an ideal model of the closed-loop behaviour.

${}^{i}\mathsf{F}(X|A,K) = {}^{i}\mathsf{F}(b|\mathsf{f},\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0)\ {}^{i}\mathsf{F}(\mathsf{f}|\hat{\mathsf{F}},\mathbf{f},\hat{\mathsf{F}}_0) = \mathsf{f}(b)\ \hat{\mathsf{F}}_0(\mathsf{f})$
is an ideal environment model within the static supporting DM task. $\hat{\mathsf{F}}$ is excluded from the condition in the second factor as the ideal description of $\mathsf{f} \in \mathbf{f}$ is independent of its selected description $\hat{\mathsf{F}}$. The first factor equals to a description of the behaviour $\mathsf{f}(b)$ in the original DM task, which is fixed in the condition.
The second factor is an ideal description of rnd $\mathsf{f} \in \mathbf{f}$ based on (2.30) based on a sole available prior guess $\hat{\mathsf{F}}_0$.

$^{i}\mathsf{F}(A|K) = \mathsf{F}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0) = \mathsf{S}(\hat{\mathsf{F}}|\mathbf{f},\hat{\mathsf{F}}_0)$ is the model of the ideal strategy. The strategy is left to its fate, Definition 2.5. This choice reflects a lack of common requirements on selecting a description $\hat{\mathsf{F}}(\mathsf{f}|\mathbf{f},\hat{\mathsf{F}}_0)$.

$^{i}\mathsf{F}(K) = \mathsf{F}(\mathbf{f},\hat{\mathsf{F}}_0)$ is an ideal model of the knowledge K, (2.30). It is left to its fate, Definition 2.5. This choice is implied by Lemma 2.2.

The static FPD, applied to the formalisation above, results in the following theorem.

Theorem 2.5 (Description of unknown rnd). *Let the DM task be given by Definitions 2.12, 2.13 and 2.14. Then, the strategy minimising KLD* $\mathsf{D}(\mathsf{F}_\mathsf{S}||^{i}\mathsf{F})$ *is the deterministic one. This strategy generates the optimal rnd* $^{opt}\hat{\mathsf{F}} = {}^{opt}\mathsf{F}$ *describing the unknown rnd* $\mathsf{f}(b) \in \mathbf{f} \subset \mathfrak{F}$, $b \in \boldsymbol{b}$, *using the knowledge (2.30), determined by the domain* $\mathbf{f}$ *and prior guess* $\hat{\mathsf{F}}_0 \equiv \mathsf{F}_0$

$$^{opt}\mathsf{F} \in \operatorname{Arg}\min_{\mathsf{F}(\mathsf{f})\in\mathfrak{F}} \int_{\mathbf{f}} \mathsf{F}(\mathsf{f}) \ln\left(\frac{\mathsf{F}(\mathsf{f})}{\mathsf{F}_0(\mathsf{f})}\right) \nu(\mathrm{d}\mathsf{f}). \tag{2.34}$$

Proof. For the DM elements specified by Definitions 2.13 and 2.14, the optimised KLD becomes linear in the optimised strategy and according to Lemma 2.2 the optimal action is the minimising argument of its version conditioned on A and K, which gets the form

$$\int_{(\boldsymbol{b},\mathbf{f})} \mathsf{f}(b)\hat{\mathsf{F}}(\mathsf{f}|K) \ln\left(\frac{\mathsf{f}(b)\hat{\mathsf{F}}(\mathsf{f}|K)}{\mathsf{f}(b)\hat{\mathsf{F}}_0(\mathsf{f})}\right) \nu(\mathrm{d}(b,\mathsf{f})) = \int_{\mathbf{f}} \hat{\mathsf{F}}(\mathsf{f}|K) \ln\left(\frac{\hat{\mathsf{F}}(\mathsf{f}|K)}{\hat{\mathsf{F}}_0(\mathsf{f})}\right) \nu(\mathrm{d}\mathsf{f}),$$

where cancelling, Fubini theorem and normalisation of rnds imply the last equality. This minimised functional coincides with (2.34), where the symbol ˆ at the prior guess and the final optimum is dropped. □

To our best knowledge, the result (2.34) has no published counterpart and represents a sort of *generalised minimum KLD principle*.

2.6 Use of Supporting DM Tasks

This section employs the solutions of the supporting DM tasks (Section 2.5) to support interaction and cooperation of an imperfect selfish participant with its neighbours. The relevant tasks solved here are: i) how to map non-probabilistic, domain-specific expert knowledge and preferences onto rnds, Section 2.6.1; ii) how to extend rnd describing only a part of behaviour to rnd describing the entire behaviour, Section 2.6.2; iii) how to convert a collection of incompletely compatible rnds provided by different participants into a single rnd representing a satisfactory compromise for all participants, Section 2.6.3. These types of tasks frequently arise within multi-participant settings when cooperating participants exchange their incomplete and incompatible rnds, which express their imprecise and partial knowledge and DM preferences.

