Multifractal Height Cross-Correlation Analysis

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Abstract

We introduce a new method for detection of long-range cross-correlations and cross-multifractality – multifractal height cross-correlation analysis (MF-HXA). MF-HXA is a multivariate generalization of the height-height correlation analysis. We show that long-range cross-correlations can be caused by a mixture of the following – long-range dependence of separate processes and additional scaling of covariances between the processes. Similar separation applies for cross-multifractality – standard separation between distributional properties and correlations is enriched by division of correlations between auto-correlations and cross-correlations. Efficiency of the method is showed on two types of simulated series – ARFIMA and Mandelbrot's Binomial Multifractal model. We further apply the method on returns and volatility of NASDAQ and S&P500 indices as well as of Crude and Heating Oil futures and uncover some interesting results.

Keywords: multifractality, long-range dependence, cross-correlations

1 Introduction

The research of long-range dependence and multifractality in various time series has grown significantly during the last years, e.g. Di Matteo (2007); Matos et al. (2008); Czarnecki et al. (2008); Grech and Mazur (2004). An efficient detection of long-range dependence and estimation of Hurst exponent is crucial for financial analysts as its presence has important implications for a portfolio selection, an option pricing and a risk management. There are several methods for the long-range dependence detection, among the most popular are the rescaled range analysis (Hurst, 1951), the modified rescaled range analysis (Lo, 1991), the rescaled variance analysis (Giraitis et al., 2003), the detrended fluctuation analysis (Peng et al., 1994) and the detrending moving average (Alessio et al., 2002). For the detection of multifractality, there are three popular methods –

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the multifractal detrended fluctuation analysis (MF-DFA) of Kantelhardt et al. (2002), the generalized Hurst exponent approach (GHE) of Alvarez-Ramirez et al. (2002); Di Matteo et al. (2003), which is based on the height-height correlation analysis of Barabasi et al. (1991), and the wavelet transform modulus maxima (WTMM) of Muzy et al. (1991). The precision of various methods has been discussed as well (Couillard and Davison, 2005; Grech and Mazur, 2005; Weron, 2002; Barunik and Kristoufek, 2010; Kristoufek, 2009).

Recently, the examination of long-range cross-correlations has become of interest as it provides more information about the examined process. Podobnik et al. Podobnik and Stanley (2008) generalized the detrended fluctuation analysis for two time series and introduced the detrended cross-correlation analysis (DCCA). Zhou Zhou (2008) further generalized the method and introduced the multifractal detrended cross-correlation analysis (MF-DXA). In this paper, we introduce two new methods, which are a generalization of the height-height correlation analysis of Barabasi et al. (1991) – the multifractal height cross-correlation analysis (MF-HXA) and its special case of the height cross-correlation analysis (HXA).

The paper is structured as follows. In Section 2, we briefly discuss the basic notions of long-range correlations and multifractality. Section 3 introduces the method of MF-HXA and discusses long-range cross-correlations and cross-multifractality in detail. In Section 4, we show the efficiency of the method on two simulated types of processes. In Section 5, we apply MF-HXA on daily returns and volatility of NASDAQ and S&P500 indices as well as of the Crude and Heating Oil. We show that these two pairs of processes posses very different properties. The long-rage correlations and cross-multifractality of the stock indices cannot be distinguished from two pairwise independent processes. On the other hand, the commodity pair shows very complex dynamics with long-range cross-correlations and cross-multifractality deviating from the pairwise independent behavior. Section 6 concludes.

2 Long-range correlations and multifractality

In this section, we present basic notions of multifractality, long-range correlations and long-range cross-correlations. As the subject is widely discussed in the recent literature, we present only a brief description. For more detailed reviews, see Beran (1994); Kantelhardt (2009); Embrechts and Maejima (2002).

A stationary process is long-range dependent if an autocorrelation function $\rho(k)$ of the said process decays as $\rho(k) \sim Ck^{2H-2}$ for lag $k \to \infty$ where parameter $0 < H < 1$ is Hurst exponent (Hurst, 1951; Mandelbrot and van Ness, 1968).

A critical value of Hurst exponent is 0.5 and suggests two possible processes – either an independent process (Beran, 1994) or a short-term dependent process (Lillo and Farmer, 2004). If $H > 0.5$, auto-covariances decay hyperbolically and are positive at all lags, the process is then called long-range dependent with positive correlations (Embrechts and Maejima, 2002) or persistent (Mandelbrot and van Ness, 1968). On the other hand, if $H < 0.5$, auto-covariances again decay hyperbolically and are negative at all lags and the process is said to be long-range dependent with negative correlations (Embrechts and Maejima, 2002) or anti-persistent (Mandelbrot and van Ness, 1968). The persistent process implies that a positive movement is statistically more likely to be followed by another positive movement or vice versa. On the other hand, the anti-persistent process implies that a positive movement is more statistically probable to be followed by a negative movement and vice versa (Vandewalle et al., 1997).

If the process can be described by a single Hurst exponent H, it is called monofractal (or unifractal). If different Hurst exponents are needed for various scales, the process exhibits crossovers and is called a multiscaling process. Further, there can be different Hurst exponents for parts of the series, which is solved by a use of the time-dependent (or local) Hurst exponent (Grech and Mazur, 2004). The most complicated is the case when there is a whole spectrum of Hurst exponents which is needed for a full description of the process made up of many complex fractal processes (Kantelhardt et al., 2002).

Both long-range dependence and multifractality can be present in the relation between two separate series. The series may be long-range dependent but can also have a long memory of a different process so that it is pairwise long-range dependent with Hurst exponent H_{xy}. Cross-correlation function $\rho_{xy}(k)$ of processes x_t and y_t then decays hyperbolically as $\rho_{xy}(k) \sim Ck^{2H_{xy}-2}$. Similarly to the standard case, if the whole spectrum of cross-correlation Hurst exponents H_{xy} is needed for the description of the cross-correlations between two processes, the relation is cross-multifractal. Further features of the long-range cross-correlations and cross-multifractality are discussed in the following sections.

