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with generalized integrated chance
constraints

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Stochastic programming problems with generalized integrated chance constraints

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If the constraints in an optimization problem are dependent on a random parameter, we would like to ensure that they are fulfilled with a high level of reliability. The most natural way is to employ chance constraints. However, the resulting problem is very hard to solve. We propose an alternative formulation of stochastic programs using penalty functions. The expectations of penalties can be left as constraints leading to generalized integrated chance constraints, or incorporated into the objective as a penalty term. We show that the penalty problems are asymptotically equivalent under quite mild conditions. We discuss applications of sample-approximation techniques to the problems with generalized integrated chance constraints and propose rates of convergence for the set of feasible solutions. We will direct our attention to the case when the set of feasible solutions is finite, which can appear in integer programming. The results are then extended to the bounded sets with continuous variables. Additional binary variables are necessary to solve sample-approximated chance-constrained problems, leading to a large mixed-integer non-linear program. On the other hand, the problems with penalties can be solved without adding binary variables; just continuous variables are necessary to model the penalties. The introduced approaches are applied to the blending problem leading to comparably reliable solutions.

Keywords: chance constraints; integrated chance constraints; penalty functions; sample approximations; blending problem

AMS Subject Classification: 90C15

1. Introduction

In practical optimization problems, the constraints need not be known precisely but can depend on a realization of a random vector. Our goal is then to ensure that the constraints are fulfilled with a high level of reliability, i.e. for realizations of the
random vector with at least a prescribed probability. The most natural way to deal with this problem is to employ chance constraints. However, solving the resulting chance-constrained problems is not easy. In general, the feasible region is not convex even if the functions are convex, and in many cases it is not even easy to check feasibility because it leads to computations of multivariate integrals. On the other hand, there are some special conditions under which the convexity is preserved, e.g. the log-concave distributions \[19\], or it is relatively easy to check the feasibility of a point, e.g. for the normal distribution. Numerical methods for computation of multivariate integrals can be helpful for verifying feasibility or solving the problems, see \[10\] and the references therein. There are various methods for numerically solving chance-constrained problems, as in \[19,20\]. For the problems with discretely distributed random variables and separable random parts, p-efficient points can be used, cf. \[18\]. For continuously distributed random variables, the methods based on supporting hyperplanes and reduced gradients are available. In a case where the underlying distribution is continuous or discrete with many realizations, the sample-approximation techniques and the mixed-integer programming reformulation can help us to solve the problem approximately, see \[1,16,17\]. With an increased sample size we can even approximate the true solution of the chance-constrained problem. However, the resulting problems are NP-hard in general because additional binary variables are necessary to model the chance constraints leading to large mixed-integer problems. A linear problem with random right-hand side is studied, e.g. in \[14\], where valid inequalities are derived for the mixed-integer linear set of feasible solutions, see also \[9\] for a review of various methods for solving general mixed-integer linear problems.

We will study other possible formulations of stochastic programming problems which are based on penalty functions. Their main advantage over the chance-constrained problems is that they can be solved easier without any additional binary variables just using continuous variables to model the penalties. On the other hand, this approach does not address finding or approximating an optimal solution of the chance constrained problem. The penalty functions help us to penalize possible violations of the random constraints with respect to the decision vector and the random parameter leading to a reliable solution.

We will direct our attention to the problems with integrated chance constraints which were originally defined as expectation type constraints using maximal penalty function, cf. \[11\], see also \[12,19\]. We will define generalized constraints using arbitrary penalty functions. We will show that the resulting problem with several integrated chance constraints is asymptotically equivalent to the problem with penalties in the objective. In \[8\], the asymptotic equivalence between the problem with one joint chance constraint and the problem with simple penalty function was shown. The approach was recently extended to a whole class of penalty functions in \[5\], and to the problems with several joint chance constraints in \[3\] which cover the joint as well as the individual chance-constrained problems as special cases. Note that no assumptions on convexity or linearity of underlying functions were necessary in the proofs. This makes both alternative approaches based on penalty functions more appropriate for stochastic integer and mixed-integer problems, where convexity is usually not present. Stability of the resulting stochastic programming problems
with respect to the changes of underlying probability measure can be studied using contamination technique, cf. [5,7].

We will also discuss applications of sample-approximation techniques to the problems with generalized integrated chance constraints. We will direct our attention to the case with a finite set of feasible solutions, which can appear in bounded integer programming, and with an infinite bounded set with continuous variables. We will extend the result to the rates of convergence for a set of feasible solutions defined by sample average approximations of generalized integrated chance constraints to the true set of feasible solutions. First results for expectation-type constraints were proposed in [23].

The introduced approaches will be applied to a particular blending problem, which was formulated for instance in [17] as a joint chance-constrained problem. We will show that the penalty-function approaches and the sample-approximation technique can be helpful in numerical solution of this stochastic optimization problem. We explore how to try to ensure that the chance constraints are fulfilled with a high level of reliability. We compare the ability to generate a feasible solution of the original joint chance-constrained problem using the sample approximations of the chance constraint directly or via sample approximation of the penalty function objective or general integrated chance constraint.