Throughout, the approximation of rnds, Section 2.5.1, is explicitly used as a guide for selection of appropriate divergence measures. Practically, it will be used more often as many intermediate results are expected to be too complex.

Recall, the small letters indicate variables coming from the original DM task.

2.6.1 Mapping Knowledge and Preferences on Rnds

Within a DM problem a participant deals with raw, application-specific, information representing its incomplete knowledge and DM preferences with respect to the closed-loop behaviour $b \in \boldsymbol{b}$, see (2.1). The *raw information* directly characterises only a part of the behaviour, p, and models other part, m, while provide no information about the rest of the behaviour, u. The following decomposition of the behaviour reflects the relation of raw information to closed-loop behaviour

$$b = (\underbrace{u}_{\text{part untreated by raw info}}, \underbrace{m}_{\text{part modelled by raw info}}, \underbrace{p}_{\text{part provided by raw info}}). \tag{2.35}$$

The *modelled* $m \in \boldsymbol{m}$ and *provided* parts $p \in \boldsymbol{p}$ of raw information can always be treated as random. Even a specific realisation, say $\underline{p}$, can formally be described by $\mathsf{f}(p) = \delta(p - \underline{p})$, where δ is Dirac delta. Mostly, the knowledge of the usual ranges the behaviour b, can be quantified by a flat positive prior rnd $\mathsf{f}_0(b)$ on $\boldsymbol{b}$. The availability of $\mathsf{f}_0(b)$ is assumed from here onwards.

Generally, the model $\mathsf{f}(m|p)$ is characterised only partially. Typical raw information includes ranges of modelled variables, their means, variances, information on expected monotonicity or known deterministic relations between them. These types of raw information can be expressed using generalised moments

$$\int_{\boldsymbol{m}} \phi(m,p)\mathsf{f}(m|p)\nu(\mathrm{d}m) \leq 0, \tag{2.36}$$

where $\phi : (\boldsymbol{m},\boldsymbol{p}) \to (-\infty,\infty)$ is a known function determined by the raw information expressed. The common examples of generalised moments are in Table 2.1. In the case when no information about p is provided, the condition is taken as superfluous, i.e. $\mathsf{f}(m|p) = \mathsf{f}(m)$. Note that raw information always concerns some modelled part, $\boldsymbol{m} \neq \emptyset$. The constraint (2.36) determines a set of rnds $\boldsymbol{\mathsf{f}}(m|p)$, which is a conditional variant of the set $\boldsymbol{\mathsf{f}}$, see (2.19). The set $\boldsymbol{\mathsf{f}}(m|p)$ is convex as the rnd $\mathsf{f}(m|p)$ enters (2.36) linearly.

Searching an unknown $\mathsf{f}(m|p) \in \boldsymbol{\mathsf{f}}(m|p)$ can be formulated as an approximation of unknown rnd using the knowledge of $\boldsymbol{\mathsf{f}}(m|p)$ and prior rnd $\mathsf{f}_0(b)$, see Section 2.5.2.

The direct application of the Theorem 2.4 gives the optimal approximation of an unknown rnd representing the raw information processed

$$^{opt}\hat{\mathsf{f}}(m|p) \propto \mathsf{f}_0(m|p)\exp[-\lambda(p)\phi(m,p)], \tag{2.37}$$

where $\propto$ denotes proportionality, $\mathsf{f}_0(m|p)$ is the rnd derived from the prior rnd $\mathsf{f}_0(b)$, and the Kuhn-Tucker multiplier $\lambda(p) \geq 0$ is chosen to satisfy (2.36). Recall, $\lambda(p) = 0$ if the constraint (2.36) is not active, [18], i.e. when (2.36) does not modify $\mathsf{f}_0(m|p)$.

Table 2.1 Generalised moments for the common examples of raw information[a].

Function $\phi : (\boldsymbol{m},\boldsymbol{p}) \to (-\infty,\infty)$	Raw Information Expressed
$1-\chi(\boldsymbol{m})$	a range of m
$\pi-\chi(\boldsymbol{m})$	a probable range of m, $\pi \in (0,1)$
$m-\mu$	a finite expected value μ of m
$(m-\mu)^2-\sigma^2$	a finite variance σ^2 of m
m_1-m_2	expected monotonicity between entries m_1, m_2 of m
$\phi(m,p)-\zeta$	a deterministic relationship ϕ between m and p valid with expected error smaller than ζ

[a] χ is an indicator function of the set in argument. Parameters π, σ, ζ are included in p.