3 Multifractal height cross-correlation analysis

We introduce the multifractal height cross-correlation analysis (MF-HXA) in this section. The connection to the generalized Hurst exponent approach (GHE) is discussed in detail as well as a crucial division of long-range cross-correlations. The last subsection discusses a detection of cross-multifractality in a pair of series.

3.1 Method

The detection of long-range dependence and the estimation of the generalized Hurst exponent $H(q)$ of Barabasi et al. (1991) is based on the q-th order height-height correlation function of time series $X(t)$, with $q > 0$, as

$$K_q(\tau) = \frac{1}{T-\tau} \sum_{t=0}^{T-\tau} (|X(t+\tau) - X(t)|^q) \tag{1}$$

which scales as

$$K_q(\tau) \propto \tau^{qH(q)}. \tag{2}$$

We generalize the method presented above and introduce the multifractal height cross-correlation analysis (MF-HXA) which can be used for the detection of long-range correlations and multifractality between two separate time series.

In the procedure, we take the first differences of time series $\{x(t)\}_{t=0}^T$ and $\{y(t)\}_{t=0}^T$ and obtain $\{\Delta x(t)\}_{t=1}^T$ and $\{\Delta y(t)\}_{t=1}^T$. The differences are further standardized by a deduction of a corresponding mean μ and a division by a corresponding standard deviation σ so that we get the new series $\{\Delta\tilde{x}(t)\}_{t=1}^T = \{\frac{\Delta x(t)-\mu_x}{\sigma_x}\}_{t=1}^T$ and $\{\Delta\tilde{y}(t)\}_{t=1}^T = \{\frac{\Delta y(t)-\mu_y}{\sigma_y}\}_{t=1}^T$. Further, the series are cumulated so that we obtain $\{X(t)\}_{t=0}^T$ and $\{Y(t)\}_{t=0}^T$ with $X(t) = \sum_{i=1}^t \Delta\tilde{x}(i)$ and $Y(t) = \sum_{i=1}^t \Delta\tilde{y}(i)$. Moreover, $X(0) = Y(0) = 0$. Generalizing Equation 1 for two time series, we obtain

$$K_{xy,q}(\tau) = \frac{1}{T-\tau}\sum_{t=0}^{T-\tau}(|[X(t+\tau)-X(t)][Y(t+\tau)-Y(t)]|^{\frac{q}{2}}) \tag{3}$$

For $q = 1$, the generalized height correlation function represents a scaling of the absolute deviations of the covariates; and for $q = 2$, it corresponds to the standard cross-correlation function. We propose the multifractal height cross-correlation analysis (MF-HXA) based on the generalization of Equation 2. Scaling relationship between $K_{xy,q}(\tau)$ and the generalized cross-correlation Hurst exponent $H_{xy}(q)$ is obtained as

$$K_{xy,q}(\tau) \propto \tau^{qH_{xy}(q)}. \tag{4}$$

For $q = 2$, the method can be used for the detection of long-range cross-correlations solely and we call it the height cross-correlation analysis (HXA). Obviously, for $\{X(t)\}_{t=0}^T = \{Y(t)\}_{t=0}^T$, MF-HXA turns into the generalized Hurst exponent approach of Alvarez-Ramirez et al. (2002), which is equivalent to the height-height correlation analysis of Barabasi et al. (1991).

3.2 Two types of cross-correlations

Similarly to Hurst exponent $H(2)$, the cross-correlation Hurst exponent $0 < H_{xy}(2) < 1$ has a critical value of 0.5 which indicates that the examined series are pairwise uncorrelated (or pairwise short-range dependent). For $H_{xy}(2) > 0.5$, the series are cross-persistent so that a positive (a negative) value of $\Delta X(t)\Delta Y(t)$ is more statistically probable to be followed by another positive (negative) value of $\Delta X(t+1)\Delta Y(t+1)$. Conversely for $H_{xy}(2) < 0.5$, the series are cross-antipersistent so that a positive (a negative) value of $\Delta X(t)\Delta Y(t)$ is more statistically probable to be followed by a negative (a positive) value of $\Delta X(t+1)\Delta Y(t+1)$. In other words, the increments of the cross-persistent series tend to move in the same direction whereas the increments of the cross-antipersistent series are more statistically likely to move in an opposite direction.

We now derive an expected value of cross-correlation Hurst exponent $H_{xy}(q)$. Using a standard definition of multifractality of Calvet and Fisher (2008), consider processes $\{X(t)\}_{t=0}^{T}$ and $\{Y(t)\}_{t=0}^{T}$ are fractal with generalized Hurst exponents $H_x(q)$ and $H_y(q)$:

$$\mathbb{E}[|(X(t)-X(t-\tau))|^q] = c_x(q)\tau^{qH_x(q)} \tag{5}$$

$$\mathbb{E}[|(Y(t)-Y(t-\tau))|^q] = c_y(q)\tau^{qH_y(q)} \tag{6}$$

From Equation 3, we are considering fractality of a joint process $Z(t) = \sum_{i=1}^{t}[X(i)-X(i-1)][Y(i)-Y(i-1)]$ for $t=1,\ldots,T$ with $Z(0)=X(0)=Y(0)=0$. Let us denote $\Delta X_\tau = X(t)-X(t-\tau)$ and $\Delta Y_\tau = Y(t)-Y(t-\tau)$. We label covariances between $|\Delta X_\tau|^q$ and $|\Delta Y_\tau|^q$ as $\sigma_{XY}(\tau,q)$ and correspondingly the correlations as $\rho_{XY}(\tau,q)$. In the same way, we label the variances of $|\Delta X_\tau|^q$ and $|\Delta Y_\tau|^q$ as $\sigma_X^2(\tau,q)$ and $\sigma_Y^2(\tau,q)$, respectively. Generally, we have

$$\mathbb{E}\left[|(\Delta X_\tau)(\Delta Y_\tau)|^{\frac{q}{2}}\right] = \mathbb{E}\left[|\Delta(X_\tau)|^{\frac{q}{2}}\right]\mathbb{E}\left[|(\Delta Y_\tau)|^{\frac{q}{2}}\right] + \sigma_{XY}\left(\tau,\frac{q}{2}\right) \tag{7}$$

which leads us to several possible scenarios and different implications for both long-range cross-correlations and cross-multifractality:

