Third-degree stochastic dominance and DEA efficiency – relations and numerical comparison

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Abstract. We propose efficiency tests which are related to the third-degree stochastic dominance (TSD). The tests are based on necessary conditions for TSD and on related mean-risk models. We test pairwise efficiency as well as portfolio efficiency with respect to full diversification of available assets.

We apply the proposed tests to 25 world financial indexes and we select the efficient ones. The test data set is divided into the periods – before financial crises and during it, and it is also considered at once.

Keywords: third-degree stochastic dominance, stochastic dominance efficiency, mean-risk efficiency, DEA efficiency.

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1 Introduction

Dealing with uncertainty on financial markets is very difficult task. The investor's decision is highly dependent on the selected criteria which should help him to select the best among available investment opportunities. Harry Markowitz, [12], introduced his mean-risk model more than 50 years ago where variance was used as the risk measure. Many other risk measures has been proposed since then. The axiomatic definition of coherent risk measures is accepted by theorists as well as by practitioners, cf. [1]. The purpose of the mean-risk models is to maximize the mean return and to minimize the risk at the same time under given constraints on portfolio composition leading to biobjective optimization problem.

Another possible method how to find the best investment opportunity is to use an utility function, cf. [13]. To compare two possible outcomes, it is necessary to choose a particular nondecreasing function which corresponds to the investor's aversion to risk and serves as the utility function, and then to find an investment opportunity with the highest expected utility.

Stochastic dominance, introduced by [6, 7], is very closely related to the utility functions. It is defined over a whole set of utility functions with desired properties and compares the portfolios with respect to the whole class. Note that the stochastically dominating random variables are also optimal with respect to particular classes of risk measures, cf. [5, 14]. Note that mean-variance efficiency does not imply stochastic dominance efficiency, see [10]. Third-degree stochastic dominance (TSD) was introduced in [16] as a natural extension of stochastic dominances of lower orders. It is suitable for investors with decreasing absolute risk aversion. Recently, qualitative stability of an investment model with TSD constraint was investigated in [2].

Data Envelopment Analysis (DEA) was introduced by [4] as a tool of selection efficient units among units with the same structure of inputs and outputs. We will formulate models which help us to select efficient investment opportunities where historical rates of return are used as the inputs and the mean and risk as the outputs. By an appropriate choice of the risk measure we can obtain an efficiency test which is consistent with TSD. Efficiency tests for dominances of lower orders were proposed in [8, 9].

The paper is organized as follows. In Section 2, the third-degree stochastic dominance is defined and the basic properties are summarized. We propose various efficiency tests in Section 3. The tests are then used to find efficient world financial indeces in Section 4.

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2 Third-degree stochastic dominance

Let $\mathcal{X}$ be a set of available investment opportunities with finite second moments. We prefer higher values to lower, i.e. we deal with profits, rates of return etc. Possible choices of the set will be discussed in the next section. We will propose two equivalent definitions of the third-degree stochastic dominance. The first was given originally by [16].

Definition 1. Let $\mathcal{U}_3$ be a class of real-valued differentiable functions with $u' > 0$, $u'' \leq 0$, and $u''' \geq 0$. The relation $X \succeq_{TSD} Y$ is equivalent to the condition that $\mathbb{E}u(X) \geq \mathbb{E}u(Y)$ holds for all utility functions $u \in \mathcal{U}_3$ for which both expectations are finite, and the strict dominance, $X \succ_{TSD} Y$, holds iff moreover there exists $u \in \mathcal{U}_3$ such that $\mathbb{E}u(X) > \mathbb{E}u(Y)$.

