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NON-LOCAL PDES WITH DISCRETE STATE-DEPENDENT DELAYS: WELL-POSEDNESS IN A METRIC SPACE

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Abstract. Partial differential equations with discrete (concentrated) state-dependent delays are studied. The existence and uniqueness of solutions with initial data from a wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used to construct a dynamical system. This subset is an analogue of *the solution manifold* proposed for ordinary equations in [H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, J. Differential Equations, 195(1), (2003) 46–65]. The existence of a compact global attractor is proven. As far as applications are concerned, we consider the well known Mackey-Glass-type equations with diffusion, the Lasota-Wazewska-Czyzewska model, and the delayed diffusive Nicholson's blowflies equation, all with *state-dependent* delays.

1. **Introduction.** The partial differential equations (PDEs) with delays have attracted a lot of attention during the last decades as many processes of the real world (like an automatically controlled furnace, bi-directional associative memory (BAM) neural networks, reaction-diffusion processes) can be described by such kind of equations. Studying these equations is based on the well-developed approaches to the ordinary differential equations (ODEs) with delays [14, 8, 1] and PDEs without delays [11, 12, 19, 18]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [2, 35, 7] and references therein; see also [36, 5, 6, 3] and to the monograph [43] that are very close to this work.

In many evolution systems arising in applications the presented delays are frequently *state-dependent* (SDDs). The theory of such equations, especially the ODEs, is rapidly developping and many deep results have been obtained up to now (see e.g. [37, 38, 39, 20, 22, 40] and also the survey paper [15] for details and references). The underlying main mathematical difficulty of the theory of PDEs with SDDs lies in the fact that the functions describing state-dependent delays are

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not Lipschitz continuous on the space of continuous functions - the main space, on which the classical theory of equations with delays is developed. This implies that the corresponding *initial value problem* (IVP) is not generally well-posed in the sense of J. Hadamard [11, 12].

The partial differential equations with state-dependent delays were first studied in [26] (the case of distributed delays, weak solutions), [16] (mild solutions, infinite discrete delay), and [27] (weak solutions, finite discrete and distributed delays). An alternative approach to the PDEs with discrete SDDs is proposed in [29].

This paper is a continuation of the work [30] and its goal is to study the approach used for ODEs with SDDs [37, 38, 15] in the case of PDEs. The main idea lies in finding a wider space $Y \supset X$ such that a solution $u : [a,b] \to Y$ be a Lipschitz function (with respect to a weaker norm of Y), and constructing a dynamical system on a subset of the space $C([a,b];Y)$. It should be emphasized that the dynamical system is constructed on a metric space that is nonlinear. More precisely, the existence and uniqueness of solutions with initial data from that wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used for constructing the aforementioned dynamical system. This subset is an analogue of *the solution manifold* proposed in [38], see also [15]. We use the same class of non-local in space variables nonlinear PDEs as in [30].

As far as applications and motivations are concerned we consider the well known Mackey-Glass-type equations with diffusion (a model in physiology), the Lasota-Wazewska-Czyzewska model in hematology and the delayed diffusive Nicholson's blowflies equation in population dynamics. The approach proposed here allows to study the models with *state-dependent* delays, which seems to be more realistic comparing with the constant delay. See examples and remark 1 below for more details.

The paper is organized as follows. The section 2 is devoted to the formulation of the model and examples. The proof of the existence and uniqueness of (strong) solutions for initial functions from a Banach space forms a main part of the section 3. In the section 4, an evolution operator S_t is constructed and its asymptotic properties in different functional spaces are investigated. The dissipativeness is obtained in a Banach space, while the existence of a global attractor is proven on a smaller metric space (the solution manifold). The choice of this smaller space is different from that proposed in [30].

2. The model with discrete state-dependent delay and preliminaries.

Consider the following non-local partial differential equation with a discrete state-dependent delay η

$$\frac{\partial}{\partial t}u(t,x) + Au(t,x) + du(t,x) = b\left([Bu(t-\eta(u_t),\cdot)](x)\right) \equiv \big(F_1(u_t)\big)(x), \quad x \in \Omega, \tag{1}$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \to L^2(\Omega)$ generates an analytic semigroup, $\Omega \subset \mathbb{R}^{n_0}$ is a smooth bounded domain, $B : L^2(\Omega) \to L^2(\Omega)$ denotes a bounded operator that will be defined later, $b : \mathbb{R} \to \mathbb{R}$ stands for a Lipschitz map, $d \in \mathbb{R}, d \geq 0$, and the function $\eta : C([-r,0]; L^2(\Omega)) \to [0,r] \subset \mathbb{R}_+$ denotes *a state-dependent discrete delay*. Let $C \equiv C([-r,0]; L^2(\Omega))$. Norms defined on $L^2(\Omega)$ and C are denoted by $||\cdot||$ and $||\cdot||_C$, respectively, and $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\Omega)$. As usually, $u_t \equiv u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-r, 0]$.

Examples. As the first application we propose the delayed diffusive equation (see (1)) with Mackey-Glass type nonlinearity given by $b(v) = \beta v\,(1+v^m)^{-1}, \beta \in \mathbb{R}, m \geq 1$. It is the diffusive model of Hematopoiesis (blood cell production) with the state-dependent delay and $-A$ being the Laplace operator with the Neumann boundary conditions. The model of Hematopoiesis when u does not depend on the spatial variable $x \in \Omega$ was first proposed by Mackey and Glass [21]. For the case of constant delay $\eta(u_t) \equiv \tau > 0$ see, e.g. [9, 41] and also [44] for $n_0 = 1$ and further references. One can easily check that the nonlinearity $b(v) = \beta v\,(1+v^m)^{-1}$ is bounded and globally Lipschitz provided $m = 2k, k \in \mathbb{N}$.

As another application we can consider the diffusive Nicholson blowflies equation with state-dependent delays [30], i.e. the equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^{n_0}$ is a bounded domain with a smooth boundary, the nonlinear (birth) function b is given by $b(v) = p\cdot ve^{-v}$. For the constant delay see, e.g. [34] and also [44] for $n_0 = 1$.

The third application is the Lasota-Wazewska-Czyzewska model in hematology covering problems involving blood cell pathologies, where the nonlinearity is given by $b(v) = p\cdot v^k e^{-v}, p > 0, k \geq 1$. For the case of constant delay ODE see, e.g. [17]. In our case the operator A is as in the Mackey-Glass equation.

