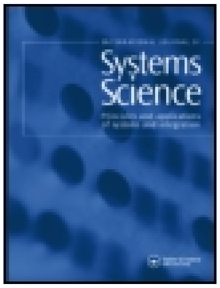

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The existence of a common quadratic Lyapunov function (CQLF) for a switched linear system guarantees its global asymptotic stability. Even if progress in finding the conditions for the existence/non-existence of a CQLF is significant, especially in switched linear systems consisting of N second-order systems or two systems of order n, the general case of N systems of order n still remains open. In this article, a sufficient condition for the non-existence of a CQLF for N systems of order n is derived. Based on the condition, a new method for determining the non-existence of a CQLF, using particle swarm optimisation, was designed and is described. Examples illustrating the proposed method are introduced at the end of this article.

Keywords: switched linear systems; stability; common quadratic Lyapunov function; particle swarm optimisation

1. Introduction

Linear dynamical systems that evolve by switching among several matrices of evolution via a commutation rule are called *switched linear systems* (SLS's). Such systems can be defined, as in Shorten, Wirth, Mason, Wulff, and King (2007), as follows:

$$\dot{x}(t) = A_{\varsigma(t)}x(t), \quad A_{\varsigma(t)} \in \mathrm{A} := \{A_1, A_2, \ldots, A_N\}, \quad (1)$$

where $A_i \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, N$ are constant and Hurwitz matrices (eigenvalues in the open left half-plane of the complex plane). The value of the $A_{\varsigma(t)}$ matrix depends on the value of the function $\varsigma(t)$: $[0, \infty) \to \{1, 2, \ldots, N\}$, called the *switching signal*, which is a piecewise constant function with a finite number of discontinuities (*switching times*) in any bounded time interval. Thus, $A_{\varsigma(t)}$ takes constant values from the set A on every interval defined between two consecutive switching times (Shorten et al. 2007; Lin and Antsaklis 2009).

It is known that the subsystems' stability does not guarantee the stability of the whole switched system under arbitrary switching (Figure 1), and therefore it is necessary to analyse the stability of the whole switched system. In the case of SLS's, an important stability issue is related to the existence of common Lyapunov functions. More particularly, the existence of a common quadratic Lyapunov function (CQLF) for a set of linear systems forming an SLS guarantees the stability of the switched system under an arbitrary switching rule (Liberzon 2003; Lin and Antsaklis 2009). Some application examples of the CQLF approach may be found in Cheng and Zhang (2006) for continuous time and Benzaouia, Akhrif, and Saydy (2010) and Benzaouia, Hmamed, Tadeo, and Hajjaji (2011) for discrete time.

There are several studies concerning the existence/non-existence conditions of a CQLF (Lin and Antsaklis 2009). However, as far as the problem of determining general conditions for the existence/non-existence of a CQLF, and their calculation by an effective method are concerned, the problem has not been solved completely to date.

In Narendra and Balakrishnan (1994) the general case, when the Lie bracket is equal to zero, is studied and it is shown that the commutativity of two matrices is a sufficient condition for their sharing of a CQLF. The analysis presented therein has resulted in an algorithm (referred here to as the NB-Algorithm) for finding such a CQLF. This algorithm has also been successfully applied to (1) where the N matrices commute pairwise. Later in Zhu, Cheng, and Qin (2007), the scope of the NB-Algorithm is expanded to the case in which the Lie brackets are represented by linear combinations of their arguments, which of course imposes certain conditions upon the combination coefficients.

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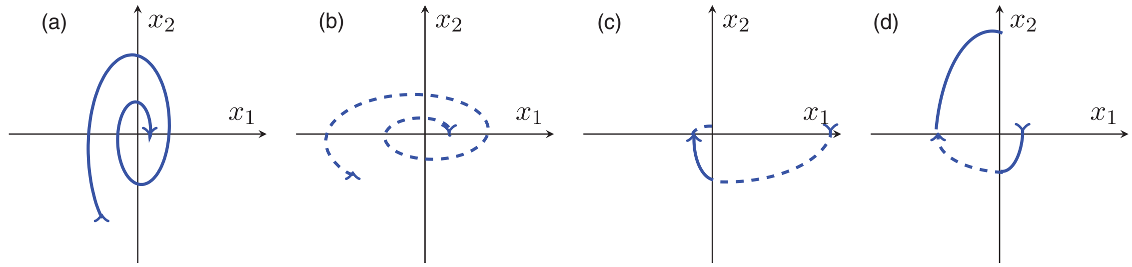

Figure 1. Example of phase diagrams of the switching between two stable second-order systems (Liberzon 2003): (a) first system; (b) second system; (c) stable switching; (d) unstable switching.

In general, establishing conditions for the existence/non-existence of a CQLF usually begins with an analysis of a pair of matrices and then tries to extend the obtained results to a set of N matrices. Approximately a decade ago, a promising conjecture regarding the positive systems was presented in Mason and Shorten (2003). Unfortunately, a recent work (Gurvits, Shorten, and Mason 2007) shows that the conjecture is not valid in the general case; it has been verified as valid just in the case of N positive systems of dimension 2×2. The SLS's formed by triangular or simultaneously triangularisable subsystems (Mori, Mori, and Kuroe 1997; Shorten and Narendra 1998; Ibeas and De la Sen 2009) are examples of a sufficient condition for the existence of a CQLF, which are non-restrictive with respect to the order or the number of systems to be analysed, but they form quite restrictive conditions with respect to the systems' structure, or existence of a common transformation matrix, respectively.

One of the most important results concerning the CQLF problem is the general solution for N second-order systems (Shorten and Narendra 2002), except for the constraint $a_{21i} \neq 0 \ \forall i \in \{1, \ldots, N\}$. In Shorten and Narendra (2002) the two main authors' results in the area are shown. First, a general and elegant solution to the case of pairs of matrices is shown; this first result is based on the stability analysis of the convex linear combination (CLC) of two systems. Second, the case of N systems of second order is approached using Helly's theorem. Although both analyses are based on different approaches, it can be said that the case of second-order systems has complete analytical solution as far as the existence problem of CQLF is concerned.

There are other types of analyses that can be applied to pairs of matrices (see, for instance, Shorten and Narendra 2003; King and Nathanson 2006; Laffey and Smigoc 2009). In King and Nathanson (2006) and Laffey and Smigoc (2009) the pairs of matrices whose difference is of rank 1 are studied. The pairs of matrices that are in the companion form (Shorten and Narendra 2003) also belong to that class of matrices. Nevertheless, in Shorten and Narendra (2003), King and Nathanson (2006) and Laffey and Smigoc (2009) a very simple algebraic sufficient condition ensuring the existence of a pairwise CQLF is established. There are even more restrictive cases, e.g. when the order and number of matrices are fixed (King and Shorten 2006), wherein a necessary and sufficient condition for the non-existence of a CQLF for pairs of third-order systems are obtained by analysing CLCs of their evolution matrices and their inverses. Several other approaches have been used to solve the CQLF problem (Shorten, Narendra, and Mason 2003; Shorten, Mason, Cairbre, and Curran 2004; Shorten et al. 2007; Moldovan and Seetharama 2009), and one can observe that most of the approaches impose certain constrains upon the order of the systems, the number of subsystems or use some other special properties (e.g. the rank of the difference, commutativity).