(i) If the examined series are pairwise independent, then it holds that $\sigma_{XY}(\tau,q)=0$ for all $q>0$. From Equations 5 - 7, we have

$$\mathbb{E}\left[|(\Delta X_\tau)(\Delta Y_\tau)|^{\frac{q}{2}}\right] = \sqrt{c_x(q)\tau^{qH_x(q)}c_y(q)\tau^{qH_y(q)}} = \tag{8}$$

$$= \sqrt{c_x(q)c_y(q)}\tau^{q\frac{H_x(q)+H_y(q)}{2}} \propto \tau^{qH_{xy}(q)}$$

which implies

$$H_{xy}(q) = \frac{H_x(q)+H_y(q)}{2} \tag{9}$$

(ii) If the examined series are pairwise uncorrelated, then it holds that $\sigma_{XY}(\tau,\frac{q}{2})=0$ for $q=2$, i.e. $\sigma_{XY}(\tau,1)=0$. Then similarly to the previous case, we have

$$H_{xy}(2) = \frac{H_x(2)+H_y(2)}{2} \tag{10}$$

(iii) If the examined series are pairwise correlated (or generally pairwise short-range dependent) as well as long-range dependent, then $\sigma_{XY}(\tau,\frac{q}{2}) \neq 0$. We know that in general $\sigma(\sum_{i=1}^{n}X_i,\sum_{j=1}^{m}Y_j) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma(X_i,Y_j)$ so that the covariances for the cumulated series, which is exactly our case, change linearly in τ. Thence, we can write

$$\mathbb{E}\left[|(\Delta X_\tau)(\Delta Y_\tau)|^{\frac{q}{2}}\right] \propto \tau^{q\frac{H_x(q)+H_y(q)}{2}} \tag{11}$$

$$\sigma_{XY}\left(\tau,\frac{q}{2}\right) \propto \tau \tag{12}$$

This implies that $K_{xy}(\tau)$ no longer scales according to the power law in τ (see Equation 4) which implies that we observe multiscaling, i.e. different scaling for different regions of τ. In effect, Equations 9 and 14 do not hold for correlated series apart from the specific case of highly correlated series. We now consider this special case.

For $q = 2$, we examine the scaling of covariances for the processes $|\Delta X_\tau|$ and $|\Delta Y_\tau|$. If the correlation between the two series is close to unity, we can use a representation of covariance in a form of product of correlation and corresponding standard deviations. We use the fact that standard deviations scale with changing τ, which covers both long-range dependent and uncorrelated series (considered separately). This leads us to

$$\mathbb{E}[|(\Delta X_\tau)(\Delta Y_\tau)|] = \mathbb{E}[|\Delta(X_\tau)|]\mathbb{E}[|(\Delta Y_\tau)|] + \rho_{XY}(1,1)\sigma_X(\tau,1)\sigma_Y(\tau,1) = \tag{13}$$
$$= \sqrt{c_x(2)c_y(2)}\tau^{H_x(2)+H_y(2)} + \rho_{XY}(1,1)b_x\tau^{H_x(2)}b_y\tau^{H_y(2)} =$$
$$= \tau^{H_x(2)+H_y(2)}\left(\sqrt{c_x(2)c_y(2)} + \rho_{XY}(1,1)b_x(2)b_y(2)\right) \propto \tau^{H_x(2)+H_y(2)}$$

which again implies[1]

$$H_{xy}(2) = \frac{H_x(2) + H_y(2)}{2} \tag{14}$$

Summarizing the three previous possibilities, we can see that apart from the two specific cases – an independent series in general or an uncorrelated series with $q = 2$ – we cannot draw any general conclusions about the expected value of the generalized cross-correlation Hurst exponent $H_{xy}(q)$. For dependent series and $q \neq 2$, there is no general pattern in the effect of correlations for different moments q on the final $H_{xy}(q)$.

Nevertheless, this is a crucial result as it shows that even for two pairwise independent long-range dependent processes, it holds that the cross-correlation Hurst exponent $H_{xy}(2) \neq 0.5$, apart from a special case $H_x(2) + H_y(2) = 1$. Therefore, $H_{xy}(2) \neq 0.5$ can be caused by long-range dependence of the two processes even if there are no "true" long-range cross-correlations. Therefore, we need to distinguish between the two types of long-range cross-correlations: (i) long-range cross-correlations caused by long-range interrelation between two series (i.e. the scaling of covariances between $|\Delta X_\tau|^{q/2}$ $|\Delta Y_\tau|^{q/2}$ as shown in Equation 7), and (ii) long-range cross-correlations which are caused by long-range dependence of the separate series.

3.3 Cross-multifractality

If a spectrum of Hurst exponents $H_{xy}(q)$ is needed to describe the relationship between two time series, the series are cross-multifractal. The generalized cross-correlation Hurst exponent $H_{xy}(q)$ is independent of q for monofractal series or it is dependent on q for multifractal series. The influence of joint distributional

[1] The following equality was shown to hold for two correlated ARFIMA processes (i.e. two ARFIMA models with different Hurst exponents but with the same error terms) by Podobnik et al. (2009).

properties implies that multifractality can be due to the cross-correlations as well as the broadness (heavy tails) of the joint-distribution (Kantelhardt, 2009). Again, the effect of correlations can be separated into two – auto-correlations and cross-correlations as discussed in the previous section.