2.6.2 Extension of Incomplete Rnds

The gained approximation $^{opt}\hat{\mathsf{f}}(m|p)$ (2.37) describes only a part of the behaviour $b \in \boldsymbol{b}$ and has to be extended to a rnd $^{e}\mathsf{f}(b) \in {}^{e}\mathbf{f}$ describing the whole behaviour. Let there exist a rnd $\mathsf{g}(b) \in \mathbf{g}$ *fully expressing the available knowledge* about relations existing within $b \in \boldsymbol{b}$. Then the targeted extension can be viewed as an approximation of the known $\mathsf{g}(b)$ by the rnd $^{e}\mathsf{f}(b) \in {}^{e}\mathbf{f}$, where $^{e}\mathbf{f}$ is constrained by the requirement

$$(\forall {}^{e}\mathsf{f} \in {}^{e}\mathbf{f}) \quad {}^{e}\mathsf{f}(m|p) = {}^{opt}\hat{\mathsf{f}}(m|p). \tag{2.38}$$

The approximation problem is formulated and solved as in Section 2.5.1.

Theorem 2.6 (Optimal extension of a rnd). *Let rnd* g *on* $b \in \boldsymbol{b}$*, fully describing all known relations within b, be given and* $^{opt}\hat{\mathsf{f}}(m|p)$ *be defined by (2.37) on a part of behaviour b, see (2.35). Then the optimal extension* $^{e}\mathsf{f}(b)$*, (2.38), of the rnd* $^{opt}\hat{\mathsf{f}}(m|p)$ *minimises KLD* $\mathsf{D}(\mathsf{g}||{}^{e}\mathsf{f})$*, and has the form*

$$^{e}\mathsf{f}(b) = \mathsf{g}(u|m,p)\ {}^{opt}\hat{\mathsf{f}}(m|p)\ \mathsf{g}(p), \tag{2.39}$$

where $\mathsf{g}(u|m,p)$ *and* $\mathsf{g}(p)$ *are rnds derived from the given* $\mathsf{g}(b)$.

Proof. According to Theorem 2.3, the extension $^{e}\mathsf{f}(b)$ should minimise KLD of $\mathsf{g}(b)$ on $^{e}\mathsf{f}(b)$, i.e.

$$\begin{aligned}
\mathsf{D}(\mathsf{g}||{}^{e}\mathsf{f}) &= \int_{(\boldsymbol{u},\boldsymbol{m},\boldsymbol{p})} \mathsf{g}(u|m,p)\mathsf{g}(m|p)\mathsf{g}(p)\ln\left(\frac{\mathsf{g}(u|m,p)\mathsf{g}(m|p)\mathsf{g}(p)}{{}^{e}\mathsf{f}(u|m,p)\,{}^{e}\mathsf{f}(m|p)\,{}^{e}\mathsf{f}(p)}\right)\nu(\mathrm{d}(u,m,p))\\
&= \int_{(\boldsymbol{m},\boldsymbol{p})} \mathsf{g}(m,p)\left[\int_{\boldsymbol{u}} \mathsf{g}(u|m,p)\ln\left(\frac{\mathsf{g}(u|m,p)}{{}^{e}\mathsf{f}(u|m,p)}\right)\nu(\mathrm{d}u)\right]\nu(\mathrm{d}(m,p))\\
&+ \int_{\boldsymbol{p}} \mathsf{g}(p)\left[\int_{\boldsymbol{m}} \mathsf{g}(m|p)\ln\left(\frac{\mathsf{g}(m|p)}{{}^{e}\mathsf{f}(m|p)}\right)\nu(\mathrm{d}m)\right]\nu(\mathrm{d}p)\\
&+ \int_{\boldsymbol{p}} \mathsf{g}(p)\ln\left(\frac{\mathsf{g}(p)}{{}^{e}\mathsf{f}(p)}\right)\nu(\mathrm{d}p)
\end{aligned}$$

The first term is an expectation of the conditional version of KLD minimised by ${}^{e}\mathsf{f}(u|m,p) = \mathsf{g}(u|m,p)$, the second term is fixed as ${}^{e}\mathsf{f}(m|p) = {}^{opt}\hat{\mathsf{f}}(m|p)$, see (2.38). The last term is KLD of $\mathsf{g}(p)$ on ${}^{e}\mathsf{f}(p)$, which is minimised by ${}^{e}\mathsf{f}(p) = \mathsf{g}(p)$. Thus (2.39) determines the targeted rnd. □

Remark 2.1 (On relationships).