Remark 1. The operator B may for example be of the following forms (linear operators)

$$[Bv](x) \equiv \int_\Omega v(y)\widetilde{f}(x,y)dy, \quad x \in \Omega, \tag{2}$$

or even simpler

$$[Bv](x) \equiv \int_\Omega v(y)f(x-y)\ell(y)dy, \quad x \in \Omega, \tag{3}$$

where $f : \Omega \to \mathbb{R}$ is a smooth function and $\ell \in C_0^\infty(\Omega)$. In the last case the nonlinear term in (1) is of the (nonlocal) form

$$\big(F_1(u_t)\big)(x) \equiv b\left(\int_\Omega u(t-\eta(u_t),y)f(x-y)\ell(y)dy\right), \quad x \in \Omega. \tag{4}$$

□

It is interesting to mention that the presence of the operator B in (1) (in our example it is a nonlocal operator) has not only mathematical but also strong biological motivations as discussed in [4] (see also a survey [10] for relevant references).

Consider the equation (1) with the initial condition

$$u|_{[-r,0]} = \varphi \tag{5}$$

and let

$$H \equiv \left\{\varphi \in C([-r,0]; D(A^{-\frac{1}{2}})) \mid \varphi(0) \in D(A^{\frac{1}{2}})\right\}. \tag{6}$$

Let further

$$||\varphi||_H \equiv \max_{s\in[-r,0]} ||A^{-\frac{1}{2}}\varphi(s)|| + ||A^{\frac{1}{2}}\varphi(0)||$$

be a norm defined on the space H and $D(A^\alpha)$ denote the domain of the operator A^α. In the sequel the following assumptions will play an important role.

(H1.η) The discrete delay function $\eta : H \to [0, r]$ is such that

$$\exists L_\eta > 0, \quad \exists q \geq 0 \text{ such that } \forall \varphi, \psi \in H \Rightarrow$$

$$|\eta(\varphi) - \eta(\psi)| \leq L_\eta \left(q||A^{\frac{1}{2}}(\varphi(0) - \psi(0))||^2 + \int_{-r}^{0} ||A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))||^2 \, d\theta \right)^{\frac{1}{2}} \quad (7)$$

(H.B) The following Lipschitz property of the operator B holds.

$$\exists L_B > 0 \text{ such that } \forall u, v \in D(A^{-\frac{1}{2}}) \Rightarrow ||Bu - Bv|| \leq L_B ||A^{-\frac{1}{2}}(u - v)|| \quad (8)$$

Remark 2. The assumption (8) concerning the operator B implies that B is compact (the case where B is the identity is excluded). This smoothing effect of the operator B is important in our study of discrete state-dependent delays. As a natural example, under the assumption that for all (almost all) $x \in \Omega \Rightarrow f(\cdot - x)\ell(\cdot) \in D(A^{\frac{1}{2}})$ and $u \in D(A^{-\frac{1}{2}})$, the term of the form (3) (a convolution) implies that

$$|\langle u, f(\cdot - x)\ell(\cdot)\rangle| \leq ||A^{-\frac{1}{2}}u|| \, ||A^{\frac{1}{2}} f(\cdot - x)\ell(\cdot)||,$$

which gives

$$\left(\int_\Omega \left| \int_\Omega u(y) f(y - x)\ell(y) dy \right|^2 dx \right)^{\frac{1}{2}} \leq ||A^{-\frac{1}{2}}u|| \left(\int_\Omega ||A^{\frac{1}{2}} f(\cdot - x)\ell(\cdot)||^2 \, dx \right)^{\frac{1}{2}}.$$

Hence, the property (H.B) (see (8)) holds with $L_B \equiv \left(\int_\Omega ||A^{\frac{1}{2}} f(\cdot - x)\ell(\cdot)||^2 \, dx \right)^{\frac{1}{2}}$. The same arguments hold (with $L_B \equiv \left(\int_\Omega ||A^{\frac{1}{2}} \widetilde{f}(x, \cdot)||^2 \, dx \right)^{\frac{1}{2}}$) for a more general term of the form (2).

The same motivation (a smoothing effect due to a nonlocal term) is used in the integral term in (7). The same convolution type example, as just discussed, can serve as a natural example of this effect, i.e. for any Lipschitz function $g : R \to [0, 1]$ consider

$$\eta(\varphi) = \int_{-r}^{0} g \left(\int_\Omega \varphi(\theta, y)\ell(y) \, dy \right) d\theta.$$

One can easily check that

$$|\eta(\varphi) - \eta(\psi)| \leq L_g \int_{-r}^{0} |\langle \varphi(\theta) - \psi(\theta), \ell \rangle| \, d\theta \leq L_g \int_{-r}^{0} ||A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))|| \cdot ||A^{\frac{1}{2}}\ell|| \, d\theta$$

$$\leq L_\eta \left(\int_{-r}^{0} ||A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))||^2 \, d\theta \right)^{\frac{1}{2}}, \quad \text{with} \quad L_\eta \equiv L_g ||A^{\frac{1}{2}}\ell|| r^{\frac{1}{2}}.$$

From the point of view of applications (the biological ones mentioned in the introduction) it is also natural to have the nonlinearity and delay function of the nonlocal type (for a detailed discussion of the nonlocality of the delay terms see, e.g. [4] and a survey [10] and references therein). □

Let now the following space

$$\mathcal{L} \equiv \left\{ \varphi \in C([-r, 0]; D(A^{-\frac{1}{2}})) \mid \sup_{s \neq t} \left\{ \frac{||A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))||}{|s - t|} \right\} < +\infty; \, \varphi(0) \in D(A^{\frac{1}{2}}) \right\}, \quad (9)$$

with the natural norm

$$||\varphi||_{\mathcal{L}} \equiv \max_{s\in[-r,0]} ||A^{-\frac{1}{2}}\varphi(s)|| + \sup_{s\neq t}\left\{\frac{||A^{-\frac{1}{2}}(\varphi(s)-\varphi(t))||}{|s-t|}\right\} + ||A^{\frac{1}{2}}\varphi(0)|| \quad (10)$$

be defined. For any segment $[a,b]\subset\mathbb{R}$ (c.f. (9)) and any Lipschitz-on-$[a,b]$ function φ, let

$$|||\varphi|||_{[a,b]} \equiv \sup\left\{\frac{||A^{-\frac{1}{2}}(\varphi(s)-\varphi(t))||}{|s-t|} : s\neq t;\ s,t\in[a,b]\right\} \quad (11)$$

denote its Lipschitz constant and let $|||\varphi||| \equiv |||\varphi|||_{[-r,0]}$. Then the following lemma holds.