Under the scenario of numeric solutions, an interesting solution to the general case of N system of order n is presented in Cheng, Guo, and Huang (2003). Therein, a method for determining the existence of a CQLF, which is based on a necessary and sufficient condition related to the positivity of a given integral, is proposed. Even though the method is stated for the general case, only the case of second-order systems is developed and proved in detail. Methods based on numeric optimisation have been designed for the computation of a CQLF. The most relevant are: a method for finding a CQLF based on the resolution of LMI systems (Boyd, El Ghaoui, Feron, and Balakrishnan 1994) by using, e.g., a Matlab toolbox (*LMI approach*); a method for finding a CQLF based on gradient (Liberzon and Tempo 2003) (*L-T approach*) and a method for finding a CQLF based on swarm intelligence (Ordóñez-Hurtado and Duarte-Mermoud 2012) (*O-D approach*). Although the existence of a CQLF is a guarantee of asymptotic stability for a given switched system, it is also important to determine when a CQLF cannot exist. In this sense such approaches fail because a final non-feasible solution does not imply that a feasible solution cannot exist, since the successful of the searching process depends on a good tuning of configuration parameters. However, there exist some methods focused to the numeric determination of the non-existence of a CQLF, such as the work of Davis and Eisenbarth (2011) based on linear programming for the simultaneous solution of polynomial systems of inequalities in second-order systems, and the work of Ordóñez-Hurtado and Duarte-Mermoud (2011) based on swarm intelligence to analyse linear combinations of pairs of systems. These approaches are complementary to the computation methods since they give conclusive information in the cases in which the computation methods do not reach their goal.

In this article a new method for determining the non-existence of a CQLF without any restriction upon the order of the systems and the number of its subsystems is proposed, which basically was obtained by developing the two following stages. First, a sufficient condition for the non-existence of a CQLF is derived from a known necessary condition for the existence of a CQLF for N systems. Then, based on the derived condition, a pair of fitness functions is designed, suitable to be optimised using particle swarm optimisation (PSO). The performance of the proposed method is validated through numerical tests. This article is organised as follows: the statement of the problem is done in Section 2, and a brief summary of the PSO technique is given in Section 3. Section 4 is devoted to explaining the technical details as to how PSO can be applied to the problem under study. Experimental results are presented in Section 5, and a concluding section is presented in Section 6.

2. Problem statement and preliminaries

Henceforth we will denote $V(x)>0$ and $P>0$ as a function or a matrix which is positive definite, and $V(x)\geq 0$ and $P\geq 0$ as a function or a matrix which is positive semidefinite. Similarly, the notation $V(x)<0$ and $P<0$ will be used for a function or a matrix which is negative definite, and $V(x)\leq 0$ and $P\leq 0$ for a function or a matrix which is negative semidefinite.

Consider the (continuous) SLS (1), and let $V(x)>0$ be a quadratic Lyapunov function candidate of the form

$$V(x) = x^T P x, \quad P>0, \quad P \in \mathbb{R}^{n\times n} \tag{2}$$

with negative definite time derivative along any non-zero system trajectory, i.e.

$$\dot{V}(x) = x^T\left(PA_i + A_i^T P\right)x < 0. \tag{3}$$

or equivalently,

$$PA_i + A_i^T P = -Q_i < 0, \quad \forall A_i \in \mathrm{A}. \tag{4}$$

Then, if a matrix $P>0$ satisfying (4) exists, the function $V(x)$ is a CQLF for all individual systems of the form

$$\Sigma_{A_i} : \dot{x}(t) = A_i x(t), \quad i = 1,2,\ldots,N, \quad \forall A_i \in \mathrm{A}, \tag{5}$$

and its existence guarantees uniform asymptotic stability of the SLS (1) under arbitrary switching (Liberzon 2003; Lin and Antsaklis 2009). Some authors (Shorten et al. 2004) distinguish between the existence of P for $Q_i>0$ ($Q_i\geq 0$), which leads to the concept of a *strong* (*weak*) *CQLF*. In this article, the term CQLF will always mean strong CQLF.

In the control literature, the issue of determining conditions for the existence/non-existence of a CQLF has been studied quite extensively, and a complete solution is known in the case of two matrices, say A_1 and A_2, of second order (Shorten and Narendra 2002). The main vehicle to achieve that result is the stability analysis of CLCs of $A_1^{\pm 1}$ and A_2, i.e.

$$\begin{aligned}\sigma_\alpha[A_1, A_2] &:= CLC[A_1, A_2] \\ &:= \alpha A_1 + (1-\alpha)A_2, \quad \alpha \in [0,1],\end{aligned} \tag{6}$$

$$\begin{aligned}\sigma_\alpha\left[A_1^{-1}, A_2\right] &:= CLC\left[A_1^{-1}, A_2\right] \\ &:= \alpha A_1^{-1} + (1-\alpha)A_2, \quad \alpha \in [0,1],\end{aligned} \tag{7}$$

where $\sigma_\alpha[A_i, A_j]$ denotes the *pencil* of A_i and A_j (Shorten and Narendra 2002), and is Hurwitz if its eigenvalues lies in the open left half-plane of the complex plane for all $\alpha\in[0,1]$ (Shorten et al. 2003). For the reader's convenience, we list below the main theorems, lemmas and propositions related to the CQLF problem, which will be used in the sequel.

Lemma 2.1 (Shorten and Narendra 2002)**:** *Let us consider the systems* $\Sigma_A : \dot{x} = Ax$ *and* $\Sigma_{A^{-1}} : \dot{x} = A^{-1}x$ *where* $A\in\mathbb{R}^{n\times n}$ *is Hurwitz. Then, any quadratic Lyapunov function for* Σ_A *is also a quadratic Lyapunov function for* $\Sigma_{A^{-1}}$.

Theorem 2.2 (Shorten and Narendra 2002)**:** *Let the system* (1) *with* $x\in\mathbb{R}^2$ *and* $N=2$ *be given. Then the following statements are equivalent*:

1. *There exists a CQLF for* (1) *with* $\mathrm{A} = \{A_1, A_2\}$.
2. *The pencils* $\sigma_\alpha[A_1, A_2]$ *and* $\sigma_\alpha\left[A_1, A_2^{-1}\right]$ *are Hurwitz.*
3. *The products* A_1A_2 *and* $A_1A_2^{-1}$ *do not have negative real eigenvalues.*

Proposition 2.3 (Liberzon 2003)**:** *The linear systems* $\dot{x} = A_1x$ *and* $\dot{x} = A_2x$, *with* $A_1, A_2\in\mathbb{R}^{2\times 2}$, *share a*

CQLF if and only if all pairwise CLC in $\mathrm{A} = \{A_1, A_2, A_1^{-1}, A_2^{-1}\}$ *are Hurwitz.*

Lemma 2.4 (Shorten et al. 2004)**:** *If the stable LTI systems* $\dot{x} = A_1 x$ *and* $\dot{x} = A_2 x$, *with* $A_1, A_2 \in \mathbb{R}^{n\times n}$, *have a CQLF, then the pencils* $\sigma_\alpha[A_1, A_2]$ *and* $\sigma_\alpha[A_1^{-1}, A_2]$ *are non-singular. Equivalently, the products* $A_1^{-1}A_2$ *and* A_1A_2 *do not have negative real eigenvalues.*

Lemma 2.5 (Horn and Johnson 1985)**:** *Let us consider* $C, D \in \mathbb{R}^{n\times n}$, $C, D > 0$ *and* $\alpha \in \mathbb{R}^+$. *Then* (i) $C + D > 0$ *and* (ii) $\alpha C > 0$.