This article is organized as follows. In Section 2, we formulate the chance-constrained problem and propose a possible formulation using the penalty functions. The expectations of the penalized constraints are left in the constraints with prescribed levels or incorporated into the objective as a penalty term. We show that the problems with penalties are asymptotically equivalent with the decreasing levels and the increasing penalty parameter. In Section 3, sample-approximation techniques for solving the problems are discussed and result on rates of convergence for the generalized integrated chance constraint are derived. Numerical comparison of the proposed approaches on a blending problem is included in Section 4. In Section 5, we summarize our results.

2. Relation between chance-constrained and penalty function problems

Let $f(x)$ be a real function on $\mathbb{R}^n$, $g_{ji}(x, \xi), i = 0, \ldots, k_j, j = 1, \ldots, m$, be real functions on $\mathbb{R}^n \times \mathbb{R}^n$ measurable in $\xi$ for all $x \in X$, and $\xi$ be a random vector on $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}$. The problem with uncertain constraints which depend on a random factor $\xi$ can be formulated as

$$\min_{x \in X} f(x)$$

s.t.

$$g_{11}(x, \xi) \leq 0, \ldots, g_{1k_1}(x, \xi) \leq 0,$$

$$\vdots$$

$$g_{m1}(x, \xi) \leq 0, \ldots, g_{mk_m}(x, \xi) \leq 0.$$  (1)

In this form, it is not obvious as to how to solve the problem. As we claim in Section 1, we would like to ensure that the constraints are fulfilled with a high level of reliability. Hence, if the distribution $P$ of the random vector is known,
we can formulate the general chance-constrained problem as follows:

$$\varphi_{CCP}^e = \min_{x \in X} f(x)$$

s.t.

$$P\left( g_{11}(x, \xi) \leq 0, \ldots, g_{1k_1}(x, \xi) \leq 0 \right) \geq 1 - \varepsilon_1,$$  \hspace{1cm} (2)

$$\vdots$$

$$P\left( g_{ml}(x, \xi) \leq 0, \ldots, g_{mk_m}(x, \xi) \leq 0 \right) \geq 1 - \varepsilon_m,$$

with an optimal solution $x_s$, where $e = (\varepsilon_1, \ldots, \varepsilon_m)$, with given levels $\varepsilon_j \in (0, 1)$. The formulation covers the joint ($k_1 > 1$ and $m = 1$) as well as the separate ($k_j = 1$ and $m > 1$) chance-constrained problems as special cases.

Below, we will consider the penalty functions $\vartheta_j : \mathbb{R}^{k_j} \to \mathbb{R}_+$, $j = 1, \ldots, m$, which are continuous nondecreasing in their components, equal to 0 on $\mathbb{R}_+^{k_j}$ and positive otherwise. Two special penalty functions are readily available: $\vartheta^{1/\alpha}(u) = \sum_{i=1}^{k_j}([u_i]^{+})^{\alpha}$, $\alpha > 0$, usually $\alpha = 1$ or $\alpha = 2$, and $\vartheta^2(u) = \max_{1 \leq i \leq k} [u_i]^2$. Both penalty functions preserve convexity of the random constraints in the decision vector. We denote the penalized constraints

$$p_j(x, \xi) = \vartheta_j(g_{j1}(x, \xi), \ldots, g_{jk}(x, \xi)) : \mathbb{R}^n \times \mathbb{R}^n' \to \mathbb{R}.$$  

Our choice is appropriate, because it holds that

$$P\left( g_{\beta i}(x, \xi) \leq 0, \quad i = 1, \ldots, k_j \right) \geq 1 - \varepsilon_j \iff P\left( p_j(x, \xi) > 0 \right) \leq \varepsilon_j.$$  \hspace{1cm} (3)

There are two possible stochastic programming formulations of the problem with a random factor using the penalty functions. We can define the problem with generalized integrated chance constraints as follows

$$\varphi_{ICC}^L = \min_{x \in X} \left\{ f(x) : \text{s.t. } \mathbb{E}[p_j(x, \xi)] \leq L_j, j = 1, \ldots, m \right\}$$  \hspace{1cm} (4)

for some prescribed bounds $L_j \geq 0$, $L = (L_1, \ldots, L_m)'$, with an optimal solution $x_{ICC}^L$. The integrated chance constraints were originally defined using the penalty function $\vartheta^2$, cf. [11,12], but any penalty function can be used in their definition.

The expectations of the penalized constraints can also be incorporated into the objective function as a penalty term leading to the problem:

$$\varphi_N = \min_{x \in X} \left[ f(x) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x, \xi)] \right]$$  \hspace{1cm} (5)

with $N$ being a positive parameter. We denote $x_N$ an optimal solution of (5). The approach for solving non-linear deterministic programs with several constraints using the penalty functions is well studied in literature. Algorithms and basic theory based on continuity and Karush–Kuhn–Tucker conditions are explained in [2,15].

A rigorous proof of the relationship between the optimal values of (2) and those of (5) for a special additive penalty function and one chance constraint was given by [8]. The approach was recently extended to a whole class of penalty functions with desirable properties and to the problems with several joint chance constraints, which was done in [3,5]. The reformulation of chance-constrained problems using the
penalties has been applied in water management, insurance and engineering, cf. [6,8,24].