Lemma 2.1. *Let the assumptions (H1.η) and (H.B) hold (see (7), (8)) and let the function $b:\mathbb{R}\to\mathbb{R}$ be Lipschitz. Then any two functions $\varphi\in\mathcal{L},\psi\in H$ (with H and $\mathcal{L}$ defined in (6) and (9)) the nonlinearity F satisfies*

$$||F_1(\varphi)-F_1(\psi)|| \le L_{F_1}\left[|||\varphi|||\right]\left(q\,||A^{1/2}(\varphi(0)-\psi(0))|| + ||A^{-1/2}(\varphi-\psi)||_C\right), \quad (12)$$

where

$$L_{F_1}[\ell] \equiv L_b L_B\sqrt{2}\max\left\{1;\ell L_\eta \max\{1;\sqrt{r}\}\right\} \quad (13)$$

and $L_{F_1}[\ell]$ is used in (12) with

$$\ell = L_\varphi \equiv |||\varphi||| \equiv \sup\left\{\frac{||A^{-1/2}(\varphi(s)-\varphi(t))||}{|s-t|} : s\neq t;\ s,t\in[-r,0]\right\}.$$

Proof of Lemma 2.1. Using the Lipschitz property of b and B (see **(H.B)**), it follows that

$$\begin{aligned}
||F_1(\varphi)-F_1(\psi)||^2 &= \int_\Omega |b([B\varphi](-\eta(\varphi),x)) - b([B\psi](-\eta(\psi),x))|^2\,dx \le \\
&\le L_b^2\int_\Omega |[B\varphi](-\eta(\varphi),x) - [B\psi](-\eta(\psi),x)|^2\,dx \\
&= L_b^2||[B\varphi](-\eta(\varphi),\cdot) - [B\psi](-\eta(\psi),\cdot)||^2 \le \\
&\le L_b^2L_B^2\,||A^{-1/2}\{\varphi(-\eta(\varphi)) - \psi(-\eta(\psi)) \pm \varphi(-\eta(\psi))\}||^2 \le \\
&\le 2L_b^2L_B^2\left(||A^{-1/2}\{\varphi(-\eta(\varphi)) - \varphi(-\eta(\psi))\}||^2 + ||A^{-1/2}(\varphi-\psi)||_C^2\right).
\end{aligned}$$

Next, $\varphi\in\mathcal{L}$ implies that there exists $L_\varphi \equiv |||\varphi||| > 0$, (see (10),(11)) such that

$$||A^{-1/2}(\varphi(s^1)-\varphi(s^2))|| \le L_\varphi|s^1-s^2|, \quad \forall s^1,s^2\in[-r,0]. \quad (14)$$

Hence, (14) and (H1.η) give

$$||F_1(\varphi)-F_1(\psi)||^2 \le$$

$$\leq 2L_b^2L_B^2\left[L_\varphi^2L_\eta^2\left(q\,||A^{1/2}(\varphi(0)-\psi(0))||^2+\int_{-r}^{0}||A^{-\frac{1}{2}}(\varphi(\theta)-\psi(\theta))||^2\,d\theta\right)+\right.$$
$$\left.+\,||A^{-\frac{1}{2}}\,(\varphi-\psi)\,||_C^2\right]\leq$$
$$\leq 2L_b^2L_B^2\left[L_\varphi^2L_\eta^2\left(q\,||A^{1/2}(\varphi(0)-\psi(0))||^2+r\,||A^{-\frac{1}{2}}(\varphi-\psi)||_C^2\,d\theta\right)+\right.$$
$$\left.+\,||A^{-\frac{1}{2}}\,(\varphi-\psi)\,||_C^2\right]\leq$$
$$\leq 2L_b^2L_B^2\max\left\{1;L_\varphi^2L_\eta^2\max\{1;r\}\right\}\left[q\,||A^{1/2}(\varphi(0)-\psi(0))||^2+||A^{-\frac{1}{2}}\,(\varphi-\psi)\,||_C^2\right].$$

The last estimate and using the formulas $\sqrt{\max\{|a|;|b|\}}=\max\{\sqrt{|a|};\sqrt{|b|}\}$ and $\sqrt{a^2+b^2\}}\leq|a|+|b|$ give (12), (13), which completes the proof. □

3. The existence and uniqueness of solutions.

As in [30] we need the following

Definition 3.1. A vector-function $u(t)\in C([-r,T];D(A^{-1/2}))\cap C([0,T];D(A^{1/2}))\cap L^2(0,T;D(A))$ with derivative $\dot u(t)\in L^\infty(0,T;D(A^{-1/2}))$ is a solution to the problem defined by (1) and (5) on $[0,T]$ if

(a) $u(\theta)=\varphi(\theta)$ for $\theta\in[-r,0]$;
(b) $\forall v\in L^2(0,T;L^2(\Omega))$ such that $\dot v\in L^2(0,T;D(A^{-1}))$ and $v(T)=0\Rightarrow$

$$-\int_0^T\langle u(t),\dot v(t)\rangle\,dt+\int_0^T\langle A^{1/2}u(t),A^{1/2}v(t)\rangle\,dt=$$
$$=\langle\varphi(0),v(0)\rangle+\int_0^T\langle F_1(u_t)-d\cdot u(t),v(t)\rangle\,dt.\quad(15)$$

Now we prove the following theorem on the existence and uniqueness of solutions.

Theorem 3.2. *Let the assumptions (H1.η) and (H.B) hold and let the function $b:\mathbb{R}\to\mathbb{R}$ be Lipschitz and bounded, i.e. $|b(s)|\leq M_b$ for all $s\in\mathbb{R}$. Let further $\varphi\in\mathcal{L}$ be a given initial condition. Then the problem defined by (1) and (5) has a unique solution on any time interval $[0,T]$ such that $\dot u\in L^2(0,T;L^2(\Omega))$.*

Remark 3. Notice that φ does not assume $\varphi\in L^2([-r,0];D(A))$. However, the definition of a solution implies that

$$u_t\in L^2([-r,0];D(A)),\quad\forall t\geq r.\quad(16)$$

□

Proof. We follow the proof of Theorem 1 in [30]. Notice that the assumption (H1.η) is slightly more general than the assumption (H.η) in [30]. This implies some changes in the proof of the uniqueness of solutions.

Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis of $L^2(\Omega)$ such that $Ae_k=\lambda_ke_k$, $0<\lambda_1<\ldots<\lambda_k\to+\infty$ and consider the Galerkin approximate solution $u^m=u^m(t,x)=\sum_{k=1}^m g_{k,m}(t)e_k$ of order m such that

$$\begin{cases}\langle\dot u^m+Au^m+du^m-F_1(u_t^m),e_k\rangle=0,\\ \langle u^m(\theta),e_k\rangle=\langle\varphi(\theta),e_k\rangle,\ \forall\theta\in[-r,0]\end{cases}\quad(17)$$

$\forall k = 1, \ldots, m$, $g_{k,m} \in C^1(0,T;\mathbb{R}) \cap L^2(-r,T;\mathbb{R})$ with $\dot{g}_{k,m}(t)$ absolutely continuous.

