AGGREGATION WITH MULTI-ATTRIBUTES: A NEW PERSPECTIVE

Roberto Ghiselli Ricci
Department of Economia Istituzioni Territorio
University of Ferrara (Italy)
ghsrrt@unife.it

Radko Mesiar
Faculty of civil Engineering
Slovak University of Technology Bratislava (Slovakia)
UTIA CAS, Pod Vodarenskou vezi 4, 182 08 Prague, Czech Republic
mesiar@math.sk

Summary

We deal with the new notion of multi-attribute aggregation operator, which incorporates many standard aggregation methods, and classify classical properties into some main groups. Different examples are provided.

Keywords: Aggregation operator; OWA operator; Multi-attribute aggregation; Weighted aggregation.

1 INTRODUCTION

Though the fusion of observed numerical input values into a single output value can be found in the majority of areas dealing with quantitative information, the theory of aggregation can be dated to the last decade only. The axiomatic framework for aggregation proposed in [7], not limited to a fixed number of inputs, was related to earlier ideas on n-ary aggregation functions discussed in [6], and we call this approach "classical". For an overview of results linked to classical aggregation functions we recommend a recent monograph ([5]). The classical approach covers huge classes of aggregation techniques with different roots, such as different kinds of means, conjunctive and disjunctive operators, and usually parametric classes of such operators are exploited to model an appropriate aggregation in each discussed domain, often based on fitting the optimal parameters to a sample space. A recursive approach proposed in [2] is based on a consecutive application of binary aggregation operators, which may differ in different steps, and thus reflect a development of an observed process. However, the basic theory of aggregation seems to be incomplete, because it does not adequately contemplate the crucial role played in many cases by some collateral parameters, called in the sequel *attributes*. Note that some particular methods based on attributes are well-known, especially various kinds of weighted aggregation ([1]) or induced aggregation ([8]), but a unified theory of multi-attribute aggregation operators is not yet known in literature. The aim of our paper is to fill this gap.

2 BASIC CONCEPTS

Throughout the paper, we assume that $\mathbb{I}$ is a nonempty real interval and we set $a := \inf \mathbb{I}$ and $b := \sup \mathbb{I}$. Note that a and b might belong to $\mathbb{I}$ or not, possibly with $a = -\infty$ or $b = +\infty$.

Let us now introduce the concept of aggregation operator in a formal way. In the following, fixed any $n \in \mathbb{N}$, let G_n be an arbitrary mapping defined on $\mathbb{I}^n$ into $\mathbb{I}$. Further, we denote by $\delta_{G_n} : \mathbb{I} \to \mathbb{I}$ the *diagonal section* of G_n defined as $\delta_{G_n} \equiv G_n|_{diag(\mathbb{I}^n)}$, where $diag(\mathbb{I}^n) := \{(x, \ldots, x) : x \in \mathbb{I}\} \subset \mathbb{I}^n$.

Definition 2.1. We say that $G_n : \mathbb{I}^n \to \mathbb{I}$ is an *aggregation function* in $\mathbb{I}^n$ if it is non-decreasing in each component and fulfills the boundary conditions

$$\inf_{\mathbf{x}\in\mathbb{I}^n} G_n(\mathbf{x}) = a \quad and \quad \sup_{\mathbf{x}\in\mathbb{I}^n} G_n(\mathbf{x}) = b. \tag{2.1}$$

From now on, we denote with $\boldsymbol{G}$ a sequence $\{G_n\}_{n\in\mathbb{N}}$ of functions such that each G_n maps $\mathbb{I}^n$ into $\mathbb{I}$. Now, we remind the classical definition of aggregation operator.

Definition 2.2. We say that $\boldsymbol{G}$: $\bigcup_{n=1}^{\infty} \mathbb{I}^n \to \mathbb{I}$ is a *classical aggregation operator* in $\mathbb{I}$ if any G_n is an aggregation function in $\mathbb{I}^n$, with $G_1 \equiv id$, where id is the identity function.

Note that the classical aggregation operators were also called *extended aggregation functions* (see [5]). A new, enlarged notion of aggregation operator is the following.

Definition 2.3. We say that $\boldsymbol{G}$: $\bigcup_{n=1}^{\infty} \mathbb{I}^n \to \mathbb{I}$ is an *aggregation operator* in $\mathbb{I}$, when every G_n is non-decreasing in each component and the following conditions are verified:

$$\liminf_{n\to\infty} \inf_{\mathbf{x}\in\mathbb{I}^n} G_n(\mathbf{x}) = a \quad \textit{and} \quad \limsup_{n\to\infty} \sup_{\mathbf{x}\in\mathbb{I}^n} G_n(\mathbf{x}) = b. \tag{2.2}$$

Remark 2.1. First of all, observe that the requirement that every $G_n \in \boldsymbol{G}$ must be an aggregation function fails. This is due to the fact that, in place of eq. (2.1), to be satisfied by any single mapping of $\boldsymbol{G}$, we have formulated a unique boundary condition, represented by eq. (2.2), valid for the whole operator. Moreover, the conventional position $G_1 \equiv id$ (the identity function) is not axiomatically imposed, even if it is allowed, because it may set strong constraints to the structure of the operator up to incompatibility with certain properties of $\mathbf{G}$ (see, for instance, [4]). Finally, for the rest of the paper, the concept of Aggregation Operator (*AgOp*, in brief) will be always implicitly referred to Definition 2.3, unless otherwise specified.