3. Particle swarm optimisation

PSO (Eberhart and Kennedy 1995a,b; Del Valle, Venayagamoorthy, Mohagheghi, Hernandez, and Harley 2008; Kameyama 2009) is a global optimisation method that computationally emulates the social behaviour of a community such as a school of fish, a flock of birds or even a crowd of people. PSO belongs to the class of modern heuristic techniques based on population, where each potential solution is called a *particle*. Establishing itself as an important technique in evolutionary computation (EC), PSO is responsible for making a population of particles evolve through several iterations (as time elapses), with the aim of finding the best possible overall solution (or at least the best approximation to it). Unlike other EC techniques, such as genetic algorithms (GA), PSO does not implement mutation or crossover operators, since its basic model always keeps the best of its evolutionary experience. The particles have the peculiarity of being able to 'fly' (move) in the multidimensional search space, without forgetting their best found positions, and being influenced by the particle that has found the best overall position. By analogy with social communities, each particle looks for the best source of profit, taking advantage of its own experience from the individual pursuit, and the collective experience resulting from the advancement of the entire swarm.

As the idea of PSO is to simulate particle motion in an d-dimensional space, where d is given by the number of unknowns of the function to be optimised, a set of basic formulas for updating both the speed and position within the search space is required. In the basic PSO (Eberhart and Kennedy 1995b), the particles evolve by means of equations

$$v_{i,d}(k+1) = v_{i,d}(k) + r_1(k)c_1\big(p_{i,d}(k) - x_{i,d}(k)\big) + r_2(k)c_2\big(g_d(k) - x_{i,d}(k)\big) \tag{8a}$$

$$x_{i,d}(k+1) = x_{i,d}(k) + v_{i,d}(k+1), \tag{8b}$$

where $v_{i,d}(k)$ and $x_{i,d}(k)$ represent, respectively, the velocity and position of the component $d \in \{1, 2, \ldots, M\}$ of the particle $i \in \{1, 2, \ldots, S\}$, at the iteration $k \in \{1, 2, \ldots, iter_{\max}\}$. Constants c_1 and c_2 are the coefficients of cognitive and social acceleration, and determine the influence of individual and group experience on the performance of each particle. Terms $r_1(k)$ and $r_2(k)$ are a pair of uniformly distributed random numbers in the range [0, 1] (i.e. $r_{1,2} \sim U[0, 1]$), used to represent the stochastic nature of any social swarm. The variable $p_{i,d}(k)$ is the d-th component of the best position of particle i, and $g_d(k)$ is the d-th component of the best global position, at iteration k.

The algorithm for basic PSO can be summarised in the following steps, in which the fitness function to be optimised is denoted by f:

(1) Initialising: set the vector of particles' positions $\mathbf{x}_i$ and the vector of particles' velocities $\mathbf{v}_i$ with uniformly random distribution; set the vector of individual best positions $\mathbf{p}_i$ and the global best position $\mathbf{g}$ as $\{\mathbf{p}_i = \mathbf{x}_i\}_{i=1}^S$ and

$$\mathbf{g} = \operatorname{argmin}\{f(\mathbf{p}_i)\}_{i=1}^S. \tag{9}$$

(2) Searching: set $k = k + 1$ and

- pick $r_{1,2} \sim U[0, 1]$, and update the particles' velocities by using (8a),
- update the particles' positions by using (8b),
- update $\mathbf{p}_i$ as follows

$$\{\mathbf{p}_i = \operatorname{argmin}(f(\mathbf{x}_i), f(\mathbf{p}_i))\}_{i=1}^S, \tag{10}$$

- update $\mathbf{g}$ by using (9).

(3) Ending: Go to Step 2 until the termination criterion is met.

A later version of PSO most commonly used is the PSOiw (PSO inertia weighted) (Shi and Eberhart 1998), which incorporates the parameter ω as the inertia weight in (8a) of the form

$$v_{i,d}(k+1) = \omega v_{i,d}(k) + c_1r_1(k)\big(p_{i,d}(k) - x_{i,d}(k)\big) + c_2r_2(k)\big(g_d(k) - x_{i,d}(k)\big) \tag{11}$$

where $\omega \in [0, 1]$ basically contributes to the convergence of the particles and the stability analysis.

As important as the proper tuning of PSO parameters is the definition of a suitable fitness function. PSO has the inherent advantage over heuristic methods such that its fitness functions are not too restrictive as in the case of deterministic techniques, making it able to manage non-differentiable functions as well, including nonlinearities and discontinuities. However, it is known that by incorporating a non-suitable fitness function to measure the goodness of the particles evaluated a poor performance can be obtained, not directly related to the method itself or to the choice of its parameters, but rather by the unsuitable design of the fitness function.

Finally, since evolutionary algorithms (EA) are a very good alternative for solving global optimisation problems with multiple maximums and minimums, discontinuities and deterministic solutions in non-polynomial time, it is clear that PSO (including their different versions and/or variants) often turns out to be a good alternative to Genetic Algorithms (GA) (Rahmat-Samii 2003; Boeringer and Werner 2004), which can also be reflected in the increasing number (close to exponential) of successful applications supported by PSO (Poli 2008; Chen and Dye 2012). Nevertheless, in spite of the all practical advantages offered by PSO in its successful applications, it is interesting to highlight the fact that convergence to the global optimum for standard PSO is not assured, even to a local optimum one, as shown in Jiang, Luo, and Yang (2007).

4. Determining the non-existence of a CQLF using PSO

In order to design a method for determining the non-existence of a CQLF using PSO, the development of two main steps is neccessary: (i) to explore the feasibility of interpreting such determination as an optimisation problem and (ii) to design a suitable fitness function to be optimised by using PSO.

By analysing the problem, it is concluded that we are not looking for an optimal solution as such, we just need to verify the crossing of a threshold defined by the non-compliance of a necessary (or necessary and sufficient) condition for the existence of a CQLF, or equivalently, the compliance of a sufficient (or necessary and sufficient) condition for the non-existence. Thus, by optimising, the goal is to prove the non-existence of a CQLF by finding a counterexample of existence. This conclusion provides the basis for designing an appropriate fitness function, and thus the construction of a method for resolving the problem of non-existence of a CQLF.