- The constraint (2.36) represents a special case of more general constraints

$$\Phi\big(\mathsf{f}(m|p)|p\big) \le 0, \tag{2.40}$$

  with functionals Φ delimiting a convex set $\mathbf{f}(m|p)$. This generalisation can be useful when a bound on KLD of the constructed rnd $\mathsf{f}(m|p)$ on another rnd is known. Elaboration of this case is out of the chapter's scope.
- Moment and ranges constraints apply either to plain variables in the behaviour or to *innovations*, i.e. deviations of the modelled random variables from their (conditional) expectations.
- Participants often exploit deterministic models resulting from the first principles and domain-specific knowledge. They are mostly expressed by a set of equations $\phi(m,p) = \varepsilon(m,p)$, where $\varepsilon(m,p)$ is a modelling error. Then the constraints (2.36) express a bound on the expectation of the modelling error.
- The application of Theorem 2.4 may lead to too complex rnd. The corresponding approximating rnd can be constructed by a direct use of Theorem 2.3.
- Generally, a vector form of (2.36) should be considered. This case, however, may have no solution when the vector constraints are incompatible and delimit an empty set $\mathbf{f}(m|p)$. To avoid this, a vector case is treated as a collection of respective scalar cases and the resulting collection of rnds is merged into a common compromise, see Section 2.6.3 and [16]. This solution decreases the computational load on the participant treating raw information.

2.6.3 Combination of Rnds

The section proposes a reliable way how to construct a single rnd (compromise) representing a collection

$$\{\mathsf{f}_\kappa(b)\}_{\kappa\in\boldsymbol{\kappa}},\ \ \boldsymbol{\kappa} = 1,2,\ldots,|\boldsymbol{\kappa}|,\ \ b\in\boldsymbol{b}, \tag{2.41}$$

of incompletely compatible rnds in $\mathfrak{F}$, which act on the same behaviour set $\boldsymbol{b}$ and originate from $|\boldsymbol{\kappa}|$ different information sources[8]. The rnds (2.41) may either be extensions of rnds representing raw knowledge and preferences (see Theorem 2.6) or be provided by the participant's neighbours. The partial incompatibility of rnds may be caused by: i) imprecise processed observations; ii) extension and approximation errors resulted from the use of Theorem 2.3 and Theorem 2.4; iii) natural differences in knowledge, preferences, and abilities of interacting participants.

It is often desirable and even inevitable to find a *compromise* which respects *all* information sources, i.e. a rnd yielding a sufficiently good approximation of each $\mathsf{f}_\kappa(b)$, $\kappa \in \boldsymbol{\kappa}$. The compromise's acceptability, i.e. a condition when the compromise is taken as a satisfactory approximation of $\mathsf{f}_\kappa(b)$, is determined by the individual sources[9].

Let a bound on the acceptable degree of compromise $\beta_\kappa \in (0,\infty)$ for the κth source be provided together with the respective κth rnd from (2.41). Assume β_κ, $\kappa \in \boldsymbol{\kappa}$, determine a non-empty set $\mathbf{f} \neq \emptyset$ of *all* possible compromises $\mathsf{f} \in \mathbf{f}$ of the collection (2.41) such that

$$\mathbf{f} : \int_{\mathbf{f}} \mathsf{D}(\mathsf{f}_\kappa||\mathsf{f})\mathsf{F}(\mathsf{f})\nu(\mathrm{df}) \leq \beta_\kappa < \infty,\ \kappa \in \boldsymbol{\kappa} = \{1,\ldots,|\boldsymbol{\kappa}|\},\ |\boldsymbol{\kappa}| < \infty, \tag{2.42}$$

where $\mathsf{F}(\mathsf{f}) \in \mathfrak{F}$ is a probabilistic description of f. Notice the order of arguments in the KLD in (2.42). Theorem 2.3 indirectly motivates this choice: as $\mathsf{f} \in \mathbf{f}$ must be a good approximation of f_κ, the divergence of f_κ on f should be optimised. This also results from [2], which is tightly connected with the formalised justification of the FPD, see for instance [12].

The available $\{\mathsf{f}_\kappa\}_{\kappa\in\boldsymbol{\kappa}}$ (2.41) reflect an unknown rnd $\mathsf{f} \in \mathbf{f}$ describing their *optimal compromise*. Theorem 2.5 provides its optimal probabilistic description.

Theorem 2.7 (Optimal compromise). *Let a collection (2.41) and respective β_κ, $\kappa \in \boldsymbol{\kappa}$ determining a non-empty set $\mathbf{f}$, (2.42), of all possible compromises $\mathsf{f} \in \mathbf{f}$ of (2.41) be given.*
Then the optimal probability that f is the optimal compromise of rnds (2.41) is given by the rnd ${}^{opt}\mathsf{F}(\mathsf{f})$, Theorem 2.5,

$${}^{opt}\mathsf{F}(\mathsf{f}) \propto \mathsf{F}_0(\mathsf{f}) \prod_{b\in\boldsymbol{b}} \mathsf{f}(b)^{\tilde{\rho}(b)},\ \tilde{\rho}(b) \equiv \sum_{\kappa\in\boldsymbol{\kappa}} \lambda_\kappa \mathsf{f}_\kappa(b). \tag{2.43}$$

It is determined by a chosen prior (flat) guess $\mathsf{F}_0(\mathsf{f}) = \hat{\mathsf{F}}_0(\mathsf{f})$ (2.30) and Kuhn-Tucker multipliers $\lambda_\kappa \geq 0$ chosen to respect inequalities in (2.42). The assumed non-emptiness of $\mathbf{f}$, which depends on the choice of β_κ, $\kappa \in \boldsymbol{\kappa}$, guarantees the existence of such λ_κ.