There are two mostly used measures to asses multifractality – a range of generalized Hurst exponents, and singularity strength and spectrum. The range ΔH is usually defined as either $\Delta H = \max\{H(q)\} - \min\{H(q)\}$ or $\Delta H = H(q_{min}) - H(q_{max})$. Since the range of $H(q)$ is an unstable measure even for unifractal processes, the latter measures are used more frequently (Kantelhardt et al., 2002). Singularity strength, or Hölder exponent, α is a characteristic measure of a series whereas singularity spectrum $f(\alpha)$ refers to a dimension of subset of the series described by α. Both α and $f(\alpha)$ are connected to the scaling exponent $\tau(q)$, which is defined[2] as $\tau(q) = qH(q)$ for a standard case and as $\tau(q)_{xy} = qH_{xy}(q)$ from Equations 2 and 4 . To obtain α and $f(\alpha)$, we generalize the procedure of Barabasi et al. (1991) for two time series.

To characterize the relationship between the series $X(t)$ and $Y(t)$, we construct a probability measure $p_t(\tau)$ connected to a hierarchy of changes of the two series. The measure is calculated as

$$p_{xy,t}(\tau) = \frac{\sqrt{|[X(t+\tau) - X(t)][Y(t+\tau) - Y(t)]|}}{\sum_{t=1}^{T-\tau} \sqrt{|[X(t+\tau) - X(t)][Y(t+\tau) - Y(t)]|}}. \tag{15}$$

As $p_{xy,t}(\tau)$ is a standard probability measure, it holds that $\sum_{t=1}^{T-\tau} p_{xy,t}(\tau) = 1$ and $p_{xy,t}(\tau) \geq 0$. We further define a generating function for two time series $\chi_{q,xy}(\tau)$, which is associated with the probability measures $p_{xy,t}(\tau)$, and corresponding generalized dimensions $D_{xy,q}$ as

$$\chi_{xy,q}(\tau) = \sum_{t=1}^{T-\tau} p^q_{xy,t}(\tau). \tag{16}$$

$$\chi_{xy,q}(\tau) \propto \tau^{(q-1)D_{xy,q}}. \tag{17}$$

Finally, we use the Legendre transformation and obtain the singularity strength α through a change of the generalized dimension $D_{xy,q}$ with varying q. The singularity spectrum $f(\alpha)$ is then obtained with a use of both α and $D_{xy,q}$. The specific relationships hold as follows

$$\alpha_{xy} = \frac{\partial[(q-1)D_{xy,q}]}{\partial q} \tag{18}$$

$$f(\alpha_{xy}) = q\alpha_{xy} - (q-1)D_{xy,q} \tag{19}$$

The above described procedure can be replaced by an alternative one. If we assume that the probability measure $p_{xy,t}(\tau)$ describes the hierarchy of both

[2]Note that definitions of α and $f(\alpha)$ differ across literature depending on definition of $\tau(q)$, which is either $\tau(q) = qH(q)$ (Di Matteo et al., 2005) or $\tau(q) + 1 = qH(q)$ (Bogachev et al., 2008). If the latter definition is used, there is unity added to both α and $f(\alpha)$.

series $X(t)$ and $Y(t)$ uniformly, we can write $p_{xy,t}(\tau) = \frac{1}{T}$ for $\tau \to 0$. Such an assumption allows to use only $H_{xy}(q)$ for the construction of the singularity strength α and the singularity spectrum $f(\alpha)$. It then holds that[3]

$$\alpha_{xy} = \frac{\partial [qH_{xy}(q)]}{\partial q} - H_{xy}(1) \tag{20}$$

$$f(\alpha_{xy}) = q\frac{\partial [qH_{xy}(q)]}{\partial q} - qH_{xy}(q) \tag{21}$$

4 Two illustrative examples

To validate MF-HXA, we present results for two randomly generated processes – two independent ARFIMA processes and two independent multifractal series based on the Mandelbrot's Binomial Multifractal model. Note that both variants of the processes are independent so that the expected cross-correlation Hurst exponents $H_{xy}(q)$ are equal to arithmetic means of $H_x(q)$ and $H_y(q)$ of the separate processes according to Equation 9.

4.1 Two ARFIMA processes

The autoregressive fractionally integrated moving average models (ARFIMA) are a generalization of the autoregressive moving average models (ARMA) of Box and Jenkins (1970) which allow for long-range dependence. With a use of a backshift operator B, ARFIMA models are represented by $(1-\sum_{i=1}^{p}\varphi_i B^i)(1-B)^d X_t = (1+\sum_{i=1}^{q}\theta_i B^i)\varepsilon_t$, where $(1-B)^d = \sum_{k=0}^{d}\frac{(-1)^k B^k \Gamma(d+1)}{\Gamma(k+1)\Gamma(d-k+1)}$ (see Baillie et al. (1996) for details). Here d is a fractional differencing parameter and it holds that $d = H - 0.5$.

In Figure 1a, we show the estimates of $H_x(q)$, $H_y(q)$ and $H_{xy}(q)$ for two independent ARFIMA processes with $H_x = 0.7$ and $H_y = 0.9$ with $T = 1000$, $\tau_{min} = 2$, $\tau_{max} = 100$ and $q = 0.1, 0.2, \ldots, 9.9, 10$. Even though the both series are monofractal, the generalized Hurst exponents range from $H_x(0.1) = 0.7442$ to $H_x(10) = 0.6579$ and from $H_y(0.1) = 0.9281$ to $H_y(10) = 0.8546$. The differences are due to a finite sample size and emphasize a need for using α and $f(\alpha)$ for the examination of multifractality. Importantly, the estimates of the generalized Hurst exponents characterizing the long-range dependence solely are close to the expected values – $H_x(2) = 0.7134$ and $H_y(2) = 0.9035$. Further, the estimates of cross-correlation Hurst exponents $H_{xy}(q)$ satisfy the relation of Equation 9 with only small deviations holds for all qs.