Now, it is neccessary finding a suitable condition. Due to the generality sought for the proposed solution, the evaluation of conditions for the case of N systems of order n is required. A first condition to be analysed is the one presented in Kamenetski and Pyatnitski (1987), it is based on the work of King and Shorten (2006).

Theorem 4.1: *Let* $\mathrm{A} = \{A_1, \dots, A_N\}$ *be a set of* $n \times n$ *Hurwitz matrices. Then* A *does NOT have a CQLF if and only if there are positive semidefinite matrices* $X_1, \dots, X_N$*, not all zero, such that*

$$\sum_{i=1}^{N} A_i X_i + X_i A_i^T = 0. \quad (12)$$

Despite Theorem 4.1 representing a necessary and sufficient condition for the non-existence of a CQLF in the general case of N matrices of order n, there are two critical issues from the optimisation point of view: (i) the necessary precision to reach the equality (12), and (ii) the dimensionality of the potential solutions. Regarding the first issue, the limited precision for all computational numeric formats does not assure that it is always possible for all arbitrary set of matrices under analysis to find experimentally a set of matrices $X_i, \dots, X_M$, in spite of the existence of this last set. In relation to the second issue, the dimensionality is given by the amount of different elements which must be found (each entry of the matrices to be computed), i.e. $N \times (n \times n)$, or at least $N \times \frac{n(n+1)}{2}$ if the matrices $X_i, \dots, X_N$ are symmetric, which implies a high computational cost.

As a second choice, a condition presented in Shorten and Cairbre (2001) (Theorem 2.1, Corollary 2.1b) can be used, which is summarised in the next theorem.

Theorem 4.2: *A necessary condition for a CQLF to exist for the system* (1) *is that the matrix pencil*

$$\sum_{i=1}^{N} \alpha_i A_i + \beta_i A_i^{-1}, \quad \forall \alpha_i, \beta_i \geq 0, \quad \sum_{i=1}^{N} \alpha_i + \beta_i, \quad (13)$$

is Hurwitz.

Theorem 4.2, which states the verification of the Hurwitz property for the polytopes of vertex matrices $\{A_i^{\pm 1}\}_{i=1}^{N}$, represents a less general condition than (12) since it is only a necessary condition (not necessary and sufficient). However, Theorem 4.2 has a pair of significant advantages: (i) the numeric precision is not a critical issue, because is the compliance of an inequality instead of an equality what is required here (the Hurwitz condition is satisfied if the real part of all its eigenvalues is less than zero) and (ii) the dimensionality of the potential solutions is only $2N$, that corresponds to the $2N$ different coefficients of each linear combination to be evaluated, which is much less than in the case of the Theorem 4.1.

From the above result, Proposition 4.3, which is the basis of the proposed method in this article, is stated next.

Proposition 4.3: *Let us consider the SLS* (1) *with* $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n \times n}$ *Hurwitz* $\forall i \in \{1, 2, \dots, N\}$ *and* $\overline{\mathrm{A}} = \{A_1^{-1}, \dots, A_N^{-1}\}$. *A sufficient condition for the* *non-existence of a CQLF for* (1) *is that at least one non-Hurwitz CLC in* $\{\mathrm{A}, \overline{\mathrm{A}}\}$ *exists.*

Proof: Let us assume that there is a CQLF of the form $V(x) = x^T P x$, $P > 0$ for (1), or equivalently set in A. From Lemma 2.1, P is also a CQLF for set $\overline{\mathrm{A}}$. Let us consider an arbitrary CLC in $\{\mathrm{A}, \overline{\mathrm{A}}\}$ defined as

$$W = CLC\left[A_i, A_i^{-1}\right]_{i=1,\ldots,N} = \sum_{i=1}^{N}\left(\alpha_i A_i + \alpha_{N+i} A_i^{-1}\right), \quad \alpha_j \geq 0, \quad \sum_{j=1}^{2N} \alpha_j = 1. \tag{14}$$

P is also a CQLF for W since it is verified that

$$PW + W^T P = -X, \tag{15}$$

with $X = \sum_{j=1}^{2N}\left[\alpha_j Q_j\right] > 0$ (Lemma 2.5). Since P satisfying (15) exists, every W must be Hurwitz (Theorem 4.2). Therefore, the existence of some W non-Hurwitz contradicts the existence of a CQLF, and this proves Proposition 4.3. □

Remark 1: The justification that Proposition (4.3) is only a sufficient condition is based on the necessary and sufficient condition for the non-existence of a CQLF in pair of matrices of third order presented in King and Shorten (2006) (Theorem 5). Such condition states the evaluation of the singularity of $\widetilde{W} = CLC\left[A_1, A_2, A_1^{-1}, A_2^{-1}, \sigma_\alpha[A_1, A_2]\right]$ (among others), and since $W \subset \widetilde{W}$, then the evaluation of the non-singularity of W is only the verification of a part of the whole condition. It can be observed that the complexity of a necessary and sufficient condition for the non-existence of a CQLF in a pair of systems dramatically increases with the order of the matrices to be analysed, since in the case of 2×2 matrices it is only necessary to evaluate the singularity of $\sigma_\alpha[A_1, A_2]$ and $\sigma_\alpha\left[A_1^{-1}, A_2\right]$. This shows that for systems of order greater than 3 the sufficient condition will be further from being a necessary and sufficient condition. However, a particular case in which Proposition 4.3 becomes a necessary and sufficient condition for the non-existence of a CQLF is when a pair of $n \times n$ matrices which difference is of rank 1 is analysed (King and Nathanson 2006), but this strongly limits the generality of the method.

Even though a numerical method based on a sufficient condition for the non-existence of a CQLF (Proposition 4.3) seems to be weak, it is interesting to analyse first the following situation before assuming such weakness. From the numerical point of view, determining the existence of a CQLF has already been faced by authors like Liberzon and Tempo (2003) and Ordóñez-Hurtado and Duarte-Mermoud (2012), and nevertheless none of these works yield conclusive results with respect to the assurance of the non-existence of a CQLF. This becomes clear in the case of finding a final solution that is not feasible (no CQLF), because this may be the consequence that such CQLF does not exist, or exists but cannot be computed due to an unsuitable tuning of the configuration parameters or a limited numeric precision. Therefore, a method like the one proposed in this article is complementary to the computation methods LMI/L-T/O-D, and constitutes a progress in the search of a complete solution to the CQLF problem from the experimental point of view.

A direct consequence of Proposition 4.3 is the generalisation of Lemma 2.4 in Shorten et al. (2004) to the case of N matrices in $\mathbb{R}^{n\times n}$, as stated in the following lemma.

Lemma 4.4: *If the SLS* (1) *has a CQLF, then every* $W = CLC\left[A_i, A_i^{-1}\right]_{i=1,\ldots,N}$ *is non-singular. Equivalently, none of the products of matrices* $A_j W$, $\forall A_j \in \mathrm{A}$, *have negative real eigenvalues.*

Proof: The necessary condition of non-singularity of every W is directly derived as a corollary from Proposition 4.3.