[8] The term *information source* denotes either outcome of processing of raw information or a cooperating participant in multiple-participant DM.

[9] Generally the acceptability defined by individual sources may lead to no compromise.

If the prior rnd $\mathsf{F}_0(\mathsf{f})$ *is selected in the conjugate Dirichlet form*

$$\mathsf{F}_0(\mathsf{f}) = \mathcal{D}_{\mathsf{f}}(\rho_0) \propto \prod_{b\in\boldsymbol{b}} \mathsf{f}^{\rho_0(b)-1}(b),\ \ \rho_0(b) > 0,\ \bar{\rho} \equiv \int_{\boldsymbol{b}} \rho_0(b)\nu(\mathrm{d}b) < \infty, \quad (2.44)$$

where $\rho_0(b)$ *is a free parameter expressing previous experience, then* ${}^{opt}\mathsf{F}(\mathsf{f})$ *(2.43) is also Dirichlet rnd*

$${}^{opt}\mathsf{F}(\mathsf{f}) = \mathcal{D}_{\mathsf{f}}(\rho)\ \ \textit{with}\ \ \rho(b) = \rho_0(b) + \tilde{\rho}(b) = \rho_0(b) + \sum_{\kappa\in\boldsymbol{\kappa}} \lambda_\kappa \mathsf{f}_\kappa(b). \quad (2.45)$$

The expectation of the rnd (2.45), that serves as an estimate $\hat{\mathsf{f}}$ *of the optimal compromise* f*, has the form*

$$\hat{\mathsf{f}}(b) = \frac{\rho_0(b) + \sum_{\kappa\in\boldsymbol{\kappa}} \lambda_\kappa \mathsf{f}_\kappa(b)}{\bar{\rho} + \sum_{\kappa\in\boldsymbol{\kappa}} \lambda_\kappa}. \quad (2.46)$$

It is a convex combination of rnds $\frac{\rho_0(b)}{\bar{\rho}}$ *and* $\{\mathsf{f}_\kappa\}_{\kappa\in\boldsymbol{\kappa}}$ *with weights* $\alpha_0 = \frac{\bar{\rho}}{\bar{\rho}+\sum_{\kappa\in\boldsymbol{\kappa}}\lambda_\kappa} > 0$ *and* $\alpha_\kappa = \frac{\lambda_\kappa}{\bar{\rho}+\sum_{\kappa\in\boldsymbol{\kappa}}\lambda_\kappa} \geq 0$, $\alpha_0 + \sum_{\kappa\in\boldsymbol{\kappa}} \alpha_\kappa = 1$.

Proof. The constraints (2.42) specifying a convex set of $\mathbf{F}$ of possible descriptions of $\mathbf{f}$ are respected by employing Kuhn-Tucker multipliers $\lambda_\kappa \geq 0$. The elementary manipulations with the corresponding Kuhn-Tucker functional transform it into KLD $\mathsf{D}(\mathsf{F}||{}^{opt}\mathsf{F})+$ constant independent of F with ${}^{opt}\mathsf{F}$ given by the formula (2.43). Properties of KLD (2.8) imply the optimality of the rnd (2.43). Reproducing property of Dirichlet rnd and its expectation can be verified by direct evaluations [13]. □

An additional problem arises when rnds from the collection are defined on a part of behaviour $b \in \boldsymbol{b}$ (see (2.35)), i.e. the compromise is searched among

$$\{\mathsf{f}_\kappa(m_\kappa|p_\kappa)\}_{\kappa\in\boldsymbol{\kappa}},\ \ \boldsymbol{\kappa} = \{1,2,\ldots,|\boldsymbol{\kappa}|\},\ \ m \in \boldsymbol{m}_\kappa \subset \boldsymbol{b}, p \in \boldsymbol{p}_\kappa \subset \boldsymbol{b} \quad (2.47)$$

$b = (u_\kappa, m_\kappa, p_\kappa)$ is an individual split (2.35) for a source $\kappa \in \boldsymbol{\kappa}$.