Remark 2.2. It is not difficult to see that eq. (2.2) is equivalent to saying that

$$\liminf_{n\to\infty} \delta^+_{G_n}(a) = a \tag{2.3}$$

and

$$\limsup_{n\to\infty} \delta^-_{G_n}(b) = b, \tag{2.4}$$

where

$$\delta^+_{G_n}(a) := \begin{cases} \delta_{G_n}(a), & \textit{if } a \in \mathbb{I}, \\ \lim\limits_{x\to a^+} \delta_{G_n}(x), & \textit{otherwise}; \end{cases}$$

and

$$\delta^-_{G_n}(b) := \begin{cases} \delta_{G_n}(b), & \textit{if } b \in \mathbb{I}, \\ \lim\limits_{x\to b^-} \delta_{G_n}(x), & \textit{otherwise}, \end{cases}$$

Example 2.1. We present now three cases of not classical *AgOps* in $\mathbb{I} = [0, 1]$. In the first, let $\boldsymbol{G}$ be defined as

$$G_n(\mathbf{x}) = \max(\mathbf{x}) \cdot \frac{\sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i},$$

where ***max*** stands for the *maximum* operator. Since $\delta_{G_n}(1) = \frac{n}{1+n}$, eq. (2.4) is trivially satisfied in the elementary form of a monotonically attained limit.

In the second, $\boldsymbol{G}$ is given by

$$G_n(\mathbf{x}) = \begin{cases} \max(\mathbf{x}) \cdot \frac{\sum_{i=1}^{n} x_i}{1+\sum_{i=1}^{n} x_i}, & \textit{if } n \textit{ is even}; \\ AM(\mathbf{x}), & \textit{otherwise}, \end{cases}$$

where ***AM*** is the *arithmetic mean*. Since

$$\delta_{G_n}(1) = \begin{cases} \frac{n}{1+n}, & \textit{if } n \textit{ is even}; \\ 1, & \textit{otherwise}, \end{cases}$$

eq. (2.4) is fulfilled as a (not monotonically attained) limit.

Finally, fixed any $c \in]0, 1[$, let $\boldsymbol{G}$ be defined as

$$G_n(\mathbf{x}) = \begin{cases} c \cdot \max(\mathbf{x}) \cdot \frac{\sum_{i=1}^{n} x_i}{1+\sum_{i=1}^{n} x_i}, & \textit{if } n \textit{ is even}; \\ AM(\mathbf{x}), & \textit{otherwise}. \end{cases}$$

In this case, we have

$$\delta_{G_n}(1) = \begin{cases} c \cdot \frac{n}{1+n}, & \textit{if } n \textit{ is even}; \\ 1, & \textit{otherwise}, \end{cases}$$

hence eq. (2.4) still holds, but not as an ordinary limit.

Now, we introduce a sharp classification of the properties which may characterize any AgOp into two classes.

Definition 2.4. We say that a property $\mathcal{P}$ for an *AgOp* $\mathbf{G}$ in $\mathbb{I}$ is *static*, when every G_n verifies $\mathcal{P}$. Otherwise, $\mathcal{P}$ is called *dynamic* when it links G_n and G_m for different indices $n, m \in \mathbb{N}$.

Definition 2.5. An n-ary function $G_n : \mathbb{I}^n \to \mathbb{I}$ is *idempotent* when

$$\delta_{G_n} \equiv id. \tag{2.5}$$

An *AgOp* $\mathbf{G}$ in $\mathbb{I}$ is *idempotent* when it fulfills eq. (2.5) for all $n \in \mathbb{N}$.

Definition 2.6. We say that an *AgOp* $\mathbf{G}$ in $\mathbb{I}$ is *self-identical* (see [9]) if, given any $\mathbf{x} \in \mathbb{I}^n$, then

$$G_n(x_1, ..., x_n) = G_{n+1}(x_1, ..., x_n, G_n(x_1, ..., x_n))$$

for all $n \in \mathbb{N}$.

Idempotency is clearly a static property, while self-identity is dynamic. However, there are cases in which a property might be formulated both under the static and the dynamic point of view.

Definition 2.7. Let $G_n : \mathbb{I}^n \to \mathbb{I}$ be an n-ary function. We say that $e \in \mathbb{I}$ is a *static neutral element* for G_n if, fixed any $i \in \mathbb{N}_n$, then

$$G_n(\mathbf{x}) = x_i \tag{2.6}$$

for any $\mathbf{x} \in \mathbb{I}^n$ verifying $x_j = e$ for all $j \in \mathbb{N}_n \setminus \{i\}$. We say that $e \in \mathbb{I}$ is a *static neutral element* for an *AgOp* $\mathbf{G}$ in $\mathbb{I}$ when it satisfies eq. (2.6) for all $n \in \mathbb{N}$.