Now, with

$$\sigma_\alpha\left[A_j, W\right] = \left[\alpha A_j + (1-\alpha) CLC\left[A_i, A_i^{-1}\right]_{i=1,\ldots,N}\right] \subseteq W$$

and in the same way $\sigma_\alpha\left[A_j^{-1}, W\right] \subseteq W$, for all $A_j \in \mathrm{A}$, by using Lemma 2.4 it follows that $A_j W$ and $A_j^{-1} W$ have no negative real eigenvalues. Finally, without loss of generality since $\sigma_\alpha\left[A_j, W\right] \subseteq \sigma_\alpha\left[A_j^{-1}, W\right]$, it follows that $A_j W$ has no negative real eigenvalues. □

Remark 2: It is interesting to observe from Lemma 4.4 that for the case of $n = 2$ the non-singularity of W ensures the existence of a pairwise CQLF in A, this being a consequence of Proposition 2.3 and the fact that $\sigma_\alpha[A_i, A_j] \subset W$ and $\sigma_\alpha\left[A_i^{-1}, A_j\right] \subset W$ for all $A_i, A_j \in A$. However, by applying Theorem 2.2, checking the eigenvalues of $A_i A_j$ and $A_i^{-1} A_j$ it is sufficient to assess non-singularity of the two pencils. The usefulness of Lemma 4.4 is revealed for $n > 2$, because in this situation the algebraic condition on the eigenvalues of $A_i A_j$ and $A_i^{-1} A_j$ is only necessary for the existence of a pairwise CQLF in A (and therefore for the whole A), but not sufficient. Therefore, an interesting fact for A_1, $A_2 \in \mathbb{R}^{n\times n}$ and $n = 2$ is that the non-singularity of $\sigma_\alpha\left[A_1^{-1}, A_2\right]$ implies the non-singularity of $\sigma_\alpha\left[A_1, A_2^{-1}\right]$ and vice versa, but this is not generally true for the case of $n > 2$.

In the method being proposed here, we define

$$\Lambda = [\alpha_1, \ldots, \alpha_{2N}] \tag{16}$$

as an arbitrary set of coefficients in the CLC (14), satisfying

$$\sum_{j=1}^{2N} \alpha_j = 1, \quad \alpha_j \geq 0 \quad \forall k, \tag{17}$$

where N is the number of matrices to be analysed. This suggests that we can define the fitness function $f(\Lambda)$ as follows

$$f(\Lambda) = \min[\mathrm{Re}[\mathrm{eig}(-W(\Lambda))]], \tag{18}$$

wheres eig(·) denotes the function that calculates the eigenvalues of a matrix. The rationale of the above fitness function is that if there is a $W(\Lambda)$ that is non-Hurwitz, i.e. if a $-W(\Lambda)$ has at least one eigenvalue with a negative real part, then there is no CQLF for A (see Proposition 4.3). Thus, the fitness function only delivers useful results by obtaining $f(\Lambda) < 0$, i.e. when crossing a threshold around zero that allows deducing the non-existence of a CQLF by a counter example of its existence.

By analysing (14), it is seen that its fit to PSO is straightforward, by relating Λ with the particles to evolve. Let us define

$$\mathbf{Particle}_i = \left[C_d^{(i)}\right], \quad i = 1, \ldots, S, \; d = 1, \ldots, M, \tag{19}$$

and then finding a function $\nu(\mathbf{Particle}_i) = \Lambda_i$ it is desired, such that ν: $R^M \to R^{2N}$ relates $[\alpha_j^{(i)}]$ to $[C_d^{(i)}]$, also satisfying (17). Moreover, since $[\alpha_j^{(i)}]$ and $[C_d^{(i)}]$ represent the same element, then $f(\mathbf{Particle}_i) \triangleq f(\Lambda_i)$. So, the fitness function is defined by using

$$f(\mathbf{Particle}_i) = \min[\mathrm{Re}[\mathrm{eig}(-W(\nu(\mathbf{Particle}_i)))]]. \tag{20}$$

Now, the construction of a specific fitness function is determined depending on how the function ν is defined. For the definition of ν, two alternatives are proposed satisfying constraint (17). The first option for ν is obtained through the following steps:

(1) Let us consider $M = 2N - 1$, and each component of the particle as if it was an angle, i.e.

$$C_d^{(i)} \in [0, 2\pi], \quad \forall d \in \{1, \ldots, 2N-1\}. \tag{21}$$

(2) To calculate the elements of Λ_i as follows:

$$\alpha_j^{(i)} = \begin{cases} \left(\sin\left(C_1^{(i)}\right)\right)^2, & j = 1. \\ \left(\sin\left(C_j^{(i)}\right)\prod_{h=1}^{j-1}\cos\left(C_h^{(i)}\right)\right)^2, & j = 2, 3, \ldots, 2N-1. \\ \left(\prod_{h=1}^{2N-1}\cos\left(C_h^{(i)}\right)\right)^2, & j = 2N. \end{cases} \tag{22}$$

By using simple trigonometric identities it follows that, with the previous representation (22), relationship (17) is satisfied. The set formed by Equations (19)–(22) will be referred to from hereon as *Fitness 1* (f_1).

A second option for defining ν is given as follows:

(1) Let us consider $M = 2N - 1$, and each component of the particle as a number in [0, 1], i.e.

$$C_d^{(i)} \in [0, 1], \quad \forall d \in \{1, \ldots, 2N-1\}. \tag{23}$$

(2) To calculate $\tilde{\alpha}_j$ of the form:

$$\tilde{\alpha}_j^{(i)} = \begin{cases} C_j^{(i)}, & j = 1, 2, \ldots, 2N-1, \\ 1 - \mathrm{mod}\left(\sum_{h=1}^{2N-1} \tilde{\alpha}_h^{(i)}, 1\right), & j = 2N, \end{cases} \tag{24}$$

where $\mathrm{mod}(a, b)$ is the modulus after $\frac{a}{b}$ division.

(3) Let us define $K = \sum_{j=1}^{2N} \tilde{\alpha}_j^{(i)}$, and calculate the elements of Λ_i in the following fashion:

$$\alpha_j^{(i)} = \frac{\tilde{\alpha}_j^{(i)}}{K}, \quad j = 1, 2, \ldots, 2N. \tag{25}$$

It can be seen that these $\alpha_j^{(i)}$'s satisfy Equation (17). The set formed by Equations (19)–(20) and (23)–(25) will be referred to as *Fitness 2* (f_2).

5. Experimental results

In this section, the results of tests used to evaluate the performance of the proposed method are shown. The implementation of this method involved the development of two main programs in Matlab, one for *Fitness 1* and the other for *Fitness 2*. The PSO ToolBox of Matlab is used as the fundamental tool, which is a set of Matlab files (.m) developed by Singh (2003) that implements the PSO algorithm for systems optimisation. For the programs developed, the definition of an auxiliary file was required, which defines the corresponding fitness function, and a modification of the PSO Toolbox main file (PSO.m) was made to suitably define the initial position of the particles. In order to compare the performance of the proposed method, comparisons with the methods presented in Shorten and Narendra (2002) and Cheng et al. (2003), and with the LMI, L-T and O-D approaches will be performed in the cases where they are applicable.