Note that the bound β_κ on the acceptable degree of compromise provided by the individual source $\kappa \in \boldsymbol{\kappa}$ concerns only the part $m_\kappa \in b$ known to the source

$$\int_{\mathbf{f}} \mathsf{D}\big(\mathsf{f}_\kappa(m_\kappa|p_\kappa)||\mathsf{f}(m_\kappa|p_\kappa)\big)\mathsf{F}\big(\mathsf{f}\big)\nu(\mathrm{d}\mathsf{f}) \leq \beta_\kappa < \infty,\ \boldsymbol{\kappa} = \{1,\ldots,|\boldsymbol{\kappa}|\}, |\boldsymbol{\kappa}| < \infty.$$

Then to find an optimal compromise representing (2.47), the individual rnds forming the collection should first be extended over the whole behaviour, see Theorem 2.6. This extension, however, requires a rnd $\mathsf{g}(b)$ describing fully the known relations on $b \in \boldsymbol{b}$, see Section 2.6.2. As such it has to be a single rnd, defined on the entire $b \in \boldsymbol{b}$, common for all $\mathsf{f}_\kappa(m_\kappa|p_\kappa)$ (2.47) . Here, an existence of such $\mathsf{g}(b)$ is assumed and a set of possible compromises is thus defined

$$\mathsf{f} : \int_{\mathsf{f}} \mathsf{D}\left({}^{e}\mathsf{f}_{\kappa}||\mathsf{f}\right)\mathsf{F}\left(\mathsf{f}\right)\nu(\mathrm{df}) \le \beta_{\kappa} < \infty,\ \kappa \in \boldsymbol{\kappa} = \{1,\ldots,|\boldsymbol{\kappa}|\},\ |\boldsymbol{\kappa}| < \infty, \qquad (2.48)$$

where ${}^{e}\mathsf{f}_{\kappa}(b) \in {}^{e}\mathbf{f}$ is an optimal extension of $\mathsf{f}_{\kappa}(m_{\kappa}|p_{\kappa})$ over $b \in \boldsymbol{b}$, see (2.39). Similarly to Theorem 2.7, a description of the optimal compromise for incompletely compatible rnds can be found.

Theorem 2.8 (Optimal compromise of incompletely compatible rnds). *Let the collection (2.47) and respective β_{κ}, $\kappa \in \boldsymbol{\kappa}$, determining a* non-empty *set* $\mathbf{f}$*, (2.48), of all possible compromises* $\mathsf{f} \in \mathbf{f}$ *of (2.47) be given.*

Then the optimal probability that $\mathsf{f}(b)$, $b \in \boldsymbol{b}$, is the optimal compromise among rnds from $\mathbf{f}$ (2.48) is given by the rnd ${}^{opt}\mathsf{F}(\mathsf{f})$ (2.43). The used prior (flat) guess is assumed to be Dirichlet rnd $\mathsf{F}_0(\mathsf{f}) = \mathcal{D}_{\mathsf{f}}(\rho_0)$ given by $\rho_0(b) > 0$ on $\boldsymbol{b}$, (2.44). The expectation $\hat{\mathsf{f}}(b)$ of the optimal description ${}^{opt}\mathsf{F}(\mathsf{f})$ fulfills the equation

$$\hat{\mathsf{f}}(b) = \alpha_0 \frac{\rho(b)}{\bar{\rho}} + \sum_{\kappa \in \boldsymbol{\kappa}} \alpha_{\kappa} \hat{\mathsf{f}}(u_{\kappa}|m_{\kappa}, p_{\kappa}) \mathsf{f}_{\kappa}(m_{\kappa}|p_{\kappa}) \hat{\mathsf{f}}(p_{\kappa}), \qquad (2.49)$$

where values α_{κ}, $\kappa \in \boldsymbol{\kappa}$, are chosen so that constraints in (2.48) are respected for $\mathsf{F}(\mathsf{f}) = {}^{opt}\mathsf{F}(\mathsf{f})$. If some of them are not active, then the corresponding αs are zero.

Proof. For a rnd $\mathsf{g}(b)$ expressing fully the available knowledge, determined by the given $\{\mathsf{f}_{\kappa}(m_{\kappa}|p_{\kappa}), \beta_{\kappa}\}_{\kappa \in \boldsymbol{\kappa}}$ and $\rho_0(b)$, $b \in \boldsymbol{b}$, the optimal extensions ${}^{e}\mathsf{f}_{\kappa}(b)$ have the form $\mathsf{g}(u_{\kappa}|m_{\kappa}, p_{\kappa})\mathsf{f}_{\kappa}(m_{\kappa}|p_{\kappa})\mathsf{g}(p_{\kappa})$, Theorem 2.6. (2.46) provides the expectation of the corresponding optimal compromise

$$\hat{\mathsf{f}}(b) = \frac{\rho_0(b) + \sum_{\kappa \in \boldsymbol{\kappa}} \lambda_{\kappa} \mathsf{g}(u_{\kappa}|m_{\kappa}, p_{\kappa}) \mathsf{f}_{\kappa}(m_{\kappa}|p_{\kappa}) \mathsf{g}(p_{\kappa})}{\bar{\rho} + \sum_{\kappa \in \boldsymbol{\kappa}} \lambda_{\kappa}}. \qquad (2.50)$$

This compromise fully expresses the available knowledge, i.e. it has to hold $\mathsf{g} = \hat{\mathsf{f}}$, and (2.50) becomes (2.49).