We can compare the formulation (4) with the following alternative expression of the penalty function problem

\[
\begin{aligned}
\min_{(x,u) \in X \times \mathbb{R}^n} \left\{ f(x) + N \sum_{j=1}^m u_j : \text{s.t. } \mathbb{E}[p_j(x, \xi)] \leq u_j, u_j \geq 0, j = 1, \ldots, m \right\}.
\end{aligned}
\]

The problems differ by incorporating the upper bounds \(u_j\) into the penalty term of the objective function. However, this term converges to zero as \(N\) goes to infinity under quite mild conditions. We can conclude that both problems are equivalent as \(L_j\) goes to zero and the penalty parameter \(N\) to infinity. We will prove this in the following theorem, including explicit bounds on the optimal values.

**Theorem 2.1** Consider the two problems (4) and (5), and assume that \(X \neq \emptyset\) is a compact set, \(f(x)\) is a continuous function, \(\vartheta_j : \mathbb{R}^k \to \mathbb{R}_+\), \(j = 1, \ldots, m\), are continuous functions, nondecreasing in their components, which are equal to 0 on \(\mathbb{R}^k_+\) and positive otherwise, denote

\[
p_j(x, \xi) = \vartheta_j(g_{j1}(x, \xi), \ldots, g_{jk}(x, \xi)), \quad j = 1, \ldots, m,
\]

and assume that

(i) \(g_{ji}(\cdot, \xi), i = 1, \ldots, k, j = 1, \ldots, m\), are almost surely continuous;

(ii) there exists a nonnegative random variable \(C(\xi)\) with \(\mathbb{E}[C(\xi)] < \infty\), such that \(|p_j(x, \xi)| \leq C(\xi), j = 1, \ldots, m\) for all \(x \in X\);

(iii) \(\mathbb{E}[p_j(x', \xi)] = 0, j = 1, \ldots, m\) for some \(x' \in X\).

For arbitrary \(\gamma \in (0, 1)\), \(N > 0\) and \(L_j \geq 0\) put

\[
L_j(x) = \mathbb{E}[p_j(x, \xi)], \quad j = 1, \ldots, m,
\]

\[
\alpha_N(x) = N \cdot \sum_{j=1}^m \mathbb{E}[p_j(x, \xi)],
\]

\[
\beta_L(x) = \left(\sum_{j=1}^m L_j\right)^{\gamma-1} \sum_{j=1}^m \mathbb{E}[p_j(x, \xi)],
\]

and let \([N^{1/(\gamma-1)}m] = (N^{1/(\gamma-1)}m, \ldots, N^{1/(\gamma-1)}m)'\) be the vector of length \(m\).

Then for any prescribed \(L_j \geq 0\) there always exists \(N\) large enough so that minimization (5) generates the optimal solutions \(x_N\), which also satisfy the integrated chance constraints (4) with the given \(L = (L_1, \ldots, L_m)'\).

Moreover, bounds on the optimal value \(\varphi_L^{INC}\) of (4) based on the optimal value \(\varphi_N\) of (5) and vice versa can be constructed

\[
\varphi\left(\sum_{j=1}^m L_j(x_N)\right)^{\gamma-1} - \beta_L(x_N)(x_N) \leq \varphi_L^{INC} \leq \varphi_N - \alpha_N(x_N),
\]

\[
\varphi_L^{INC} + \alpha_N(x_N) \leq \varphi_N \leq \varphi_L^{INC} + \beta_N([N^{1/(\gamma-1)}m] + \beta_N([N^{1/(\gamma-1)}m)(x_N)^{INC}([N^{1/(\gamma-1)}m]).
\]
with
\[
\lim_{N \to +\infty} \alpha_N(x_N) = \lim_{N \to +\infty} L_j(x_N) = \lim_{L_{\max} \to 0_+} \beta_L(x_{\text{ICC}}^L) = 0
\]
for any sequences of the optimal solutions \(x_N\) and \(x_{\text{ICC}}^L\) where \(L_{\max}\) denotes the maximal component of the vector \(L\).

**Proof** We denote
\[
\delta_N = \sum_{j=1}^{m} \mathbb{E}[p_j(x_N, \xi)]
\]
for a sequence \(x_N\) of the optimal solutions of the problem (5). Our assumptions and general properties of the penalty function method [2, Theorem 9.2.2] ensure that, for any sequence \(x_N\) of the optimal solutions, \(\delta_N \to 0_+\) and also \(\alpha_N(x_N) = N\delta_N \to 0\) as \(N \to \infty\). Then
\[
\mathbb{E}[p_j(x_N, \xi)] \to 0, \quad \text{as } N \to \infty, \ j = 1, \ldots, m.
\]
Based on this result, it is obvious that for \(N\) large enough the terms \(\mathbb{E}[p_j(x_N, \xi)]\) are arbitrarily small, which implies that the optimal solution \(x_N\) is feasible for the integrated chance-constrained problem with \(L_j\) equal or greater than \(\mathbb{E}[p_j(x_N, \xi)]\).