The content of this section includes the configuration of PSOiw, and a set of experimental results. As initial examples, sets of stable matrices that do not share a CQLF will be analysed, the fact that can be analytically deduced. Then we analyse the case in which no analytical support for the existence/non-existence of a CQLF can be given.

5.1. *PSO configuration*

When using PSOiw, the choice of inertia weight implies a trade-off between commitment of exploration and exploitation (Del Valle et al. 2008). One of the typical configurations achieving this trade-off involves the use of an inertia weight $\omega(k)$ decreasing linearly with respect to k, with $\omega(0)=\omega_{\max}=0.9$ and $\omega(iter_{\max})=\omega_{\min}=0.4$, together with $c_1=c_2=2$. Usually, the population size S (total number of particles) depends on the application; between 10 and 50 for simple problems, and up to 100 or more for complex problems (Del Valle et al. 2008). On this basis, in the subsequent experiments we used a population size that is equal to or greater than twice the particles' dimension, defined as

$$S = 4N \geq 2M = 2(2N-1) = 4N-2,$$

where N is the number of matrices to be analysed. As termination criteria, two were chosen to work simultaneously: (i) crossing the threshold $f(Particle) < 0$ and (ii) achieving the maximum number of iterations ($k > iter_{\max}$). Two types of initialisation for the positions of the particles are chosen: (i) random initialisation and (ii) predefined initialisation. For the random case a random uniform distribution is used, while for the predefined case the swarm has initial positions that allow to start the optimisation process by analysing only CLCs between pairs of elements (with equal weighting).

Finally, we present the progress of the optimisation process for every $0.1 * iter_{\max}$ iterations, where *fGBest* is the best fitness value found.

5.2. *Example 1: three matrices in* $\mathbb{R}^{2\times 2}$ *(no CQLF)*

We first choose A_1, A_2 and A_3 defined as three stable matrices, the same was used in Shorten and Narendra (2003) in Example 5.4:

$$A_1 = \begin{bmatrix} 0 & 5 \\ -30 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 5 \\ -26 & -1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -6 & 27 \\ -150 & -1 \end{bmatrix}.$$

By using the LMI, L-T and O-D approaches, no conclusive information is obtained (it is not possible to find a feasible solution), raising the fact whether a CQLF does not exist or it exists but an unsuitable tuning of the configuration parameters has been used. By applying the Cheng approach (Cheng et al. 2003), the following results are obtained:

$$\Theta_{A_1} = (0, 0.04), \quad \Theta_{A_2} = (0, 0.0476), \quad \Theta_{A_3} = (0, 0.0081),$$

$$\Theta = \Theta_{A_1} \cap \Theta_{A_2} \cap \Theta_{A_3} = (0, 0.0081),$$

$$\int_{t\in\Theta} (V(t) - L(t))\mathrm{d}t = 0, \tag{26}$$

and since the value of the integral (26) is not positive, the fact that the 3-tuple of systems does not share a CQLF it is shown with this approach. However, by applying the proposed method we obtain non-conclusive results having a positive value as a final output independently of the fitness functions or position initialisations used, as shown in Table 1.

It is interesting to highlight the fact that in this example it is possible to analytically assure that every pair of systems shares a CQLF (Theorem 2.2), and however this is not enough for the three systems to share a CQLF because there is an empty intersection of the three associated ellipses (Figure 2). It is clear with this that by using the proposed method it is not possible to obtain conclusive results, since what is done is only the experimental verification of a necessary condition for the non-existence of a CQLF in pairs of systems (Lemma 4.4), in an aggregated form on the same functional.

5.3. *Example 2: five matrices in* $\mathbb{R}^{3\times 3}$ *(no CQLF)*

Now we choose A_1, A_2, A_3 and A_4 as four stable upper triangular matrices, randomly obtained as

$$A_1 = \begin{bmatrix} -1.1764 & -2.2016 & -30.4614 \\ 0 & -28.6391 & 1.8565 \\ 0 & 0 & -0.8325 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} -21.2914 & 1.8888 & 8.2560 \\ 0 & -11.2562 & 3.4612 \\ 0 & 0 & -6.3290 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.3489 & 0.7536 & 8.8639 \\ 0 & -5.3062 & 6.0803 \\ 0 & 0 & -25.3543 \end{bmatrix},$$
$$A_4 = \begin{bmatrix} -10.9267 & 5.1315 & -11.2256 \\ 0 & -10.4192 & -13.8875 \\ 0 & 0 & -0.8438 \end{bmatrix},$$

which have a CQLF (Shorten and Narendra 1998). Additionally, we choose a generic Hurwitz matrix A_5,

Table 1. Sample of the PSO-based method applied to Example 1, with different particle position initialisation (PPI).

Using f_1				Using f_2			
Randomised PPI		Predefined PPI		Randomised PPI		Predefined PPI	
Iteration	fGBest	Iteration	fGBest	Iteration	fGBest	Iteration	fGBest
20	0.0006977	20	0.003882	20	0.075553	20	0.046997
40	0.00050502	40	0.0014851	40	0.046589	40	0.03228
60	0.00036326	60	0.0011693	60	0.013142	60	0.025
80	0.00035951	80	0.0011693	80	0.0020465	80	0.012635
100	0.00027751	100	0.0011693	100	0.0017759	100	0.0060399
120	0.0001687	120	0.0011612	120	0.0015656	120	0.0039067
140	0.000132	140	0.0011612	140	0.0010153	140	0.0034564
160	0.00011075	160	0.0011268	160	0.00086993	160	0.0029783
180	0.00011075	180	0.0011268	180	0.00016213	180	0.0018561
200	0.00011075	200	0.0011268	200	0.00015034	200	0.0018558
Final eigenvalues		Final eigenvalues		Final eigenvalues		Final eigenvalues	
−0.000111		−0.01422		−0.00015		−0.01093	
−0.002897		−0.00113		−0.02947		−0.00185	

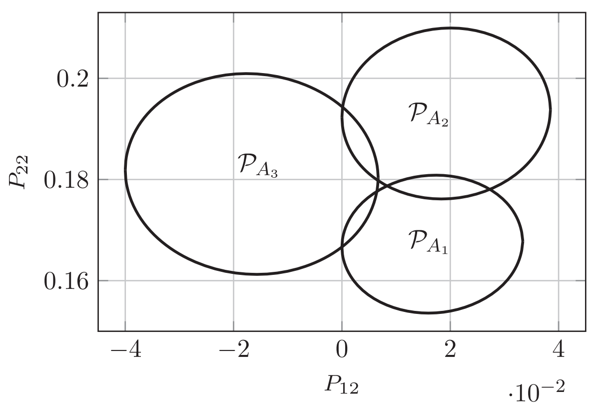

Figure 2. Sets of Lyapunov functions $\mathcal{P} = \left[\begin{smallmatrix} 1 & P_{12} \\ P_{12} & P_{22} \end{smallmatrix}\right]$ for A_1, A_2 and A_3 (Shorten and Narendra 2003) from Example 1.

also randomly obtained as

$$A_5 = \begin{bmatrix} -14.7578 & 6.3562 & -15.4226 \\ -9.9717 & 1.0197 & -4.2403 \\ 5.1944 & 5.5201 & -9.9615 \end{bmatrix}$$

such that $\mathrm{A} = \{A_i\}_{i=1,\ldots,5}$ satisfies the conditions of Lemma 2.4 on the eigenvalues of A_iA_j and $A_iA_j^{-1}$ for all $A_i, A_j \in A$. By applying the LMI, L-T and O-D approaches no conclusive results are obtained (final unfeasible solution is found), and since in this example it cannot be used the Cheng approach (Cheng et al. 2003) (only developed for second order systems), question of whether the proposed method is able to give conclusive results remains unanswered.