It remains to check existence of $\hat{\mathsf{f}}$ solving (2.49). Let us try to solve (2.49) by successive approximations starting from an initial guess ${}^{n}\hat{\mathsf{f}}(b) > 0$ on $\boldsymbol{b}$, for $n = 0$. The assumed positivity of $\rho_0(b)$ and positivity of α_0, obvious from (2.46), imply that ${}^{n}\hat{\mathsf{f}}(b) > 0$ on $\boldsymbol{b}$ for all iterations n. This implies that the right-hand side of (2.49) evaluated for $\hat{\mathsf{f}} = {}^{n}\hat{\mathsf{f}}$ provides a value in (0,1) for each b and n. Thus there is a converging subsequence of the sequence $({}^{n}\hat{\mathsf{f}}(b))_{n \ge 0}$ and its limit $\hat{\mathsf{f}}(b)$ is a fixed point of the equation (2.49). □

Theorem 2.8 gives a way how to merge rnds coming from different informational sources having at disposal only fragmental information about the entire behaviour. It effectively solves sharing of raw information and probabilistic knowledge and preferences within multiple-participants' DM.

To ensure participants' cooperation, the optimal compromise $\hat{\mathsf{f}}(b)$ is supposed to be projected back to the respective participants[10] by computing $\hat{\mathsf{f}}(m_{\kappa}|p_{\kappa})$. This way

[10] Information sources.

of sharing of raw information can be performed in algorithmic way, which decreases a computational load on the imperfect participants.

Remark 2.2 (On relationships).

- The proposed merging offers an efficient tool for solving, otherwise extremely hard, problems of decentralised decision making [3]. Importantly, the individual participant is *not* forced to model its neighbours as required in theory of incomplete (Bayesian) games, [10].
- The exploitation of the projection $\hat{\mathsf{f}}(m_\kappa|k_\kappa)$, given to the participant as the processed raw information offered by its neighbours, is an additional DM task. It can be (relatively simply) solved by the participant or by an upper cooperative level. Both cases are out of the scope of Chapter.
- The compromise $\hat{\mathsf{f}}(b)$ can be exploited by the upper level participants within a hierarchical scheme [31], whenever problem complexity allows it.
- The knowledge (2.48) with $\mathsf{F} = {}^{opt}\mathsf{F}$ is parameterised by $\rho_0(b)$, (2.44). The rnd F_0 serves for a soft delimitation of $\mathbf{f}$ and its choice is simple. A fair selection of β_κ guaranteeing non-emptiness of $\mathbf{f}$ is open. It is conjectured that, without additional reasons for preferring information coming from some sources, all β_κ, should be equal to the smallest common value for which the solution exists.
- The presented compromise $\hat{\mathsf{f}}$ (2.49) extends and refines its predecessors [16], [25]. It: i) replaces supra-Bayesian approach [8] by generalised minimum KLD principle, Theorem 2.5; ii) includes non-constant $\rho_0(b)$, which allows to apply the result also to original behaviours with *countable* number of realisations.

2.7 Concluding Remarks and Open Questions

A feasible support of interaction and cooperation of imperfect selfish participants within multiple participant dynamic DM is addressed. The efficient support is especially of importance for interacting participants exchanging their incomplete and incompatible models, which express the participants' imprecise domain-specific knowledge and DM preferences. The proposed approach respects the participant's inability to devote unlimited cognitive and computational resources to decision making, as well as its intention to follow own DM goal. The methodology allows a sort of soft cooperation even for non-collaborative selfish participants.

Chapter defines typical subtasks arisen within participants' interaction, formulates them as independent supporting DM tasks and uses fully probabilistic design to their solution. This solution is then employed for: i) mapping domain-specific expert knowledge and preferences onto probabilistic description; ii) extending probabilistic models describing only fragmental knowledge; iii) merging a collection of incompletely compatible models provided by different participants into a single one representing a acceptable compromise for all participants.

Further studies will be primarily pursued to analyse the proposed methodology and assumptions made, and to verify whether our results are competitive with the alternative approaches. Conceptually, it is inevitable to clarify whether the load of DM tasks related to the exploitation of the merged knowledge and preference descriptions can be structurally controlled. A comparison with a descriptive approach modelling natural/societal system will be advantageous and may give a deeper insight onto intuitive engineering solutions used. The most challenging and hard problem will be to analyse emergent behaviour of a network of interacting participants, which use the proposed approach.