The system (17) is a system of (ordinary) differential equations in $\mathbb{R}^m$ with a concentrated (discrete) state-dependent delay for the unknown vector function $U(t) \equiv (g_{1,m}(t), \ldots, g_{m,m}(t))$ (for the corresponding theory see [38, 39] and also a recent review [15]).

The key difference between equations with state-dependent and state-independent (concentrated) delays is that the first type of equations is not well-posed in the space of continuous (initial) functions. To get a well-posed initial value problem, it is better [38, 39, 15] to use a smaller space of Lipschitz continuous functions or even a smaller subspace of $C^1([-r,0];\mathbb{R}^m)$.

The condition $\varphi \in \mathcal{L}$ implies that the function $U(\cdot)|_{[-r,0]} \equiv P_m\varphi(\cdot)$, which defines initial data, is Lipschitz continuous as a function from $[-r,0]$ to $\mathbb{R}^m$. Here P_m is the orthogonal projection onto the subspace $span\,\{e_1, \ldots, e_m\} \subset L^2(\Omega)$. Hence, we can apply the theory of ODEs with discrete state-dependent delay (see e.g. [15]) to get the local existence and uniqueness of solutions to (17).

Next, we will get an a priory estimate to prove the continuation of solutions u^m to (17) on any time interval $[0,T]$ and then use it for the proof (by the method of compactness, see [19]) of the existence of strong solutions to (1) and (5). To that end, multiply the first equation in (17) by $\lambda_k g_{k,m}$ and sum for $k = 1, \ldots, m$ to get

$$\frac{1}{2}\frac{d}{dt}||A^{1/2}u^m(t)||^2 + ||Au^m(t)||^2 + d \cdot ||A^{1/2}u^m(t)||^2 = \langle P_m F(u_t^m), Au^m(t)\rangle \leq$$
$$\leq \frac{1}{2}||P_m F(u_t^m)||^2 + \frac{1}{2}||Au^m(t)||^2.$$

As the function b is bounded, $||F(u_t^m)||^2 \leq M_b^2|\Omega|$ (here $|\Omega| \equiv \int_\Omega 1\,dx$), which gives

$$\frac{d}{dt}||A^{1/2}u^m(t)||^2 + ||Au^m(t)||^2 \leq M_b^2|\Omega|. \tag{18}$$

Integrating (18) with respect to t and using the relationships $\varphi(0) \in D(A^{1/2})$, $u^m(0) = P_m\varphi(0) \in D(A^{1/2})$, $||A^{1/2}u^m(0)|| = ||A^{1/2}P_m\varphi(0)|| \leq ||A^{1/2}\varphi(0)||$, we get an a priory estimate

$$||A^{1/2}u^m(t)||^2 + \int_0^t ||Au^m(\tau)||^2\,d\tau \leq ||A^{1/2}\varphi(0)||^2 + M_b^2|\Omega|\,T, \quad \forall m, \forall t \in [0,T]. \tag{19}$$

The above relationship (19) means that

$$\{u^m\}_{m=1}^\infty \text{ is a bounded set in } L^\infty(0,T;D(A^{1/2})) \cap L^2(0,T;D(A)).$$

Using this fact and (17), it follows that

$$\{\dot{u}^m\}_{m=1}^\infty \text{ is a bounded set in } L^\infty(0,T;D(A^{-1/2})) \cap L^2(0,T;L^2(\Omega)).$$

Hence, the family $\{(u^m;\dot{u}^m)\}_{m=1}^\infty$ is a bounded set in

$$Z_1 \equiv \left(L^\infty(0,T;D(A^{1/2})) \cap L^2(0,T;D(A))\right) \times$$
$$\times \left(L^\infty(0,T;D(A^{-1/2})) \cap L^2(0,T;L^2(\Omega))\right). \tag{20}$$

Therefore, there exist a subsequence $\{(u^k;\dot{u}^k)\}$ and an element $(u;\dot{u}) \in Z_1$ such that

$$\{(u^k;\dot{u}^k)\} \text{ *-weak converges to } (u;\dot{u}) \text{ in } Z_1. \tag{21}$$

The proof that any *-weak limit is a strong solution is standard. To prove the property $u(t) \in C([0,T]; D(A^{1/2}))$, we use the well-known (see also [18, thm. 1.3.1])

Theorem 3.3. *(Proposition 1.2 in [31]). Let V denote a dense Banach space that is continuously embedded in a Hilbert space X and let $X = X^*$ so that $V \hookrightarrow X \hookrightarrow V^*$. Then the Banach space $W_p(0,T) \equiv \{u \in L^p(0,T;V) : \dot{u} \in L^q(0,T;V^*)\}$ (here $p^{-1} + q^{-1} = 1$) is contained in $C([0,T]; X)$.*

In our case $X = D(A^{1/2})$, $V = D(A)$, $V^* = L^2(\Omega)$, $p = q = 1/2$.

Now we prove the uniqueness of solutions. Using the fact that $\varphi \in \mathcal{L}$, the definition 3.1 of a solution v, and $\dot{v}(t) \in L^\infty(0,T; D(A^{-1/2}))$ (see (21)), it follows that for any such a solution v and any $T > 0$, there exists $L_{v,T} > 0$, such that

$$||A^{-1/2}(v(s^1) - v(s^2))|| \le L_{v,T}|s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, T]. \tag{22}$$

In the light of (11), let $L_{v,T} \equiv |||v|||_{[-r,T]}$.