Table 2 shows the results of applying the method proposed, which confirm experimentally the non-existence of a CQLF (i.e. a negative final output is obtained). By exhaustive search and graphical methods it was found that $\sigma_\alpha\left[A_5^{\pm 1}, A_2^{\pm 1}\right]$, $\sigma_\alpha\left[A_5^{\pm 1}, A_3^{\pm 1}\right]$, $\sigma_\alpha\left[A_5^{\pm 1}, A_4^{\pm 1}\right]$ and $\sigma_\alpha\left[A_5^{\pm 1}, A_1^{1}\right]$ are Hurwitz, but $\sigma_\alpha\left[A_5^{\pm 1}, A_1^{-1}\right]$ are non-Hurwitz although $A_5A_1^{-1}$ and $A_5^{-1}A_1^{-1}$ contain non-negative real eigenvalues, analytically showing that the set of systems does not share a CQLF (Proposition 4.3).

It is interesting to note that the particular results for this case show that *Fitness 2* presents faster convergence compared to *Fitness 1*. In this case, the initialisation of the swarm plays a central role depending on which fitness function is used. Predefined initialisation is shown to be beneficial for *Fitness 1*, while a random initialisation is shown to be beneficial for *Fitness 2*.

5.4. *Example 3: five matrices in* $\mathbb{R}^{4\times 4}$ *(no CQLF)*

In this case the set of matrices to be analysed consists of

$$A_1 = \begin{bmatrix} -10.68 & 20.44 & -1.34 & -4.68 \\ -15.72 & -6.74 & -7.78 & -6.97 \\ -5.47 & 12.22 & -12.89 & -1.58 \\ 2.75 & -6.74 & -4.29 & 0.74 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -20.55 & -23.93 & -3.83 & 9.85 \\ 20.12 & 2.30 & -7.64 & 4.24 \\ -12.88 & 23.25 & 0.99 & -15.16 \\ -11.78 & -12.68 & -5.12 & -3.69 \end{bmatrix},$$

Table 2. Sample of the PSO-based method applied to Example 2, with different particle position initialisation (PPI).

Using f_1				Using f_2			
Randomised PPI		Predefined PPI		Randomised PPI		Predefined PPI	
Iteration	fGBest	Iteration	fGBest	Iteration	fGBest	Iteration	fGBest
20	0.60993	1	−1.2552	8	−0.3206	20	0.51751
40	0.60993					24	−0.40567
45	−0.21958						
Final eigenvalues		Final eigenvalues		Final eigenvalues		Final eigenvalues	
−3.9829		−6.1156		0.3206		−4.6153	
0.2196		1.2552		−4.7074		0.4057	
−1.9150		0.0429		−4.3139		−1.4826	

Table 3. Sample of PSO-based method applied to Example 3, with different particles position initialisation (PPI).

Using f_1				Using f_2			
Randomised PPI		Predefined PPI		Randomised PPI		Predefined PPI	
Iteration	fGBest	Iteration	fGBest	Iteration	fGBest	Iteration	fGBest
1	−2.7342	1	−1.4437	1	−3.2387	1	−2.7042
Final eigenvalues		Final eigenvalues		Final eigenvalues		Final eigenvalues	
−9.4955		−4.3755 + 9.7904i		−4.7457 + 8.7296i		2.7042	
−5.8707 + 7.5219i		−4.3755 − 9.7904i		−4.7457 − 8.7296i		−9.1585	
−5.8707 − 7.5219i		−4.9084		−7.7287		−5.7044 + 6.5177i	
2.7342		1.4437		3.2387		−5.7044 − 6.5177i	

$$A_3 = \begin{bmatrix} -9.05 & -0.68 & 0.35 & -3.65 \\ -22.45 & -9.43 & 6.64 & -7.88 \\ 1.63 & -11.13 & -13.10 & -7.73 \\ -8.86 & 2.51 & -4.56 & -7.59 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -6.34 & -5.83 & 2.80 & -9.74 \\ 7.04 & -16.54 & -5.36 & -5.74 \\ -8.39 & 16.87 & 0.40 & 2.23 \\ 23.75 & 20.73 & -11.64 & 2.81 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} -17.92 & -7.37 & -6.62 & 2.41 \\ 0.43 & -13.17 & -0.29 & -22.42 \\ 7.09 & 13.22 & 4.98 & 6.64 \\ 4.80 & 8.95 & 6.10 & 2.47 \end{bmatrix},$$

which were also randomly obtained.

Here again the Cheng approach cannot be used, and by using the LMI, L-T and O-D approaches it is not possible to find a CQLF for the analysed system. However, Table 3 shows the results of applying the proposed method, corroborating the non-existence of a CQLF by obtaining a negative value for the output of the optimisation process.

It is seen that in this case the results are not sensitive to the fitness function chosen or the swarm initialisation employed, but this fact is not surprising because of the compliance of many analytic conditions for the non-existence of a CQLF that can be verified for this example, e.g.

- $A_1A_2^{-1}$, $A_2A_4^{-1}$ and $A_4^{-1}A_5$ contain negative real eigenvalues,
- $A_2 + A_3^{-1}$ is unstable.

Based on this information, it follows analytically (Proposition 4.3, Lemma 4.4) that several pairs of the proposed set of matrices do not share a CQLF, and therefore the whole set cannot share a CQLF.

Having checked three successful cases for which there is an analytical guarantee of the non-existence of a CQLF, we explored the case of sets of matrices for which there is no *a priori* information about the non-existence of a CQLF, except that they are Hurwitz matrices.

5.5. *Example 4: ten matrices in $\mathbb{R}^{5\times5}$ (undetermined existence of CQLF)*

Unlike in the previous examples, a set of 10 stable matrices in $\mathbb{R}^{5\times5}$ were chosen for which it is unknown

Table 4. Sample of PSO-based method applied to Example 4, with different particles position initialisation (PPI).