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References

1. Athans, M.: The role and use of the stochastic linear-quadratic-Gaussian problem in control system design. IEEE Transactions on Automatic Control 16(6), 529–552 (1971)
2. Bernardo, J.M.: Expected information as expected utility. The Annals of Statistics 7(3), 686–690 (1979)
3. Bernstein, D., Amato, C., Hansen, E., Zilberstein, S.: Policy iteration for decentralized control of Markov decision processes. J. of Artificial Intelligence Research 34, 89–132 (2009)
4. Bertsekas, D.: Dynamic Programming and Optimal Control, 2nd edn. Athena Scientific, Nashua (2001)
5. Chajewska, U., Koller, D., Parr, R.: Making rational decisions using adaptive utility elicitation. In: Proc. of the Seventeenth National Conference on Artificial Intelligence, pp. 363–369 (2000)
6. Cooke, N.: Varieties of knowledge elicitation techniques. Int. J. of Human-Computer Studies 41, 801–849 (1994)
7. Fishburn, P.: Utility Theory for Decision Making. J. Wiley, New York (1970)
8. Genest, C., Zidek, J.: Combining probability distributions: A critique and annotated bibliography. Statistical Science 1(1), 114–148 (1986)
9. Guy, T.V., Kárný, M.: Stationary fully probabilistic control design. In: Filipe, J., Cetto, J.A., Ferrier, J.L. (eds.) Proc. of the Second Int. Conference on Informatics in Control, Automation and Robotics, pp. 109–112. INSTICC, Barcelona (2005)
10. Harsanyi, J.: Games with incomplete information played by Bayesian players, I–III. Management Science 50(12) (2004) (supplement)
11. Kárný, M.: Towards fully probabilistic control design. Automatica 32(12), 1719–1722 (1996)
12. Kárný, M.: Bayesian paradigm and fully probabilistic design. In: Preprints of the 17th IFAC World Congress. IFAC (2008)
13. Kárný, M., Böhm, J., Guy, T.V., Jirsa, L., Nagy, I., Nedoma, P., Tesař, L.: Optimized Bayesian Dynamic Advising: Theory and Algorithms. Springer, London (2006)
14. Kárný, M., Guy, T.: Preference elicitation in fully probabilistic design of decision strategies. In: Proc. of the 49th IEEE Conference on Decision and Control. IEEE, Los Alamitos (2010)
15. Kárný, M., Guy, T.V.: Fully probabilistic control design. Systems & Control Letters 55(4), 259–265 (2006)
16. Kárný, M., Guy, T.V., Bodini, A., Ruggeri, F.: Cooperation via sharing of probabilistic information. Int. J. of Computational Intelligence Studies, 139–162 (2009)

17. Kárný, M., Kroupa, T.: Axiomatisation of fully probabilistic design. Information Sciences (2010) (under revision)
18. Kuhn, H., Tucker, A.: Nonlinear programming. In: Proc. of 2nd Berkeley Symposium, pp. 481–492. University of California Press, Berkeley (1951)
19. Kullback, S., Leibler, R.: On information and sufficiency. Annals of Mathematical Statistics 22, 79–87 (1951)
20. Lee, C., Budhiraja, A., Chen, L.: A survey of numerical methods for nonlinear filtering problems. Physica D: Nonlinear Phenomena 230, 27–36 (2007)
21. Meditch, J.: Stochastic Optimal Linear Estimation and Control. Mc. Graw Hill, New York (1969)
22. Peterka, V.: Bayesian system identification. In: Eykhoff, P. (ed.) Trends and Progress in System Identification, pp. 239–304. Pergamon Press, Oxford (1981)
23. Savage, L.: Foundations of Statistics. Wiley, New York (1954)
24. Schneeweiss, C.: Distributed Decision Making. Springer, Heidelberg (2003)
25. Sečkárová, V.: Supra-Bayesian approach to merging of incomplete and incompatible data. In: Guy, T., Karny, M., Wolpert, D. (eds.) Decision Making with Multiple Imperfect Decision Makers. UTIA (2010)
26. Shore, J., Johnson, R.: Axiomatic derivation of the principle of maximum entropy and the principle of minimum cross-entropy. IEEE Transactions on Information Theory 26(1), 26–37 (1980)
27. Si, J., Barto, A., Powell, W., Wunsch, D. (eds.): Handbook of Learning and Approximate Dynamic Programming. Wiley-IEEE Press, Danvers (2004)
28. Su, X., Khoshgoftaar, T.: A survey of collaborative filtering techniques. In: Advances in Artificial Intelligence, pp. 27–36 (2007)
29. Vladimirov, V.: Generalized Functions in Mathematical Physics. Mir Publishers, Moscow (1979)
30. Šindelář, J., Vajda, I., Kárný, M.: Stochastic control optimal in the Kullback sense. Kybernetika 44(1), 53–60 (2008)
31. Wolpert, D., Kulkarni, N.: From wireless networks to sensor networks and onward to networked embedded control. In: Proc. 2008 NASA/ESA Conference on Adaptive Hardware and Systems, NASA (2008)