The following trivial convergence holds
\[
\mathbb{E}[p_j(x_{\text{ICC}}^L, \xi)] \to 0, \quad \text{as } L_j \to 0_+, \ j = 1, \ldots, m.
\]
Accordingly, for any \(\gamma \in (0, 1)\)
\[
\beta_L(x_{\text{ICC}}^L) = \left( \sum_{j=1}^{m} L_j \right)^{\gamma-1} \sum_{j=1}^{m} \mathbb{E}[p_j(x_{\text{ICC}}^L, \xi)] \\
\leq \left( \sum_{j=1}^{m} L_j \right)^{\gamma} \to 0, \quad \text{as } L_{\max} \to 0_+.
\]
If we set
\[
L_j(x_N) = \mathbb{E}[p_j(x_N, \xi)], \quad j = 1, \ldots, m,
\]
then the optimal solution \(x_N\) of the expected value problem is feasible for the integrated chance-constrained program with \(L(x_N) = (L_1(x_N), \ldots, L_m(x_N))\).

Hence, we obtain the inequality
\[
\varphi_N = f(x_N) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x_N, \xi)] \\
\geq f(x_{L(x_N)}) + N \cdot \sum_{j=1}^{m} \mathbb{E}[p_j(x_N, \xi)] \\
= \varphi_{L(x_N)} + \alpha_N(x_N).
\]
Finally,
\[
\varphi_{L}^{ICC} = \left( \varphi_{L}^{ICC} + \left( \sum_{j=1}^{m} L_j \right)^{y-1} \sum_{j=1}^{m} \mathbb{E} \left[ p_j(x_{L}^{ICC}, \xi) \right] \right) - \left( \sum_{j=1}^{m} L_j \right)^{y-1} \sum_{j=1}^{m} \mathbb{E} \left[ p_j(x_{L}^{ICC}, \xi) \right] \geq \varphi_{(\sum_{j=1}^{m} L_j)^{y-1}} - \beta_{L}(x_{L}^{ICC}).
\]
Using the previous inequalities we can obtain the bounds on the optimal values. This completes the proof.

Note that the theorem does not make any statement on the convergence of the optimal solutions, but it relates the optimal values for certain values of the levels and the penalty parameter. We will investigate the behaviour of the optimal solutions in the numerical study in Section 4.

Remark 1 No assumptions on convexity of underlying functions and sets were necessary in the proof. Hence, the approach can be used for mixed-integer stochastic programs where convexity is not usually present [4].

Remark 2 Assumption (iii) may be very strong because it requires existence of a point for which the random constraints are fulfilled for almost all realizations of the random vector $\xi$.

3. Sample approximations using Monte-Carlo techniques

In this section, we will address the sample-approximation technique and derive rates of convergence for the problems with generalized integrated chance constraints. The sample-average approximations were applied to the expected value constrained problems in [23]. We will extend this result to the case with several generalized integrated chance constraints.

Usually, the sample approximation of the chance-constrained problems leads only to a feasible solution of the original problem. Using the results of [21], the problem can be formulated as a large mixed-integer non-linear program. However, due to the increasing number of binary variables, it may be very difficult to solve the resulting problem, even using special solvers for the mixed-integer problems. Hence, it may be interesting to investigate the ability to generate a feasible solution of the chance-constrained problem, i.e. a highly reliable solution, using the penalty function problems where no additional integer variables are necessary. Our approach is summarized in Table 1.

Sample-approximation techniques for chance-constrained problems were investigated in [1,13,17] and were generalized for the case with several joint chance constraints in [3]. We can also refer to [22] for the main results on the sample average approximation techniques for the stochastic programs with expectation type objective, which cover the problems with penalties in the objective.

In all of the problems, it is, in general, not clear as to how to choose the parameters ensuring the reliability levels of the original chance-constrained problem. It is unclear even when solving the sample-approximated chance-constrained problem. We will show that by solving all the problems, we are able to obtain highly reliable solutions of the original problem. It is obvious that the necessary sample size and values of the parameters depend on the particular problem.
First results on parameter settings can be found in [24] where an engineering problem of beam design was solved using chance constraints and penalties in the objective.

Now, we turn our attention to the sample-approximation technique applied to the generalized integrated chance constraints. Let $C_1, \ldots, C_S$ be an independent Monte-Carlo sample of the underlying distribution of the random vector $\xi$. We will denote the sets of feasible solutions of the original and the sample-approximated problem as

$$X_L = \{ x \in X : p_j(x) := E[p_j(x, \xi)] \leq L_j, j = 1, \ldots, m \},$$

$$X^S_L = \{ x \in X : p_j^S(x) := \frac{1}{S} \sum_{s=1}^{S} p_j(x, \xi^s) \leq L_j, j = 1, \ldots, m \}.$$

In [23] the set of feasible solutions of the original problem was relaxed. We will consider the case when the approximated constraints are relaxed and the original constraints remain unchanged. Hence we deal with the probability $P(X_L \subseteq X^S_{L-\tau})$ that the original set of feasible solutions is contained in the relaxed sample-approximated set and we will show that it increases exponentially with increasing sample size. The same results are valid for the probability $P(X^S_{L-\tau} \subseteq X_L)$. Using the rate of convergence, we are able to estimate sample size necessary to ensure that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability and vice versa. We will direct our attention to the case when the set of feasible solutions is finite or bounded infinite, where additional assumptions are necessary. We will denote by $|X|$ the cardinality of the set $X$.