Consider any two solutions u and v of (1), (5) (not necessarily with the same initial function). The standard variation-of-constants formula $u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-\tau)} F(u_\tau)\, d\tau$ and the estimate $||A^\alpha e^{-tA}|| \le \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}$ (see e.g. [7, (1.17), p.84]) give

$$\begin{aligned}
&||A^{1/2}(u(t) - v(t))|| \\
&\le e^{-\lambda_1 t}||A^{1/2}(u(0) - v(0))|| + \int_0^t ||A^{1/2}e^{-A(t-\tau)}||\, ||F(u_\tau) - F(v_\tau)||\, d\tau \\
&\le e^{-\lambda_1 t}||A^{1/2}(u(0) - v(0))|| + \int_0^t \left(\frac{1/2}{t-\tau}\right)^{1/2} e^{-1/2}\, ||F(u_\tau) - F(v_\tau)||\, d\tau,
\end{aligned} \tag{23}$$

as $||A^{1/2}e^{-A(t-\tau)}|| \le \left(\frac{1/2}{t-\tau}\right)^{1/2} e^{-1/2}$ and similarly,

$$||A^{-1/2}(u_t - v_t)||_C \le ||A^{-1/2}(u_0 - v_0)||_C + \int_0^t ||F(u_\tau) - F(v_\tau)||\, d\tau.$$

The last estimate and (23) give (just the case when $q = 1$ is shown for the purpose of clarity)

$$\begin{aligned}
&||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C \le e^{-\lambda_1 t}||A^{1/2}(u(0) - v(0))|| + \\
&\quad + ||A^{-1/2}(u_0 - v_0)||_C + \int_0^t \left\{1 + (2e(t-\tau))^{-1/2}\right\} ||F(u_\tau) - F(v_\tau)||\, d\tau.
\end{aligned} \tag{24}$$

It follows, from Lemma 2.1, that

$$||F(u_t) - F(v_t)|| \le L_{F_1,v,T}\left(q\, ||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C\right), \tag{25}$$

where $L_{F_1,v,T}$ is defined in the same way as L_{F_1} in (13), just with $\ell = L_{v,T}$ instead of L_φ - see (13) and (22).

$$L_{F_1,v,T} \equiv L_{F_1}\left[L_{v,T}\right] \equiv L_b L_B \sqrt{2} \max\left\{1; L_{v,T} L_\eta \max\{1; \sqrt{r}\}\right\}. \tag{26}$$

It should be emphasized how the Lipschitz constant $L_{v,T} \equiv |||v|||_{[-r,T]}$ of a strong solution v is taken into account in (26) (see (22) and (11)).

Let

$$g(t) \equiv ||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C.$$

Then the relationships (24) and (25) lead to the following estimate

$$g(t) \le g(0) + \int_0^t \left\{1 + (2e(t-\tau))^{-1/2}\right\} L_{F_1,v,T} \cdot g(\tau)\, d\tau$$

Lemma 3.4 (Gronwall). *Let $u, \alpha \in C[a,b], \beta(t) \ge 0, \beta$ is integrable on $[a,b]$ and*

$$u(t) \le \alpha(t) + \int_a^t \beta(\tau) u(\tau)\, d\tau, \quad a \le t \le b$$

Then

$$u(t) \le \alpha(t) + \int_a^t \beta(\tau)\alpha(\tau) \exp\left\{\int_\tau^t \beta(s)\, ds\right\} d\tau, \quad a \le t \le b$$

Moreover, if α is non-decreasing, then

$$u(t) \le \alpha(t) \exp\left\{\int_a^t \beta(s)\, ds\right\}, \ a \le t \le b.$$

It follows, from the above lemma and equality $\int_0^t (t-\tau)^{-1/2} d\tau = 2t^{1/2}$, that

$$g(t) \le g(0) \exp\left\{L_{F_1,v,T} \int_0^t \left\{1 + (2e(t-s))^{-1/2}\right\} ds\right\} \le$$
$$\le g(0) \exp\left\{L_{F_1,v,T}\left(t + \sqrt{\frac{2t}{e}}\right)\right\},$$

which implies, $\forall t \in [0,T]$, that

$$||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C \le$$
$$\le E_{F_1,v,T}\left(||A^{1/2}(u(0) - v(0))|| + ||A^{-1/2}(u_0 - v_0)||_C\right), \quad (27)$$

where

$$E_{F_1,v,T} \equiv \exp\left\{L_{F_1,v,T} \cdot \left(T + \sqrt{\frac{2T}{e}}\right)\right\}, \quad (28)$$

see (26) for the definition of $L_{F_1,v,T} \equiv L_{F_1}\left[L_{v,T}\right]$. This proves the uniqueness of the solution to (1) and (5), and completes the proof of the theorem. □

4. **Asymptotic properties of solutions.** This section is devoted to studies of the asymptotic behavior of solutions in different functional spaces. We define first (in a standard way) the evolution semigroup $S_t : \mathcal{L} \to \mathcal{L}$ (the space $\mathcal{L}$ is defined in (9)) by the formula

$$S_t\varphi \equiv u_t, \quad t \ge 0, \quad (29)$$

where $u(t)$ is a unique solution to the problem (1) and (5) (see definition 3.1).

The estimate (27) means the continuity of the evolution operator S_t in the norm of the space H (see (6)), i.e.

$$||S_t\varphi - S_t\psi||_H \le E_{F_1,v,T}\,||\varphi - \psi||_H \text{ for all } t \in [0,T]. \quad (30)$$

The aim now is to get a more precise estimate, e.g. the continuity of S_t in the norm of the space $\mathcal{L}$ (see (9), (10)). Consider the definition of the Galerkin approximate solution (see (17)). It gives

$$||A^{-1/2}(\dot{u}^m(t)-\dot{v}^m(t))|| \le ||A^{1/2}(u^m(t)-v^m(t))|| + d||A^{-1/2}(u^m(t)- \\ - v^m(t))|| + ||F_1(u_t^m)-F_1(v_t^m)||$$

and Lemma 2.1 implies

$$||A^{-1/2}(\dot{u}^m(t)-\dot{v}^m(t))|| \le (1+d+L_{F1})\{||A^{1/2}(u^m(t)-v^m(t))||+ \\ + ||A^{-1/2}(u_t^m - v_t^m)||_C\}.$$

An analogous estimate for a solution to the problem (1) and (5), can be obtained from (21) and the following

Theorem 4.1. [45, Chapter V, Theorem 9, p.125] *Let X be a Banach space. Then any *-weak convergent sequence $\{w_k\}_{n=1}^{\infty} \in X^*$ *-weak converges to an element $w_\infty \in X^*$ and $\|w_\infty\|_{X^*} \le \liminf_{n\to\infty} \|w_n\|_{X^*}$.*

More precisely,

$$\text{ess sup}_{t\in[0,T]}||A^{-1/2}(\dot{u}(t)-\dot{v}(t))|| \le (1+d+L_{F1}) \sup_{t\in[0,T]} \{||A^{1/2}(u(t)-v(t))||+ \\ + ||A^{-1/2}(u_t - v_t)||_C\}$$

The last estimate and relationship (27) imply

$$\text{ess sup}_{t\in[0,T]}||A^{-1/2}(\dot{u}(t)-\dot{v}(t))|| \le \\ \le (1+d+L_{F1})E_{F_1,v,T}\left(||A^{1/2}(u(0)-v(0))|| + ||A^{-1/2}(u_0-v_0)||_C\right) \quad (31)$$