Below we will give two arguments why to consider the third-degree stochastic dominance instead of dominances of lower orders. The index of absolute risk aversion is usually defined as $ara(x) = -\frac{u''(x)}{u'(x)}$. It can be interpreted as the normalized relative change in marginal utility due to a change in wealth and it relates to instantaneous aversion to risk. The utility functions for which $ara'(x) < 0$ are usually referred as decreasing absolute risk aversion (DARA) utility functions. For example, with constant absolute risk aversion, our risk-taking behavior is the same regardless of the size of the wealth. The necessary but not sufficient condition for decreasing absolute risk aversion is that $u''' > 0$, because it holds

$$ara'(x) = \frac{-u'''(x)u'(x) + (u''(x))^2}{(u'(x))^2} < 0,$$

The second heuristic motivation why to consider the condition on the third derivative of the utility functions can be found in [10] and says: if we denote w the initial wealth and $X \in \mathcal{X}$ a random variable with finite third moment $\mathbb{E}X^3$, we can expand the utility function $u(w + X)$ into Taylor series at the point $w + \mathbb{E}X$, compute its expected value and we approximately obtain

$$\mathbb{E}[u(w + X)] \cong u(w + \mathbb{E}X) + \frac{u''(w + \mathbb{E}X)}{2!}\sigma_X^2 + \frac{u'''(w + \mathbb{E}X)}{3!}\nu_X^3,$$

where σ_X^2 is the variance of X and ν_X^3 its third central moment. If the other factors are held constant, then the higher σ_X^2, the lower the expected utility of an investor is, and the higher the skewness, the higher the expected utility. Hence, under our assumptions on $\mathcal{U}_3$ the investor dislikes variance and likes positive skewness. Fortunately, in practical applications of the third-degree stochastic dominance we do not need the restrictive condition $\mathbb{E}X^3 < \infty$.

Now we propose an alternative definition which we will use in the next section. We consider the cumulative distribution functions which are derived from the distribution function $F_X^{(1)} = F_X$ of $X \in \mathcal{X}$:

$$F_X^{(k)}(\eta) = \int_{-\infty}^{\eta} F_X^{(k-1)}(\xi)d\xi, \ \forall \eta \in \mathbb{R}, \ k = 2, 3,$$

Definition 2. Let $X, Y \in \mathcal{X}$ be two random variables on $(\Omega, \mathcal{F}, P)$ with the distribution functions F_X, F_Y. We say that X dominates Y in the sense of third-degree stochastic dominance, denoted by $X \succeq_{TSD} Y$, if and only if the following two conditions hold:

$$\begin{aligned} F_X^{(3)}(\eta) &\leq F_Y^{(3)}(\eta), \ \forall \eta \in \mathbb{R}, \\ \mathbb{E}X &\geq \mathbb{E}Y. \end{aligned}$$

We say that X strictly dominates Y in the sense of third-degree stochastic dominance, denoted by $X \succ_{TSD} Y$, if and only if $X \succeq_{TSD} Y$ and $Y \succeq_{TSD} X$ does not holds.

Note that the condition which compares the expectations is not necessary if the supports of the compared random variables are unbounded, see [15].

3 Efficiency tests

In this section we propose several tests which should help us to identify efficient investment opportunities. Pairwise efficiency as well as portfolio efficiency allowing full diversification across the assets are taken into consideration. Using pairwise comparisons, an asset is classified as efficient if there is no other asset that strictly dominates the asset with respect to the criteria. These efficiency may be more useful for financial indeces. Since investors may combine the assets, tests for portfolio efficiency allowing full diversification across the assets are of interest too.

We consider n assets and denote R_i the rate of return of i-th asset. The following two choices of the set of investment opportunities will be used:

1. $\mathcal{X}^P = \{R_i, i = 1, \ldots, n\}$, which corresponds to investment into one single asset, and enables us to test pairwise efficiency,
2. $\mathcal{X}^{FD} = \{\sum_{i=1}^n R_i x_i : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$, which enables diversification of our portfolio across all assets, hence we will use it to test efficiency with respect to full diversification.

Another choices of the set are also possible, e.g. allowing short sales, and will be aimed in future research. We will show how the efficiency tests can be constructed in general or based on discretely distributed returns. Let r_i^t, $t = 1, \ldots, T$, be the t-th realizations of the i-th asset return R_i. It can be computed as: $r_i^t = \frac{P_i^t}{P_i^{t-1}} - 1$ where P_i^t, P_i^{t-1} is the price of the i-th asset at the end of the t-th, $(t-1)$-st time period, respectively.