3.1. Finite $|X|$  

**Theorem 3.1** Let

(i) the set of feasible solutions be finite, i.e. $|X| < \infty$,

(ii) the moment generating function of the difference $p_j(x, \xi) - p_j(x)$ be finite.
Then for small $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$,

(a) the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially fast with increasing sample size, and it holds that

$$P(X_L \subseteq X_{L+t}^S) \geq 1 - m|X| \exp \left\{ - S \min_{j,x} \frac{\tau_j^2}{2\sigma_{jx}^2} \right\},$$

where $\sigma_{jx}^2 = \text{Var} [p_j(x, \xi) - p_j(x)]$ and the minimum is taken over $x \in X$ and $j \in \{1, \ldots, m\}$.

(b) it is possible to estimate the sample size $S$ such that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability $1 - \delta$, i.e.

$$S \geq \frac{1}{\min_{j,x} \tau_j^2 / 2\sigma_{jx}^2} \ln \frac{m|X|}{\delta}.$$

Proof For $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, it holds that

$$1 - P(X_L \subseteq X_{L+t}^S) = P(\exists_{j \in \{1, \ldots, m\}} \exists x \in X : p_j(x) \leq L_j$$
$$\leq \sum_{j=1}^m \sum_{x \in X} P(p_j^S(x) - p_j(x) > \tau_j)$$
$$\leq \sum_{j=1}^m \sum_{x \in X} \exp \left\{ - S I_{jx}(\tau_j) \right\},$$

where $I_{jx}$ denotes the large deviations rate function, which is the Fenchel dual to the logarithm of the moment generating function of the difference $p_j(x, \xi) - p_j(x)$, which is assumed to be finite, i.e.

$$I_{jx}(\tau_j) = \sup_{t \in \mathbb{R}} \left\{ t\tau_j - \ln E \left[ e^{t(p_j(x, \xi) - p_j(x))} \right] \right\}.$$ 

For a small $\tau_j$ we can use the following estimate

$$I_{jx}(\tau_j) \geq \frac{\tau_j^2}{2\sigma_{jx}^2},$$

where $\sigma_{jx}^2 = \text{Var} [p_j(x, \xi) - p_j(x)]$. Using the estimate we get

$$1 - P(X_L \subseteq X_{L+t}^S) \leq \sum_{j=1}^m \sum_{x \in X} \exp \left\{ - S \frac{\tau_j^2}{2\sigma_{jx}^2} \right\}$$
$$\leq m|X| \exp \left\{ - S \min_{j,x} \frac{\tau_j^2}{2\sigma_{jx}^2} \right\},$$

where the minimum is taken over $x \in X$ and $j \in \{1, \ldots, m\}$. Using the previous upper bound it is possible to estimate the sample size $S$ such that the feasible solutions of the original problem are feasible for the relaxed sample-approximated problems with a high probability.
3.2. Bounded $|X|$

If the set of feasible solutions is infinite but bounded, we must add an assumption on Lipschitz continuity of the penalized constraints. Then we must differentiate between two cases: when the Lipschitz modulus is fixed or random.

We can use the result valid for the finite case, because for a given $\nu > 0$ there exists a finite set $X_0$ with $|X_0| \leq D^0/\nu^m$ such that for any $x \in X$ there is a $x' \in X_0$ such that $\|x - x'\| \leq \nu$. The choice of $\nu$ will be discussed in the proofs below.

3.2.1. Fixed Lipschitz modulus

**Theorem 3.2** Let

(i) the set of feasible solutions $X$ be bounded, not necessarily finite, with diameter

$$D = \sup_{x, x' \in X} \|x - x'\|,$$

(ii) the functions $p_j(x, \xi)$ be Lipschitz continuous on $X$ moduli $M_j > 0$ which do not depend on $\xi$, i.e.

$$|p_j(x, \xi) - p_j(x', \xi)| \leq M_j \|x - x'\| \quad \forall x, x' \in X, \forall \xi \in \Xi, \forall j,$$

(iii) the moment generating function of $p_j(x, \xi) - p_j(x)$ be finite.

Then for small $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$,

(a) the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially with increasing sample size, and it holds that

$$P(X_L \subseteq X_{L+\tau}^S) \geq 1 - m \left( \frac{D}{\nu} \right)^n \exp \left\{ -n \min_{j,x} \frac{(\tau_j - 2M_j\nu)^2}{2\sigma_{jx}^2} \right\},$$

where $\sigma_{jx}^2 = \text{Var}[p_j(x, \xi) - p_j(x)]$ and the minimum is taken over $x \in X_0$ and $j \in \{1, \ldots, m\}$. The constant $\nu$ is chosen such that $\tau_j - 2M_j\nu$ is small $\forall j$.