4.2 Mandelbrot's Binomial Multifractal series

The Mandelbrot's Binomial Multifractal (MBM) is the simplest multifractal measure (Mandelbrot et al., 1997). Let $m_0 > 0$, $m_1 > 0$ and $m_0 + m_1 = 1$ and

[3] For a detailed derivation, see the Appendix of Barabasi et al. (1991)

let us work on interval [0,1]. In the first stage, the mass of 1 is divided into two subintervals [0,1/2] and [1/2,1], when there is m_0 in the first subinterval and m_1 in the second one. In the following stage, each subinterval is again halved and its mass is divided between the smaller subintervals in ratio $m_0 : m_1$. After k stages, we obtain a series of 2^k values. Note that the values are deterministically given as there is no noise added in the simplest version of the method. For an interval $[z, z+2^{-k}]$, the value μ has a value of $\mu[z, z+2^{-k}] = m_0^{k\varphi_0} m_1^{k\varphi_1}$, where φ_0 and φ_1 stand for the relative frequencies of numbers 0 and 1 in a binary development of $2^k z$, respectively.

In Figure 1b, we show the estimates $H_x(q)$, $H_y(q)$ and $H_{xy}(q)$ for two independent MBM models with $m_0 = 0.2$ and $m_0 = 0.4$, respectively. We generated the series with 10 steps and obtained $T = 1024$ observations and kept other parameters the same so that MF-HXA is run with $\tau_{min} = 2$, $\tau_{max} = 100$ and $q = 0.1, 0.2, \dots, 9.9, 10$. The variation of $H_x(q)$, $H_y(q)$ and $H_{xy}(q)$ is much stronger than in the case of monofractal ARFIMA models. The values range from $H_x(0.1) = 0.8645$ to $H_x(10) = 0.4305$ and from $H_y(0.1) = 0.8175$ to $H_y(10) = 0.6819$ for the respective processes. Importantly, Equation 9 holds for all qs.

5 Application

To show the usefulness of MF-HXA, we apply the method on two different types of financial assets – the US stock indices (NASDAQ and S&P500) and the commodity prices (Crude and Heating Oil). We choose such pairs as we expect strong correlations and therefore potential long-range cross-dependence and multifractality. Indeed, NASDAQ and S&P500 are highly correlated (the correlation coefficient for logarithmic returns is $\rho_{xy}(0) = 0.8652$) as well as the pair of the oil commodities (with $\rho_{xy}(0) = 0.6476$). For the stock index prices, we analyze a period between 1.1.2000 and 31.12.2009 (2531 observations). For the commodity prices, we use a nearest-future basis prices based on the Commodity Research Bureau for a period between 1.11.1993 and 16.2.2010 (4115 observations). Note that all the series contain some extreme events as well as strong trends and reversals. For the stock indices, the examined period contains a long-term decreasing trend in the beginning of the 2000s as well as the current financial crisis. For the oil prices, the series contains several strong speculative trends, to name the most evident one – the bubble that started in 2007, peaked in July 2008 (when the price of both oil commodities almost tripled in 18 months) and reversed into a strong bullish market (returning below the levels of the beginning of 2007).

We research on the potential long-range dependence and cross-correlations in returns and volatility. As a measure of volatility, we take the absolute returns, which is standard in the financial literature and also intuitive as returns can be taken as a product of a sign and a magnitude (absolute return). Basic descriptive statistics are summarized in Table 1. All the examined assets share similar properties – average return close to zero, negative skewness and excess kurtosis. Such a deviation from normality is supported by both Jarque-Bera and Shapiro-Wilk tests, which reject normality at all meaningful significance levels. Stationarity is supported by ADF and KPSS test – strong rejection of a unit root and inability to reject stationarity. With a use of a standard Q-statistic, we reject that both returns and volatility show no significant autocorrelations. Importantly, the autocorrelations are much stronger for volatility than for returns. Such results indicate potential long-range dependence in volatility of the examined series. All the previous results hold for all the examined series.

To examine possible long-range (cross-)dependence and (cross-)multifractality in both returns and volatility of the series, we apply MF-HXA with $\tau_{min} = 1$ and $\tau_{max} = 100$ for $q = 0.1, 0.2, \dots, 9.9, 10$. We choose τ_{min} and τ_{max} to have enough values for the final regression according to Equation 4. A step of 0.1 of different qs ensures that the evolution of the generalized Hurst exponents, corresponding α and $f(\alpha)$ is smooth and well interpreted.

Figures 2 - 5 show the estimates of the generalized Hurst exponents for returns and volatility of NASDAQ, S&P500, Crude Oil, Heating Oil and corresponding joint processes.

The generalized Hurst exponents for the returns vary for NASDAQ from $H_{NASD}(0.1) = 0.5662$ to $H_{NASD}(10) = 0.3958$ and for S&P500 from $H_{S\&P}(0.1) = 0.5063$ to $H_{S\&P}(10) = 0.3544$. The cross-correlated Hurst exponents vary from $H_{xy}(0.1) = 0.5365$ to $H_{xy}(10) = 0.3817$. Long-range dependence Hurst exponents $H(2)$ are estimated as $H_{NASD}(2) = 0.5131$, $H_{S\&P}(2) = 0.4992$ and $H_{xy}(2) = 0.5137$. Therefore, there are no signs of persistence in returns for any of the series. As for multifractality, the specific values of $H_{xy}(q)$ behave according to Equation 9 so that the cross-multifractality of NASDAQ and S&P cannot be distinguished from two pairwise independent processes.