Hence, see (11),

$$|||u-v|||_{[0,T]} \le (1+d+L_{F1})E_{F_1,v,T}\left(||A^{1/2}(u(0)-v(0))|| + ||A^{-1/2}(u_0-v_0)||_C\right)$$

From that and (27), it follows that

$$||u_t - v_t||_{\mathcal{L}} \le (2+d+L_{F1})E_{F_1,v,T}\,||u_0-v_0||_{\mathcal{L}},\ \forall t\in[0,T], \quad (32)$$

which finally means that for any $T \ge 0$ there exists a constant $C_T > 0$ such that $\forall t \in [0,T]$ it gives

$$||u_t - v_t||_{\mathcal{L}} = ||S_t\varphi - S_t\psi||_{\mathcal{L}} \le C_T||\varphi-\psi||_{\mathcal{L}},\ \forall \varphi,\psi \in \mathcal{L} \quad (33)$$

The last inequality means the continuity of the evolution operator S_t in the norm of the space $\mathcal{L}$ (see (29) and compare with (30)).

Remark 4. It should be noted that the evolution operator and, more generally, the time-shift is not a (strongly) continuous mapping in the norm of the space $\mathcal{L}$ (see (9)). This can be illustrated by the following simple (scalar) example.
Consider the space

$$\mathcal{L}ip\,([-r,T];\mathbb{R}) \equiv \left\{v:[-r,T]\to\mathbb{R} : \sup\left\{\frac{|v(s)-v(t)|}{|s-t|}, s\ne t; s,t\in[-r,T]\right\} < \infty\right\}$$

and analogously define the space $\mathcal{L}ip\,([-r,0];\mathbb{R})$ with the natural norm

$$||v||_{\mathcal{L}ip} \equiv \max_{\theta\in[-r,0]}|v(\theta)| + \sup\left\{\frac{|v(s)-v(t)|}{|s-t|}, s\ne t;\ s,t\in[-r,0]\right\}.$$

The (strong) continuity of the time-shift means that

$$\forall v \in \mathcal{L}ip\,([-r,T];\mathbb{R}) \text{ and } \forall t \in [0,T] \Longrightarrow \lim_{h\to 0} ||v_{t+h} - v_t||_{\mathcal{L}ip} = 0. \qquad (34)$$

Obviously, when $t = 0$ one considers $h \to 0^+$, while for $t = T$, the case $h \to 0^-$ should be investigated.

To prove the claim, we must show that (34) does not hold, i.e.

$$\exists v \in \mathcal{L}ip\,([-r,T];\mathbb{R}) \text{ and } \exists t_0 \in [0,T] \quad \text{for which} \quad \lim_{h\to 0} ||v_{t_0+h} - v_{t_0}||_{\mathcal{L}ip} \neq 0. \quad (35)$$

Thus, consider the case $t_0 = 0, h \to 0^+$ and the function

$$v(t) \equiv \begin{cases} 0, & t \in [-r,0] \\ t, & t \in (0,T] \end{cases}.$$

It can be seen that $v_{t_0} = v_0 \equiv 0$ and

$$v_{t_0+h} = v_{t_0+h}(\theta) = \begin{cases} 0, & \theta \in [-r,-h] \\ h+\theta, & \theta \in (-h,0] \end{cases}.$$

Hence, $||v_{t_0+h} - v_{t_0}||_{\mathcal{L}ip} = ||v_{t_0+h}||_{\mathcal{L}ip} = h+1$ and finally $\lim\limits_{h\to 0+} ||v_{t_0+h} - v_{t_0}||_{\mathcal{L}ip} = \lim\limits_{h\to 0+} (h+1) = 1 \neq 0$, which means that (34) does not hold. In the space $\mathcal{L}$, we would proceed analogously. □

Remark 5. In the same way as in the previous remark one can show that the time-shift is **not** a (strongly) continuous mapping in the topology of $L^\infty(-r,0)$. One could consider the function $\widetilde{v}(t) \equiv \begin{cases} 0, & t \in [-r,0] \\ 1, & t \in (0,T] \end{cases}$ and $t_0 = 0$ to show that $\lim\limits_{h\to 0+} ||\widetilde{v}_h - \widetilde{v}_0||_{L^\infty(-r,0)} = 1 \neq 0$. By the way, $\widetilde{v} = \frac{d}{dt}v$, where, as usually, the time-derivative is understood in the sense of distributions. □

The above remarks show that despite of the existence and uniqueness of solutions in the space $\mathcal{L}$ and even strong continuity of the evolution operator S_t in the norm of $\mathcal{L}$ (see (33)), the pair $(S_t;\mathcal{L})$ does *not* form a *dynamical system* since S_t is not strongly continuous as a mapping of time variable.

The methods, developed for ordinary delay equations in [38] suggest to restrict our considerations to a smaller subset of the space of Lipschitz functions. In this paper we follow this suggestion and consider the evolution operator S_t on the following subset of $\mathcal{L}$

$$X \equiv \left\{ \begin{array}{ll} \varphi \in C^1([-r,0];D(A^{-1/2})) & \text{such that} \\ \varphi(0) \in D(A^{1/2})) \text{ and } \dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi) & \end{array} \right\} \subset \mathcal{L}. \qquad (36)$$

Here the equality $\dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi)$ is understood as an equality in $D(A^{-1/2})$.

Remark 6. The set X is an analogue of the *solution manifold* introduced in [38] for the case of ODEs with state-dependent delays. □

To show that the set X is invariant under the evolution operator S_t, we first have to establish an additional smoothness property of the solutions of problem (1), (5).

Lemma 4.2. *For any $\varphi \in C^1([-r,0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$), the solution to (1), (5) (which is given by Theorem 3.2) has the property (c.f. Theorem 3.3 and Theorem 3.2)*

$$\dot{u} \in C([0,T]; D(A^{-1/2})), \quad \forall T > 0. \tag{37}$$

Remark 7. We do not assume $\varphi(0) \in D(A)$, just $\varphi(0) \in D(A^{1/2})$, so we cannot directly use [24, Theorem 3.5, p.114].