(b) we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set and vice versa with a high probability, equal to $1 - \delta$

$$S \geq \frac{1}{\min_{j,x} (\tau_j - 2M_j\nu)^2 / 2\sigma_{jx}^2} \left( \frac{m}{\delta} + n \ln \frac{D}{\nu} \right).$$

**Proof** For $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, it holds that

$$1 - P(X_L \subseteq X_{L+\tau}^S) = P(\exists j \in \{1, \ldots, m\} \exists x \in X : p_j(x) \leq L_j \& p_j^S(x) > L_j + \tau_j)$$

$$\leq P(\exists j \in \{1, \ldots, m\} \exists x \in X_0 : p_j(x) \leq L + M_j\nu$$

$$& p_j^S(x) > L + \tau_j - M_j\nu)$$

$$\leq \sum_{j=1}^m \sum_{x \in X_0} P(p_j^S(x) - p_j(x) > \tau_j - 2M_j\nu)$$

$$\leq \sum_{j=1}^m \sum_{x \in X_0} \exp\{-S\lambda_j(\tau_j - 2M_j\nu)\}.$$
If \( u \) is chosen such that \( |C_2^8|/C_0^2M_ju \) are small, we can use the following bound
\[
I_{jx}(\tau_j - 2M_ju) \geq \frac{(\tau_j - 2M_ju)^2}{2\sigma_{jx}^2},
\]
where \( \sigma_{jx}^2 = \text{Var}[p_j(x, \xi) - p_j(x)] \). Then we obtain
\[
1 - P(X_L \subseteq X_{L+1}^S) \leq |X_o| \sum_{j=1}^m \exp \left\{ - S \left( \frac{(\tau_j - 2M_ju)^2}{2\sigma_{jx}^2} \right) \right\}
\leq m \left( \frac{D}{\nu} \right)^n \exp \left\{ - S \min_{j,x} \frac{(\tau_j - 2M_ju)^2}{2\sigma_{jx}^2} \right\}.
\]
Finally, we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set with a high probability.

### 3.2.2. Random Lipschitz modulus

**Theorem 3.3** Let

(i) the set of feasible solutions \( X \) be bounded with a diameter \( D = \sup_{x,x' \in X} \|x - x'\| \),

(ii) \( p_j(x, \xi) \) be Lipschitz continuous on \( X \) moduli \( M_j(\xi) > 0 \) which depend on \( \xi \), i.e.
\[
|p_j(x, \xi) - p_j(x', \xi)| \leq M_j(\xi) \|x - x'\| \quad \forall x, x' \in X, \forall \xi \in \Xi, \forall j,
\]

(iii) \( M_j = \mathbb{E}[M_j(\xi)] < \infty \ \forall j, \)

(iv) \( \) the moment generating functions of \( p_j(x, \xi) - p_j(x) \) and \( M_j(\xi) - M_j \) \( \) be finite. \( \)

Then for \( \tau_j > 0 \) small, \( \tau = (\tau_1, \ldots, \tau_m) \),

(a) the probability that the set of feasible solutions is contained in the relaxed sample-approximated set of feasible solutions increases exponentially with increasing sample size, and it holds that
\[
P(X_L \subseteq X_{L+1}^S) \geq 1 - m \left( 1 + \frac{D^n}{\nu^n} \right) \exp \left\{ - Sd(\tau) \right\},
\]
where
\[
\sigma_{jx}^2 = \text{Var}[p_j(x, \xi) - p_j(x)],
\]
\[
\sigma_{M_j}^2 = \text{Var}[M_j(\xi) - M_j],
\]
\[
d(\tau) = \min \left\{ \min_{j,x} \frac{\tau_j^2}{8\sigma_{jx}^2}, \min_j \frac{\tau_j^2}{8\sigma_{M_j}^2} \right\},
\]
the minimum is taken over \( x \in X_o \) and \( j \in \{1, \ldots, m\} \) and the constant \( \nu = \max_j \tau_j/(4M_j + \tau_j) \).

(b) we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original
feasibility set and vice versa with a high probability, equal to $1 - \delta$

$$S \geq \frac{1}{d(\tau)} \left( \ln \frac{m}{\delta} + \ln \left( 1 + \frac{D^n}{v^\tau} \right) \right).$$

**Proof**  Denote $M_j^S = 1/S \sum_{x=1}^S M_j(\xi^x)$ the sample-approximated Lipschitz modulus. For $\tau_j > 0$, $\tau = (\tau_1, \ldots, \tau_m)$, it holds that

$$1 - P(X_L \leq X_{L+\tau}) = P(\exists j \in [1, \ldots, m] : p_j(x) \leq L_j \& p_j^S(x) > L_j + \tau_j)$$

$$\leq P(\exists j \in [1, \ldots, m] : p_j(x) \leq L + M_j \nu$$

$$\& p_j^S(x) > L + \tau_j - M_j^S \nu)$$

$$= P(\exists j \in [1, \ldots, m] : p_j^S(x) - p_j(x) > \tau_j - (M_j^S + M_j \nu).$$

In this case, such $\nu$ should be chosen that the following implications hold for any $j$

$$M_j^S - M_j > \frac{\tau_j}{2} \Rightarrow (M_j^S + M_j) \nu > \frac{\tau_j}{2}.$$