For the commodity futures, the generalized Hurst exponents for the returns vary for Crude Oil from $H_{Crude}(0.1) = 0.4608$ to $H_{Crude}(10) = 0.5424$ and for Heating Oil from $H_{Heat}(0.1) = 0.4703$ to $H_{Heat}(10) = 0.1943$. The cross-correlated Hurst exponents vary from $H_{xy}(0.1) = 0.4665$ to $H_{xy}(10) = 0.5569$. Further, long-range dependence Hurst exponents are estimated as $H_{Crude}(2) = 0.5044$, $H_{Heating}(2) = 0.4535$ and $H_{xy}(2) = 0.5114$. Such results show weak signs of anti-persistence in the Heating Oil returns whereas the Crude Oil and the joint process show no signs of long-range dependence. Nevertheless, the cross-correlation Hurst exponent $H_{xy}(2)$ deviates from the average Hurst exponents of the separate processes indicating additional scaling in the covariances of the joint process. The evolution of generalized Hurst exponents shows even more interesting results. First, $H_{Crude}(q)$ shows a non-monotone dependence on q indicating unifractality of the process. Second, $H_{Heat}(q)$ is strongly dependent and decreasing in q, indicating strong multifractality. Third, $H_{xy}(q)$ almost copies the evolution of $H_{Crude}(q)$ indicating strong effect of both linear and non-linear correlations between the processes on the cross-multifractality.

We now turn to the analysis of the volatilities. For NASDAQ and S&P500, the results are quite similar – the generalized Hurst exponents vary with q with very high long-range dependence Hurst exponents $H(2)$ around 0.9. More specifically, the generalized Hurst exponents vary from $H_{NASD}(0.1) = 0.9484$

to $H_{NASD}(10) = 0.7632$ for NASDAQ and $H_{S\&P}(0.1) = 0.9154$ to $H_{S\&P}(10) = 0.7983$ for S&P500. The evolution of the generalized Hurst exponents for the joint process are, in the same way as for the process of returns, in hand with Equation 9. Thus again, the dynamics of the long-range cross-correlation between volatilities of NASDAQ and S&P500 cannot be distinguished from a pair-wise independent process.

The results are very different for the volatilities of Crude and Heating Oil. Both commodities show strong long-range dependence with $H(2)$ around 0.8. However, the behavior of the generalized Hurst exponent is very different for the two. For the Crude Oil, the generalized Hurst exponent is non-monotone in q and varies only weakly q indicating unifractality of the process of the absolute returns. On the other hand, the generalized Hurst exponents for the Heating Oil vary strongly and in a monotonous manner in q from $H_{Heat}(0.1) = 0.8137$ to $H_{Heat}(10) = 0.3946$. The generalized Hurst exponent of the joint process deviate from the average for all moments q indicating additional scaling in the covariances of the processes of the Crude and Heating Oil. Similarly to the behavior of the returns of the commodities, the generalized Hurst exponents $H_{xy}(q)$ almost overlap with $H_{Crude}(q)$ for all qs.

To separate the effects of linear and non-linear correlations at a zero lag from the long-range correlations and cross-correlations, we present the results for shuffled series as well. By shuffling, all potential autocorrelations and long-range cross-correlations are destroyed. Moreover, if the shuffled series are considered multifractal, such multifractality is caused by a distributional properties solely.

For the stock indices, the most obvious differences arise in the dependence of the generalized Hurst exponents on q. As for S&P500, the dependence is very similar as for the original series, whereas for NASDAQ and the joint process, the variability of $H(q)$ decreased. This indicates that the multifractality of S&P500 is mainly caused by a distributional broadness whereas the multifractality of NASDAQ and the joint process is due to correlations and cross-correlations as well. Interestingly, similar statements hold for both returns and absolute returns dynamics.

The results for the shuffled series of the commodity indices are quite different. Most importantly, the generalized Hurst exponents of the joint process deviate from the average of the generalized Hurst exponents of the separate processes for $q > 3$ for returns and for $q > 1.7$ for volatility. This can be assigned to a scaling of covariances for different moments q. Moreover, the increasing deviation of the generalized Hurst exponent with q implies stronger correlations at extreme events, i.e. at the tails of the joint distribution.

To further examine the multifractality and cross-multifractality, we present the singularity strengths α and singularity spectra $f(\alpha)$ based on Equations 15 – 19. The results for returns, volatilities and corresponding shuffled series are illustrated in Figures 6 – 9. The characteristics of the relations between α and $f(\alpha)$ correspond well the basic examination of the behavior of the generalized Hurst exponents with respect to q.

The singularity spectra of the returns of NASDAQ, S&P500 and the joint process almost overlap while the important differences are obvious from the spectra of the shuffled series. The difference between the shuffled and original series supports the previous findings – the multifractality of the returns of S&P500 is mainly due to the distributional properties whereas NASDAQ and the joint process can be characterized by the correlations multifractality as well.

For the commodity returns, the singularity spectra again support the previous findings. On one hand, we get a degenerate spectra for the Crude Oil and the joint process indicating unifractality. On the other hand, the Heating Oil is multifractal while both types of multifractality are present. However, we cannot say anything about the correlations of the extreme events through a singularity spectra analysis.

An examination of the singularity spectra of the volatilities again support the most important findings based on the generalized Hurst exponents behavior. Whereas the stock indices behave similarly, we again observe very different characteristics for the commodities – the singularity spectra almost overlap for the Crude Oil and the joint process, which corresponds well to the previous results.

6 Conclusions

In the paper, we introduced a new method for the detection of long-range cross-correlations and cross-multifractality – the multifractal height cross-correlation analysis (MF-HXA). We showed that long-range cross-correlations can be caused by long-range dependence of separate processes and/or by additional dependence between the two series caused by scaling of the covariances. Similarly for cross-multifractality, the causes can be separated into three groups – multifractality due to the joint-distributional properties and due to correlations, which can be further divided into the auto-correlations and the cross-correlations.