Proof of Lemma 4.2. By Theorems 3.3, 3.2, for any $\varphi \in C^1([-r,0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$), there exists a unique solution $u(t) \in C([-r,T]; D(A^{-1/2})) \cap C([0,T]; D(A^{1/2}))$. This property and Lemma 2.1 then imply the continuity of the function

$$p(t) \equiv F_1(u_t) \in C([0,T]; L^2(\Omega)). \tag{38}$$

Consider the following auxiliary *linear system without delay*

$$\begin{cases} \dot{v}(t) + Av(t) + dv(t) = p(t), \quad t \geq 0, \\ v(0) = \varphi(0) \in D(A^{1/2}) \end{cases} \tag{39}$$

In the same way as in (17), the Galerkin approximate solution $v^m = v^m(t,x) = \sum_{k=1}^m g_{k,m}(t)e_k$ of order m to (39) can be defined such that

$$\begin{cases} \langle \dot{v}^m + Av^m + dv^m - p(t), e_k\rangle = 0, \quad t \geq 0, \\ \langle v^m(0), e_k\rangle = \langle \varphi(0), e_k\rangle, \quad \forall k = 1,\dots,m. \end{cases} \tag{40}$$

where $g_{k,m} \in C^1(0,T;\mathbb{R}) \cap L^2(-r,T;\mathbb{R})$ and $\dot{g}_{k,m}(t)$ is absolutely continuous.

The difference between approximate solutions u^m and v^m lies in that v^m are solutions just to *linear* system (40). So, for any two approximate solutions v^n and v^m (solutions to (40) of different orders n and m), one has $g_{k,n}(t) \equiv g_{k,m}(t)$, which is denoted by $g_k(t)$.

Multiply (40) by $\lambda_k g_k$ and sum for $k = n+1, ..., n+p$ (p is any positive integer) to get

$$\langle \dot{v}^{n+p}(t) - \dot{v}^n(t), A(v^{n+p}(t) - v^n(t))\rangle + ||A(v^{n+p}(t) - v^n(t))||^2 +$$
$$+ d\langle v^{n+p}(t) - v^n(t), A(v^{n+p}(t) - v^n(t))\rangle = \langle (P_{n+p} - P_n)p(t), A(v^{n+p}(t) - v^n(t))\rangle$$

It should be recalled that, see the proof of Theorem 3.2, P_m is the orthogonal projection onto the subspace *span* $\{e_1,\dots,e_m\} \subset L^2(\Omega)$. Hence,

$$\frac{1}{2}\frac{d}{dt}||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 + ||A(v^{n+p}(t) - v^n(t))||^2 + d||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 \leq$$
$$\leq ||(P_{n+p} - P_n)p(t)|| \cdot ||A(v^{n+p}(t) - v^n(t))|| \leq \frac{1}{2}||(P_{n+p} - P_n)p(t)||^2 +$$
$$+ \frac{1}{2}||A(v^{n+p}(t) - v^n(t))||^2$$

which gives

$$\frac{d}{dt}||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 + ||A(v^{n+p}(t) - v^n(t))||^2 \leq ||(P_{n+p} - P_n)p(t)||^2.$$

Integrating the last estimate results ($\forall t \in [0,T]$) in

$$||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 + \int_0^t ||A(v^{n+p}(\tau) - v^n(\tau))||^2 \, d\tau \le$$
$$\le ||A^{\frac{1}{2}}(v^{n+p}(0) - v^n(0))||^2 + \int_0^t ||(P_{n+p} - P_n)p(\tau)||^2 \, d\tau \le$$
$$\le ||(P_{n+p} - P_n)A^{\frac{1}{2}}\varphi(0))||^2 + \int_0^T ||(P_{n+p} - P_n)p(\tau)||^2 \, d\tau.$$

Summing up, the above estimate, the fact that $\varphi(0) \in D(A^{1/2}))$, the strong convergence $||I - P_n|| \to 0$ for $n \to \infty$, and (38) imply that

$$\textit{the sequence } \{v^n\}_{n=1}^\infty \textit{ is a Cauchy sequence in } C([0,T]; D(A^{\frac{1}{2}})). \tag{41}$$

Now our goal is to show that the sequence $\{\dot{v}^n\}_{n=1}^\infty$ is a Cauchy sequence in $C([0,T]; D(A^{-1/2}))$. So, multiply first (40) by $\lambda_k^{-\frac{1}{2}}$ to get $\lambda_k^{-\frac{1}{2}}\dot{g}_k(t) = -\lambda_k^{\frac{1}{2}}g_k(t) - d\lambda_k^{-\frac{1}{2}}g_k(t) + \langle\lambda_k^{-\frac{1}{2}}p(t), e_k\rangle$. This gives $\lambda_k^{-1}(\dot{g}_k(t))^2 \le 3\lambda_k(g_k(t))^2 + 3d^2\lambda_k^{-1}(g_k(t))^2 + 3|\langle\lambda_k^{-\frac{1}{2}}p(t), e_k\rangle|^2$. The sum for $k = n+1, ..., n+p$ reads

$$||A^{-\frac{1}{2}}(\dot{v}^{n+p}(t) - \dot{v}^n(t))||^2 \le 3||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 +$$
$$+ 3d^2||A^{-\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 + \frac{3}{\lambda_{n+1}}||(P_{n+p} - P_n)p(t)||^2 \le$$
$$\le 3\left(1 + \frac{d^2}{\lambda_{n+1}^2}\right)||A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))||^2 + \frac{3}{\lambda_{n+1}}||I - P_n||^2||p(t)||^2.$$

The last estimation together with (41) give that

$$\text{the sequence } \{\dot{v}^n\}_{n=1}^\infty \text{ is a Cauchy sequence in } C([0,T]; D(A^{-\frac{1}{2}})). \tag{42}$$

Thus, there exists a unique solution $v(t)$ ($v \equiv \lim_{n\to\infty} v^n$) to the linear system (39), which satisfies $v \in C([0,T]; D(A^{\frac{1}{2}}))$ and $\dot{v} \in C([0,T]; D(A^{-\frac{1}{2}}))$.

On the other hand, the nonlinear delay system (1), (5) with the initial function φ has also a unique solution. From the construction of $p(t)$ (see (38)), it follows that $u(t) \equiv v(t)$ for all $t \in [0,T]$, which gives (37) and completes the proof of Lemma 4.2. □

Lemma 4.2 particularly shows that the set X, defined by (36), is invariant under the evolution operator S_t (see (29)). This fact allows to define an evolution operator (denoted again by S_t) $S_t : X \to X$ in the same way as in (29). Now, if the natural norm

$$||\varphi||_X \equiv \max_{s\in[-r,0]} ||A^{-1/2}\varphi(s)|| + \max_{s\in[-r,0]} ||A^{-1/2}\dot{\varphi}(s)|| + ||A^{1/2}\varphi(0)||$$

on X is taken into account, then Theorem 3.2, Lemma 4.2, and Theorem 3.3 give the continuity of S_t with respect to t in the norm of X. Hence, $(S_t; X)$ defines a dynamical system.