3.1 Mean-risk efficiency

Let $\mathcal{R} : \mathcal{X} \to \mathbb{R}$ denote a risk measure which is a function of available investment opportunities and which quantifies the corresponding risk as a real number.

Definition 3. We say that $X \in \mathcal{X}$ strictly dominates $Y \in \mathcal{X}$ in the sense of mean-risk criterion, denoted $X \succ_{\mathbb{E},\mathcal{R}} Y$, if $\mathbb{E}X \geq \mathbb{E}Y$ and $\mathcal{R}(X) \leq \mathcal{R}(Y)$ with at least one strict inequality.

Definition 4. We say that $X \in \mathcal{X}$ is mean-risk efficient if there exists no $Y \in \mathcal{X}$ such that $Y \succ_{\mathbb{E},\mathcal{R}} X$.

By an appropriate choice of the risk measure we can get various efficiency tests. Our tests are based on Theorem 1 in [14], which states necessary conditions for $X \succeq_{TSD} Y$: $\mathbb{E}X \geq \mathbb{E}Y$ and $\mathbb{E}X - lsd(X) \geq \mathbb{E}Y - lsd(Y)$, where lsd denotes the lower semideviation. For $X \in \mathcal{X}$, it is defined as

$$lsd(X) = \left(\mathbb{E}[X - \mathbb{E}X]_-^2\right)^{1/2},$$

where $[\cdot]_-^2 = (\min\{0, \cdot\})^2$. If we consider discretely distributed returns, we get for the i-th asset

$$lsd(R_i) = \left(\frac{1}{T}\sum_{t=1}^T [r_i^t - \overline{r}_i]_-^2\right)^{1/2},$$

where $\overline{r}_i = \frac{1}{T}\sum_{t=1}^T r_i^t$.

If at least one of the following conditions holds with strict inequality, then $Y \succ_{\mathbb{E},lsd} X$:

$$\mathbb{E}Y \geq \mathbb{E}X, \quad lsd(X) \geq lsd(Y). \tag{1}$$

Using the following program for $\alpha \in (0, 1]$ we can obtain portfolios which are mean-lsd efficient and consistent with TSD, cf. [14]:

$$\begin{aligned} \max \sum_{i=1}^n \overline{r}_i x_i + \alpha \frac{1}{T} \sum_{t=1}^T z_t^2 & \\ \sum_{i=1}^n x_i(\overline{r}_i - r_t^t) &\leq z_t, \\ \sum_{i=1}^n x_i &= 1, \\ x_i, z_t &\geq 0, \end{aligned} \tag{2}$$

where z_t are auxiliary decision variables which help us to model the positive parts.

3.2 TSD efficiency

Definition 5. We say that $X \in \mathcal{X}$ is efficient with respect to TSD if there exists no $Y \in \mathcal{X}$ such that $Y \succ_{TSD} X$.

To compare two random variables we will use the alternative expression of the integrated distribution function, see [5, 14]:

$$F_X^{(3)}(\eta) = \frac{1}{2}\int_{-\infty}^{\eta} [\eta - \xi]_+^2 dF_X(\xi).$$

where $[\cdot]_+^2 = (\max\{0, \cdot\})^2$. We can test $Y \succ_k X$ by investigating if the following conditions hold with at least one strict inequality in

$$\begin{aligned} F_X^{(3)}(\eta) - F_Y^{(3)}(\eta) &\geq 0, \forall \eta, \\ \mathbb{E}Y - \mathbb{E}X &\geq 0. \end{aligned} \tag{3}$$