It is enough to set

$$\nu_j = \frac{\tau_j}{4M_j + \tau_j}$$

and $\nu = \max_j \nu_j$. Then, if the moment generating functions of $p_j(x, \xi) - p_j(x)$ and $M_j(\xi) - M_j$ are finite, we can use the following estimates

$$P(\exists j \in [1, \ldots, m] : p_j^S(x) - p_j(x) + (M_j^S + M_j) \nu > \tau_j)$$

$$\leq \sum_{j=1}^m P(M_j^S > M_j + \tau_j/2) + \sum_{j=1}^m P(\exists x \in X_u : p_j^S(x) - p_j(x) > \tau_j/2)$$

$$\leq \sum_{j=1}^m P(M_j^S - M_j > \tau_j/2) + \sum_{j=1}^m \sum_{x \in X_u} P(p_j^S(x) - p_j(x) > \tau_j/2)$$

$$\leq \sum_{j=1}^m \exp\{-SI_{M_j}(\tau_j/2)\} + \sum_{j=1}^m \sum_{x \in X_u} \exp\{-SI_{I_j}(\tau_j/2)\},$$

where $I_{M_j}, I_{I_j}$ are the corresponding large deviation functions. If $\tau_j$ are small, we can use the estimates

$$I_{I_j}(\tau_j/2) \geq \frac{\tau_j^2}{8\sigma_{I_j}^2}, \quad I_{M_j}(\tau_j/2) \geq \frac{\tau_j^2}{8\sigma_{M_j}^2},$$

where $\sigma_{M_j}^2 = \text{Var}[M_j^S - M_j]$. We set

$$d(\tau) = \min \left\{ \min_{j,x} \frac{\tau_j^2}{8\sigma_{I_j}^2}, \min_j \frac{\tau_j^2}{8\sigma_{M_j}^2} \right\}.$$
Then

\[ 1 - P(X_L \leq X_{L+\epsilon}) \leq m(1 + |X_0|)\exp\{-Sd(\tau)\}. \]

Finally, we can get an estimate for the sample size which is necessary to ensure that the relaxed sample-approximated feasibility set is contained in the original feasibility set with a high probability.

4. Numerical comparison

In this section, we deal with blending problem which was originally defined as a joint chance-constrained problem. The costs of the fertilizers are minimized subject to the constraints on minimal nutrients necessary to increase the production of a crop. However, the first fertilizer has an uncertain nutrient content.

The problem was solved using the sample-approximation method in [17]. We will show that by solving the sample-approximated problems with expectation of the penalized random constraints we can also obtain reliable solutions of the underlying problem.

The blending problem is formulated as jointly chance-constrained program

\[
\phi^{CCP}_\epsilon = \min x_1 + x_2 \\
\text{s.t.} \\
P(\xi_1 x_1 + x_2 \geq 7, \xi_2 x_1 + x_2 \geq 4) \geq 1 - \epsilon, \\
x_1 \geq 0, \quad x_2 \geq 0,
\]

where \( \epsilon \in (0, 1) \) and the random components \((\xi_1, \xi_2)\) are independent and have uniform distributions on the intervals \([1, 4]\) and \([1/3, 1]\). The explicit solution can be obtained, cf. [17], depending on the choice of the level \( \epsilon \in [0.5, 1] \):

\[
x_1^* = \frac{9}{11 - 9(1 - \epsilon)}, \quad x_2^* = \frac{41 - 36(1 - \epsilon)}{11 - 9(1 - \epsilon)}, \quad \phi^{CCP}_\epsilon = \frac{50 - 36(1 - \epsilon)}{11 - 9(1 - \epsilon)}.
\]

Optimal solutions for particular choices of the level \( \epsilon \) are given in Table 2.

Using a penalty function \( \vartheta: \mathbb{R}^2 \to \mathbb{R}_+ \) we can obtain the problem with a generalized integrated chance constraint

\[
\min x_1 + x_2 \\
\text{s.t.} \\
\mathbb{E}\left[\vartheta(7 - \xi_1 x_1 - x_2, 4 - \xi_2 x_1 - x_2)\right] \leq L, \\
x_1 \geq 0, \quad x_2 \geq 0
\]

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for some level $L \geq 0$. If we incorporate the expectation of the penalized constraints into the objective, we get

$$\min_{x_1, x_2} \quad x_1 + x_2 + N \cdot \mathbb{E}\left[\vartheta(7 - \xi_1 x_1 - x_2, 4 - \xi_2 x_1 - x_2)\right]$$

s.t.
$$x_1 \geq 0, \quad x_2 \geq 0,$$

for some $N > 0$. The reformulation using penalty objective was also considered in [5] to study stability of the optimal value with respect to the changes of the underlying probability measure using contamination technique. In a case where the penalty term is small enough, the problem with penalty objective can serve as a good approximation of the chance-constrained problem. This was confirmed by the proposed numerical experiment.