To show the usefulness of the method, we applied MF-HXA on returns and volatility of NASDAQ, S&P500, Crude Oil and Heating Oil. We showed that the two pairs of series are characterized by very different behavior. Whereas the long-range cross-correlations and cross-multifractality between the stock indices cannot be distinguished from a pairwise independent processes, the relationship is more complex for the oil commodities where we find strong scaling in covariances of the processes.

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Table 1: Descriptive statistics of NASDAQ and S&P500 returns

	NASDAQ	S&P500	Crude Oil	Heating Oil
Mean	-0.0002	-0.0001	0.0004	0.0003
SD	0.0181	0.0136	0.0258	0.0269
Skewness	-0.1269	-0.1262	-0.2355	-0.2957
Kurtosis	6.7762	11.5119	8.7022	14.830
Jarque-Bera	1510	7644	5611	24047
p-value	0.0000	0.0000	0.0000	0.0000
Shapiro-Wilk	0.9510	0.9056	0.9426	0.9258
p-value	0.0000	0.0000	0.0000	0.0000
$\rho(1)$	-0.0073	-0.0791	-0.025	0.0272
$Q(20)$	59.30	100.28	44.66	35.78
p-value	0.0000	0.0000	0.001	0.016
$\rho(1)_{abs}$	0.1957	0.2496	0.1451	0.2163
$Q(20)_{abs}$	3135	4785	1361	1386
p-value	0.0000	0.0000	0.0000	0.0000
KPSS	0.3112	0.1215	0.0451	0.0445
5% critical value	0.463	0.463	0.463	0.463
ADF	-17.1554	-17.495	-48.1484	-20.0234
p-value	0.0000	0.0000	0.0000	0.0000

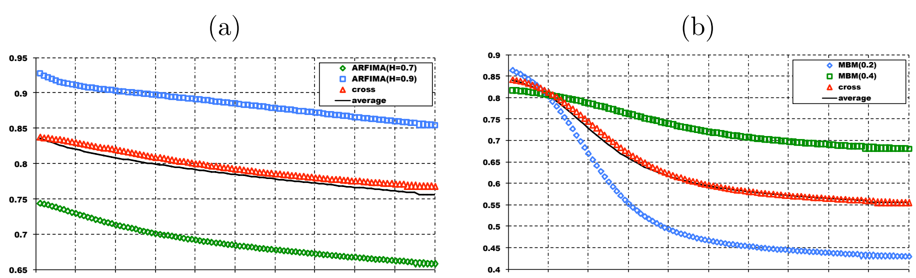

Figure 1: (a) Estimates of $H_x(q)$, $H_y(q)$ and $H_{xy}(q)$ (y-axis) for two ARFIMA processes with $H_x = 0.7$ and $H_y = 0.9$ for different qs (x-axis); (b) Estimates of $H_x(q)$, $H_y(q)$ and $H_{xy}(q)$ (y-axis) for two MBM with $m_0 = 0.2$ and $m_0 = 0.4$, respectively, for different qs (x-axis).

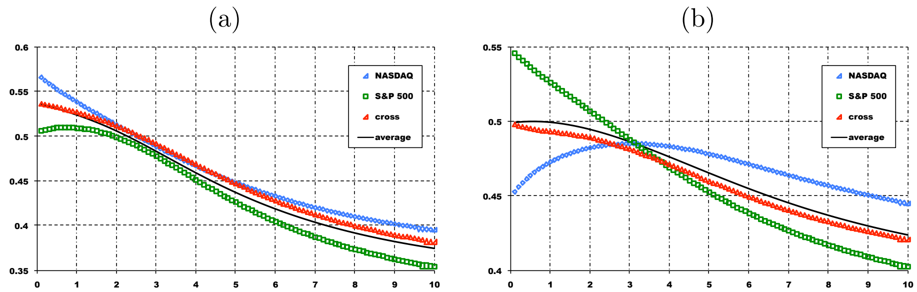

Figure 2: Estimates of $H_{NASD}(q)$, $H_{S\&P}(q)$ and $H_{xy}(q)$ (y-axis) for returns of NASDAQ and S&P500 for $q = 0.1, 0.2, \ldots, 10$ (x-axis) for original (a) and shuffled data (b).

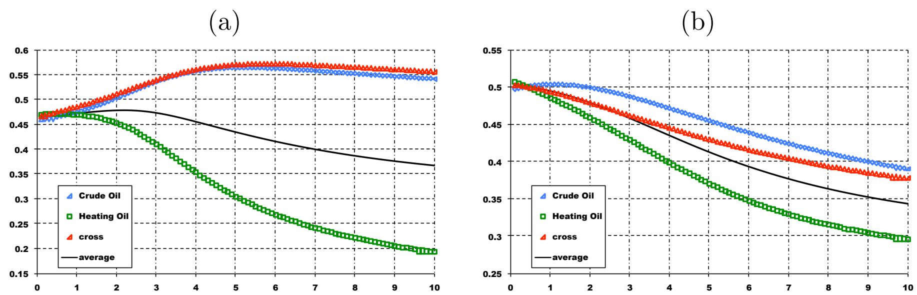

Figure 3: Estimates of $H_{NASD}(q)$, $H_{S\&P}(q)$ and $H_{xy}(q)$ (y-axis) for returns of Crude and Heating Oil for $q = 0.1, 0.2, \ldots, 10$ (x-axis) for original (a) and shuffled data (b).