3.3 DEA efficiency test

In this section, we will propose new DEA portfolio efficiency test with respect to the third-degree stochastic dominance. Any asset is compared with all portfolios which can be mixed from all considered assets, i.e. full diversification is enabled. The test for a benchmark $b \in \{1, \ldots, n\}$ can be formulated in general as follows:

$$\begin{aligned} &\max \delta^m + \delta^r \\ \sum_{i=1}^n x_i \mathbb{E}R_i &= \mathbb{E}R_b + \delta^m, \\ lsd^2\Big(\sum_{i=1}^n x_i R_i\Big) &\leq lsd^2(R_b) - \delta^r, \\ \sum_{i=1}^n x_i &= 1, \\ x_i, \delta_m, \delta_r &\geq 0, \end{aligned}$$

where the mean and lower semideviation of the benchmark are compared with portfolio mean and risk. If the optimal value is equal to 0, then the benchmark asset is said to be efficient, otherwise it is not efficient. Similar three tests were proposed and compared in [11]. Note that the proposed test as well as our test state only necessary condition for TSD-efficiency.

For discretely distributed random returns, we obtain the following quadratic programming problem which can be easily solved by standard solvers:

$$\begin{aligned} &\max \delta^m + \delta^r \\ \sum_{i=1}^n \overline{r}_i x_i &= \overline{r}_b + \delta^m, \\ \sum_{i=1}^n x_i(\overline{r}_i - r_t^t) &\leq z_t, \\ \frac{1}{T}\sum_{t=1}^T z_t^2 &\leq lsd_b^2 - \delta^r, \\ \sum_{i=1}^n x_i &= 1, \\ x_i, z_t, \delta_m, \delta_s &\geq 0. \end{aligned} \tag{4}$$

4 Stock indices efficiency – empirical study

We consider the following 25 world financial (stock) indices which are listed on Yahoo Finance:

- **America** (5): MERVAL BUENOS AIRES, IBOVESPA, S&P TSX Composite index, S&P 500 INDEX RTH, IPC,
- **Asia/Pacific** (11): ALL ORDINARIES, SSE Composite Index, HANG SENG INDEX, BSE SENSEX, Jakarta Composite Index, FTSE Bursa Malaysia KLCI, NIKKEI 225, NZX 50 INDEX GROSS, STRAITS TIMES INDEX, KOSPI Composite Index, TSEC weighted index,
- **Europe** (8): ATX, CAC 4, DAX, AEX, SMSI, OMX Stockholm PI, SMI, FTSE 100,
- **Middle East** (1): TEL AVIV TA-100 IND.

In our analysis we describe each index by its weekly rates of returns. We divided the returns into three datasets:

- before crises (B): September 11, 2006 - September 15, 2008
- during crises (D): September 16, 2008 - September 20, 2010
- whole period (W).

We choose September 16, 2008 to divide the data because all financial indices strongly fell down in week starting with this day. The descriptive statistics of the returns can be found in [3], where the same dataset was analyzed using different techniques. It can be observed that almost all returns are negatively skewed. Moreover, comparing the before crises data with during crises data we found that the during crises returns usually have higher standard deviation and kurtosis.

Table 1 shows efficient indeces according to the tests introduced in the previous section: pairwise tsd (1), pairwise mean-lsd (2), full diversification mean-lsd (3), and DEA (4). The pairwise comparison using the integrated distribution functions $F^{(3)}$ was implemented in Matlab using optimization toolbox. The quadratic programming tests were solved using the modelling system GAMS 23.0 and the solver Cplex 12.0.

The pairwise mean-lsd test selects most of efficient indeces and all the indeces selected by another tests are among them. The efficient indeces selected by mean-lsd model and DEA test are the same. It can be also seen that the returns observed during crises influence the tests based on the whole period more than the returns obtained before crises.

	P-TSD			P-ML			F-ML / DEA		
	B	D	W	B	D	W	B	D	W
IBOVESPA				X			X		
S&PTSX Composite index				X					
S&P 500 INDEX,RTH	X			X					
IPC				X		X			
BSE SENSEX				X		X			
Jakarta Composite Index					X	X		X	X
FTSE Bursa Malaysia KLCI		X	X		X	X			
NZX 50 INDEX GROSS	X			X		X			
TSEC weighted index					X				

Table 1: Efficient indeces (B - before crises, D - during crises, W - whole period)

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