We generated 100 samples for each sample size (100, 150, 200, 500) and solved the resulting problems using GAMS 23.2 and IBM ILOG CPLEX 12.1. The sample sizes were chosen according to the recommendations given in [17] (100–200) and one larger sample size (500) was taken into consideration. We used two penalty functions in both problems: $\vartheta^{1,1}(u_1, u_2)=[u_1]^+[u_2]^+$ and $\vartheta^2(u_1, u_2)=\max\{[u_1]^+, [u_2]^+\}$.

To solve the sample-approximated chance constrained problem, additional binary variables $y_s$ are necessary

$$\min_{x_1, x_2} \quad x_1 + x_2$$

s.t.
$$7 - \xi_1^s x_1 + x_2 \leq C(1 - y_s),$$
$$4 - \xi_2^s x_1 + x_2 \leq C(1 - y_s),$$
$$\frac{1}{S} \sum_{s=1}^{S} y_s \geq 1 - \gamma,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad y_s \in \{0, 1\}, s = 1, \ldots, S,$$

where $C$ is a large enough constant. On the other hand, in the penalty problems no additional binary variables are necessary and only continuous variables are needed to model positive parts in the penalty functions. We obtained a sample-approximated problem with generalized integrated chance constraint using the penalty function $\vartheta^{1,1}$:

$$\min_{x_1, x_2} \quad x_1 + x_2$$

s.t.
$$7 - \xi_1^s x_1 + x_2 \leq u_{1s},$$
$$4 - \xi_2^s x_1 + x_2 \leq u_{2s},$$
$$\frac{1}{S} \sum_{s=1}^{S} u_{1s} + u_{2s} \leq L,$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad u_{1s}, u_{2s} \geq 0, s = 1, \ldots, S,$$
for some level $L > 0$, sample-approximated problem with integrated chance constraint using the penalty function $\vartheta^2$

$$\min x_1 + x_2$$

s.t.

$$7 - \xi_1^i x_1 + x_2 \leq u_s,$$
$$4 - \xi_2^i x_1 + x_2 \leq u_s,$$
$$\frac{1}{S} \sum_{s=1}^{S} u_s \leq L,$$
$$x_1 \geq 0, \ x_2 \geq 0, \ u_s \geq 0, \ s = 1, \ldots, S,$$

for some level $L > 0$, a sample-approximated problem with penalty objective using the penalty function $\vartheta^{1.1}$

$$\min x_1 + x_2 + \frac{N}{S} \sum_{s=1}^{S} u_{1s} + u_{2s}$$

s.t.

$$7 - \xi_1^i x_1 + x_2 \leq u_{1s},$$
$$4 - \xi_2^i x_1 + x_2 \leq u_{2s},$$
$$x_1 \geq 0, \ x_2 \geq 0, \ u_{1s}, u_{2s} \geq 0, \ s = 1, \ldots, S,$$
for a penalty parameter $N > 0$, and sample-approximated problem with penalty objective using the penalty function $\varphi^2$

$$\min x_1 + x_2 + \frac{N}{S} \sum_{s=1}^{S} u_s$$

s.t.
$$7 - \xi_1^s x_1 + x_2 \leq u_s,$$
$$4 - \xi_2^s x_1 + x_2 \leq u_s,$$
$$x_1 \geq 0, \ x_2 \geq 0, \ u_s \geq 0, \ s = 1, \ldots, S$$

for a penalty parameter $N > 0$.

Due to the simple structure of the problem we can compute the exact probability of fulfilling the constraints for obtained solutions of the sample-approximated problems. Mean and minimal probabilities are contained in Tables 3–7. The tables
also contain mean optimal solutions and mean optimal values. In Tables 6 and 7, there are also means of the penalty terms. As we can see, the penalty terms actually decrease with increasing $N$ and penalize the violations of the random constraints.

For instance, the obtained solutions for sample size $S = 200$ and choice of parameters $\gamma = 0.001$, $L = 0.0001$, $N = 100$ are reliable for at least 95% and, on average, for almost 99%. Of course, the setting of parameters for a particular sample size depends on the concrete problem. In our blending problem, there is no significant difference in results caused by choosing any of the penalty functions.

### 5. Conclusion

In this article, we dealt with several possible stochastic programming formulations of a problem with constraints depending on a random factor. Chance-constrained
problems and problems with penalties in the constraints or in the objective were considered. We focused on the problems with expectation-type constraints which generalize the so-called integrated chance constraints. We showed that these problems are, under quite mild conditions, asymptotically equivalent to those with penalties in the objective. This is a result complementary to [3,5,8] where similar equivalence was proven between the chance-constrained problems and the problems with a penalty objective.

We generalized results on the exponential rates of convergence for sample-approximated integrated chance-constrained problems. Based on these rates, it is possible to estimate sample sizes which are necessary to ensure that the feasible solutions of the original problem are also feasible for a relaxed sample-approximated problem. Moreover, no additional binary variables were necessary to be introduced in order to solve the penalty problems.

In the numerical study, we obtained promising results that the sample-approximated penalty problems can provide highly reliable solutions. This is possible without using mixed-integer formulation, avoiding to the binary variables by employing continuous variables to model the penalty terms.

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Acknowledgements
This work was supported by the grants GA CR P402/10/1610 and SVV 261315/2010.

References


Table 7. Blending problem – penalty objective $\theta^2$.

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