Using f_1				Using f_2			
Randomised PPI		Predefined PPI		Randomised PPI		Predefined PPI	
Iteration	fGBest	Iteration	fGBest	Iteration	fGBest	Iteration	fGBest
40	0.0018	1	0.0023	40	0.0736	40	0.0015
73	−0.0032	62	−0.0066	80	0.0588	80	0.0015
				120	0.0435	120	0.0015
				160	0.0206	160	0.0007
				200	0.0075	174	−0.0005
				225	−0.0007		
Final eigenvalues		Final eigenvalues		Final eigenvalues		Final eigenvalues	
−0.2259		−0.2279		−0.5896		−0.7579	
−0.1201 + 0.0573i		−0.1184 + 0.0631i		−0.2882		−0.2447	
−0.1201 − 0.0573i		−0.1184 − 0.0631i		−0.1256		−0.1144	
0.0032		0.0066		0.0007		0.0005	
−0.0104		−0.0130		−0.0140		−0.0115	

a priori whether or not a CQLF exists. These matrices are generated from a stable matrix and a stochastic disturbance of the type $A_i = A + R_i$, $i \in \{1, 2, \ldots, 10\}$, where R_i are 5th order matrices whose components are pseudo random numbers with normal distribution, of mean 0 and variance 1. Matrix A is defined as

$$A = \begin{bmatrix} -5.6255 & -3.6453 & 4.0045 & -26.1274 & -5.2049 \\ -1.3169 & -2.5247 & 1.8567 & -17.9511 & -4.1578 \\ 3.5778 & -2.6112 & -0.7498 & -2.3677 & 5.6897 \\ 10.8137 & 6.6703 & -0.970 & -0.4835 & 11.4901 \\ -30806 & 5.2276 & -7.6581 & 14.9509 & -9.0751 \end{bmatrix},$$

which was also chosen randomly. This example may represent the case where parametric variations on a nominal matrix produce a family of matrices, in which it is unknown whether or not that family shares a CQLF. The use of the Cheng approach is again discarded, and the LMI, L-T and O-D approaches are not able of finding a CQLF for the set of systems. Table 4 shows the results of applying the proposed method for the set of matrices obtained with the explained procedure. Once again, convergence to a negative number ensures that the set of matrices that was analysed does not share a CQLF. In this case it is observed that *Fitness 1* has a tendency to outperform *Fitness 2*, but the benefits of a random or predefined initialisation of the population are not very significant.

Although both fitness functions achieved the optimisation goal, non-conclusive results could be obtained by using $iter_{\max} < 100$ with *Fitness 2*, since the final value would be a positive number even though it is now known that the set of matrices under analysis does not share a CQLF.

5.6. *Example 5: twenty matrices in $\mathbb{R}^{6\times 6}$ (undetermined existence of CQLF)*

Finally, a set of 20 Hurwitz matrices in $\mathbb{R}^{6\times 6}$ was used, which consists of 19 arbitrary upper triangular matrices (i.e. share a CQLF as may be seen in Shorten and Narendra (1998)) and the arbitrary Hurwitz matrix in companion form:

$$A_{20} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0642 & -0.9107 & -4.5204 & -9.8177 & -10.3470 & -5.2047 \end{bmatrix}.$$

Analytically it is known that a CQLF can be found for $\{A_i\}_{i=1}^{19}$, but this is not enough to assure the existence of a CQLF for $A = \{A_i\}_{i=1}^{20}$. By applying the LMI, L-T and O-D approaches it is not possible to find a CQLF for A, and discarding the use of the Cheng approach we are again in a situation of uncertainty.

Table 5. Sample of PSO-based method applied to Example 5, with different particles position initialisation (PPI).

Using f_1				Using f_2			
Randomised PPI		Predefined PPI		Randomised PPI		Predefined PPI	
Iteration	fGBest	Iteration	fGBest	Iteration	fGBest	Iteration	fGBest
80	0.1882	80	0.0801	6	−0.2297	10	−0.0005
160	0.0504	160	0.0801				
240	0.0196	240	0.0552				
320	0.0181	320	0.0547				
400	0.0181	400	0.0546				
480	0.0181	480	0.0546				
560	0.0181	505	−0.4997				
640	0.0181						
720	0.0181						
800	0.0181						
Final eigenvalues		Final eigenvalues		Final eigenvalues		Final eigenvalues	
−1.8715		−6.1034		0.2296		0.0005	
−0.3121		−1.7480 + 1.7841i		−1.0298		−1.0048	
−0.0181		−1.7480 − 1.7841i		−1.6637		−1.6142	
−0.8687		0.4997		−3.2947 + 0.1456i		−3.2188 + 1.1629i	
−1.1835		−0.2952 + 1.3676i		−3.2947 − 0.1456i		−3.2188 − 1.1629i	
−0.8282		−0.2952 − 1.3676i		−4.8349		−4.6435	

Experimental results presented in Table 5 show that the optimisation process involving *Fitness 1* with random initial positions converges in this particular case to a positive value. Again this information is not useful because, on one hand, it is the verification of a sufficient condition and, on the other hand, PSO does not perform a full search, even in their latest versions (Del Valle et al. 2008, and later) since this is an infinite dimension problem. However, when using *Fitness 1* with a predefined initialisation, convergence to a negative number is obtained. In contrast, when using *Fitness 2*, the convergence to a negative value is achieved regardless of the initialisation procedure, and faster than *Fitness 1*.

6. Conclusions

This article reports the development of a method for determining the non-existence of a CQLF for an SLS based on PSO. The designed method was tested for cases of different dimensionality (order and number of matrices), exhibiting, in general, good performance that tends to worsen when increasing the dimensionality of the problem, affecting the convergence speed and the convergence itself.

When the output of the optimisation process is a non-positive number, the method is able to assure with certainty equal to 1 that the set of matrices under analysis do not share a CQLF. But, since we are verifying a sufficient condition for the non-existence of a CQLF, when the output of the optimisation process is a positive number the proposed method is not able to guarantee conclusively whether or not a CQLF exists for the set of matrices under analysis. Nevertheless, the proposed method constitutes an important progress in the way of solving numerically the CQLF problem, especially considering that the complexity of the problem increases in higher order systems, in which the proposed solution proved to be able of giving conclusive information where other reported approaches did not succeed.

PSOiw exhibited good results in this particular study such as the ability to find a feasible solution with fast convergence (depending on the fitness used). However, it may be beneficial from the technical point of view to use a newer version of PSO (e.g. Del Valle et al. 2008; Kameyama 2009) and from the analytical point of view to use another scheme of inertia weight variation, or to employ another swarm initialisation in order to improve convergence. Since the advantages of the initialisations used are not so clear (showed to be highly dependent on systems and/or fitness function used), it could be interesting to explore the potential benefits of using the presented ones in Clerc (2008).

Given the nature of the proposed method, the comparison with traditional techniques such as LMI or gradient was performed in the context of a complementary method. However, GA and differential evolution (DE) could be directly used instead of PSO as optimisation tool, but this falls outside the scope of this article.

Finally, it is noted that other necessary and/or sufficient conditions for the existence/non-existence of a CQLF could be analysed in order to extend the results obtained in this study, by using the proposed method but considering the particularities of the new conditions. Along the same line, a method based on PSO to determine the existence of a CQLF can be designed as was done in Ordóñez-Hurtado and Duarte-Mermoud (2012). Investigations on these topics are currently underway.

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