Now we will pay attention to the long-time asymptotic behavior of the constructed evolution semigroup $S_t : X \to X$.

Theorem 4.3. *Using the above notation and under the assumptions of Theorem 3.2, the dynamical system (S_t, X) is dissipative. If, in addition, $q = 0$ in*

(H1.η), *then* (S_t, X) *possesses a compact global attractor* $\mathcal{A}$, *which is a bounded set in the space* $C^1([-r,0];D(A^{-1/2}))\cap C([-r,0];D(A^{\alpha}))$, $\alpha\in(\frac{1}{2},1)$.

Proof of Theorem 4.3. It will be shown first that (S_t, X) is a dissipative dynamical system. To that end, the below proposition is needed.

Lemma 4.4. [30, Lemma 1] *Let all the assumptions of Theorem 3.2 hold and let* $\alpha\in(\frac{1}{2},1)$. *Then there exists a bounded subset* $\mathcal{B}V_\alpha$ *of the space* $C^1([-r,0];D(A^{-\frac{1}{2}}))\cap C([-r,0];D(A^{\alpha}))$, *which absorbs any strong solution to the problem (1) and (5) for any initial function* $\varphi\in\mathcal{L}$.

Second, to apply the classical theorem on the existence of a global attractor (see, for example [2, 35, 7]), we show that (S_t, X) is asymptotically compact. Consider therefore any solution $u(t)$ to the problem (1) and (5) with $\varphi\in\mathcal{B}V_\alpha$ as an initial function. We will show that for any $\delta>r>0$ and any $T>\delta$ the set $\mathcal{U}\equiv\{u_t=S_t\varphi\,|\,\varphi\in\mathcal{B}V_\alpha,\ t\in[\delta,T]\}$ is relatively compact in X.

Recall that the set $\mathcal{B}V_\alpha$ is a ball in $C^1([-r,0];D(A^{-1/2}))\cap C([-r,0];D(A^{\alpha}))$ (for more details see [30]) and notice that, by Corollary 4 from [32], the set $\mathcal{B}V_\alpha$ is relatively compact in $C([-r,0];D(A^{-1/2}))$ (see also [32, lemma 1]). It remains to show that $\{\dot u(t)\,|\,\varphi\in\mathcal{B}V_\alpha,\ t\in[\delta-r,T]\}$ is equi-continuous in $C([\delta-r,T];D(A^{-1/2}))$.

Theorem 4.5. [24, Corollary 4.3.3 and Theorem 4.3.5]. *Let A be an infinitesimal generator of an analytic semigroup* $\{T(t)\}_{t\geq 0}$. *If* $f\in L^1((0,T);Y)$ *is locally Hölder continuous on* $(0,T]$, *then for every* $x\in Y$ *the initial value problem*

$$\dot u(t)=Au(t)+f(t), t>0;\quad u(0)=x$$

has a unique solution u. *If* $f\in C^\theta([0,T];Y)$, *then for every* $\delta>0$, $Au\in C^\theta([\delta,T];Y)$ *and* $\dot u\in C^\theta([\delta,T];Y)$.

Here $C^\theta([0,T];Y)$ denotes the family of all Hölder continuous functions on $[0,T]$ with the exponent $\theta\in(0,1)$. In this case, $Y=L^2(\Omega)$.

In order to apply Theorem 4.5 to our case, we have to show that $p(t)=F_1(u_t)\in C^\theta([\delta-r,T];L^2(\Omega))$ (c.f. (38),(39)). Therefore, consider $t\in[\delta-r,T]$ and

$$\begin{aligned}||p(t+h)-p(t)||&=||F_1(u_{t+h})-F_1(u_t)||\leq\\ &\leq L_{F_1}[\ell_{\mathcal{B}V_\alpha}]\max_{s\in[-r,0]}||A^{-1/2}(u(t+h+s)-u(t+s))||\leq L_{F_1}[\ell_{\mathcal{B}V_\alpha}]\,\ell_{\mathcal{B}V_\alpha}|h|\end{aligned}$$

where $L_{F_1}[\ell_{\mathcal{B}V_\alpha}]$ is the constant defined in Lemma 2.1 with $\ell_{\mathcal{B}V_\alpha}$ such that $|||\psi|||\leq\ell_{\mathcal{B}V_\alpha}$ $\forall\psi\in\mathcal{B}V_\alpha$ (the existence of such $\ell_{\mathcal{B}V_\alpha}$ follows from Theorem 4.1). Here, $q=0$ is used.

The last inequality shows that $p:[\delta-r,T]\to L^2(\Omega)$ is Lipschitz continuous, which is the situation to which Lemma 4.4 can be applied. It should also be noted that the family $\{p(t)\}$, for all initial $\varphi\in\mathcal{B}V_\alpha$, is uniformly Lipschitz, i.e. all the Lipschitz constants are lower or equal to $L\equiv L_{F_1}[\ell_{\mathcal{B}V_\alpha}]\cdot\ell_{\mathcal{B}V_\alpha}$. Then by Theorem 4.1, it is guaranteed (see the proof) that the family $\{\dot u(t)\,|\,\varphi\in\mathcal{B}V_\alpha,\ t\in[\delta-r,T]\}$ is uniformly Hölder continuous, and thus equi-continuous in $C([\delta-r,T];L^2(\Omega))$.

Theorem 4.6. [32, lemma 1] *Let B be a Banach space. A set F of* $C([0,T];B)$ *is relatively compact if and only if*

(i) $F(t)\equiv\{f(t):f\in F\}$ *is relatively compact in* B, $0<t<T$,

(ii) F is uniformly equicontinuous, i.e. $\forall \varepsilon > 0, \exists \eta$ *such that* $\|f(t_2) - f(t_1)\|_B \leq \varepsilon, \forall f \in F, \forall 0 \leq t_1 \leq t_2 \leq T$ *such that* $|t_2 - t_1| \leq \eta$

Applying Theorem 4.6 completes the proof of Theorem 4.3. □

In all the applications considered above (Mackey-Glass-type equations with diffusion, the Lasota-Wazewska-Czyzewska model in hematology and the delayed diffusive Nicholson's blowflies equation) the functions b are bounded and globally Lipschitz, so for any delay function η satisfying $(H1.\eta)$ and any (nonlocal) operator B, satisfying $(H.B)$ (as discussed in Remark 2), the conditions of Theorem 3.2 and Theorem 4.3 are satisfied. As a result, we conclude that the initial value problem (1) and (5) is well-posed in X and the dynamical system (S_t, X) has a global attractor (Theorem 4.3).

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