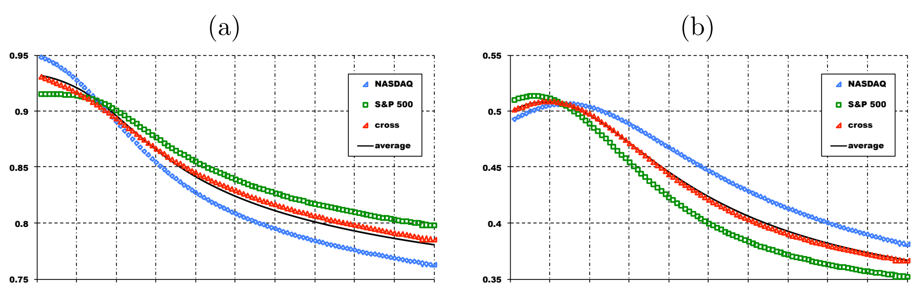

Figure 4: Estimates of $H_{NASD}(q)$, $H_{S\&P}(q)$ and $H_{xy}(q)$ (y-axis) for absolute returns of NASDAQ and S&P500 for $q = 0.1, 0.2, \ldots, 10$ (x-axis) for original data (a) and shuffled data (b).

(a) (b)

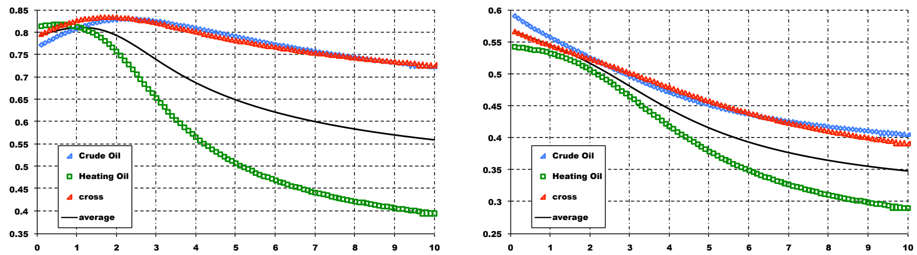

Figure 5: Estimates of $H_{NASD}(q)$, $H_{S\&P}(q)$ and $H_{xy}(q)$ (y-axis) for absolute returns of Crude and Heating Oil for $q = 0.1, 0.2, \ldots, 10$ (x-axis) for original data (a) and shuffled data (b).

(a) (b)

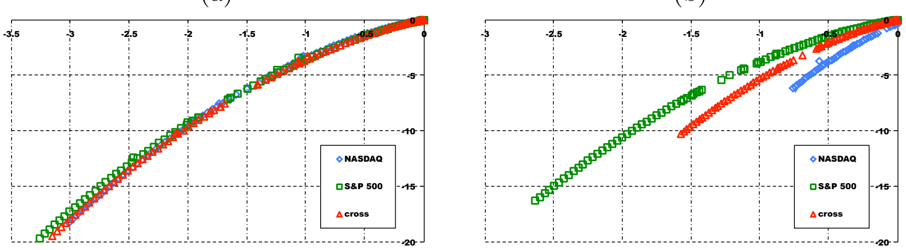

Figure 6: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for returns of NASDAQ, S&P500 and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled data (b).

(a) (b)

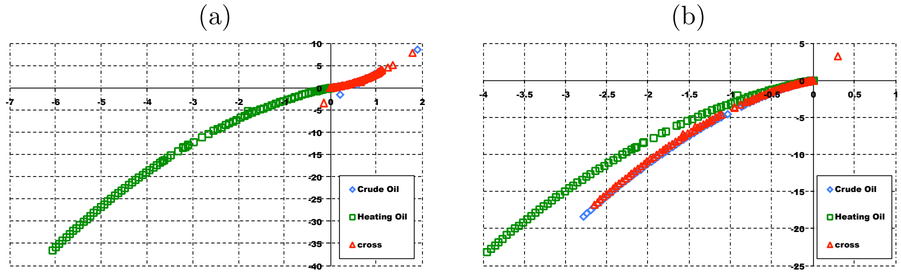

Figure 7: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for returns of Crude Oil, Heating Oil and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled data (b).

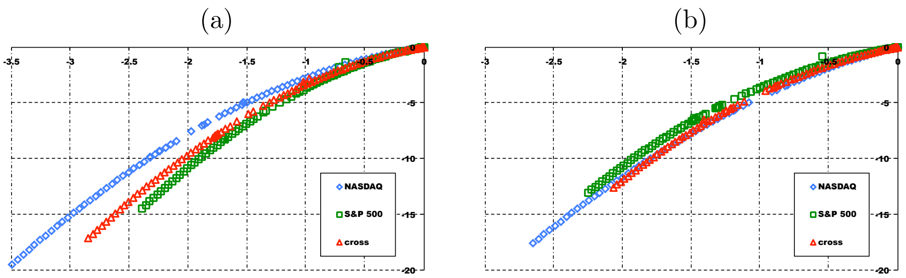

Figure 8: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for absolute returns of NASDAQ, S&P500 and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled (b).

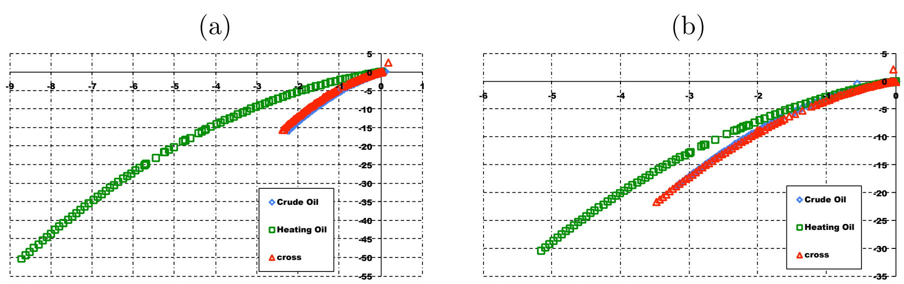

Figure 9: Singularity strengths α (x-axis) and singularity spectra $f(\alpha)$ (y-axis) for absolute returns of Crude Oil, Heating Oil and combined for $q = 0.1, 0.2, \ldots, 10$ for original (a) and shuffled data (b).