Observe that, for $n = 1$, eq. (2.6) is nothing but the condition $G_1 \equiv \text{id}$, and if $n = 2$ we recover the classical algebraic property $G_2(x, e) = G_2(e, x) = x$.

Definition 2.8. We say that $e \in \mathbb{I}$ is a *dynamic neutral element* for an *AgOp* $\mathbf{G}$ in $\mathbb{I}$ if, for every $i \in \mathbb{N}_{n+1}$ and for any $\mathbf{x} \in \mathbb{I}^{n+1}$ such that $x_i = e$, then

$$G_{n+1}(\mathbf{x}) = G_n(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) \quad (2.7)$$

for all $n \in \mathbb{N}$.

It is possible to provide examples which show that, in general, these two forms of "neutrality" are each other independent.

Finally, we specify the range of application of a property $\mathcal{P}$, independently of the static or dynamic case. For the rest of the section, let $q \in \mathbb{N}$.

Definition 2.9. We say that $\mathcal{P}$ is a q-property for an *AgOp* $\mathbf{G}$ in $\mathbb{I}$, when it does not concern the first $G_1, \dots, G_{q-1}$ mappings of the operator. The index q may be omitted in the classical case, i.e. when $q = 1$.

Example 2.2. Let $\mathbf{G}$ be defined on $\mathbb{I} = [0, 1]$ as

$$G_n(\mathbf{x}) = \begin{cases} \frac{1}{q} \sum_{i=1}^{n} x_i, & \textit{if } n \le q; \\ AM(\mathbf{x}), & \textit{otherwise.} \end{cases}$$

It is easy to see that $\mathbf{G}$ is a q-idempotent *AgOp*.

Example 2.3. Let $\alpha : [0, 1] \to [0, 1]$ be any non-decreasing mapping, not reducible to the identity, such that $\alpha(t) = t$ on the boundary. Let $\mathbf{G}$ be defined on the real unit interval as

$$G_n(\mathbf{x}) = \begin{cases} \max(\alpha(\mathbf{x})), & \textit{if } n > 1; \\ id, & \textit{otherwise}, \end{cases}$$

where $\alpha(\mathbf{x}) := (\alpha(x_1), \dots, \alpha(x_n))$ for any n-tuple $\mathbf{x}$. It is not difficult to see that $\mathbf{G}$ is a classical *AgOp* (by Definition 2.2) in $[0, 1]$ admitting $x = 0$ as 2-dynamic neutral element. Note that, unlike the classical case, even if $G_1 \equiv id$, zero is not a 2-static neutral element.

3 A MULTI-ATTRIBUTE AGOP

In this section, we introduce a new concept of aggregation operators, based upon the idea that in some cases any input value may be accompanied by a set of attributes which enter the process of fusion of the data in the sense that anyone of them has the property to influence, monotonically or not, the final result of the aggregation.

In the sequel, fixed $p \in \mathbb{N}$, let $\mathbb{I}_k$ be a nonempty real interval for any $k \in \mathbb{N}_p$, and we set $a_k := \inf \mathbb{I}_k$ and $b_k := \sup \mathbb{I}_k$, with the same degrees of freedom as for the boundaries of $\mathbb{I}$. We denote with $\mathbf{d}^k := (d_1^k, \dots, d_n^k)$ any vector belonging to $\mathbb{I}_k^n$ for each $k \in \mathbb{N}_p$.

Definition 3.1. We say that $D(p) := \mathbb{I}_1 \times \mathbb{I}_2 \times \dots \times \mathbb{I}_p$ is a p-dimensional *set of multi-attributes* and denote by $\mathbf{d} := (\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_n)$ any *multi-vector* belonging to $D^n(p)$, where $\mathbf{d}_j := (d_j^1, \dots, d_j^p)$ for each $j \in \mathbb{N}_n$. When $p = 1$, we will adopt the simplified notation $\mathbf{d} = (d_1, \dots, d_n)$.

We also need to introduce the family of all sequences of multi-attributes, denoted by $\mathcal{F} = \{\mu = \{\mathbf{d}_{(n)}\}_{n \in \mathbb{N}} \subset \bigcup_{n=1}^{\infty} D^n(p)\}$. Consistently with the previous notations, we will adopt the symbols $\mathbf{d}_{j\,(n)}$ and $d_{j\,(n)}^k$ respectively for the j-th p-tuple of attributes of $\mathbf{d}_{(n)}$ and for the k-th attribute of $\mathbf{d}_{(n)}$ related to the j-th argument of $\mathbf{x}$.

From now on, we denote with $\boldsymbol{F}=\{F_n\}_{n \in \mathbb{N}}$ any sequence of functions such that each F_n maps $\mathbb{I}^n \times D^n(p)$ into $\mathbb{I}$.

Remark 3.1. Suppose we have n values $x_1, \dots, x_n$ to aggregate using a mapping F_n. If x_j properly represents an argument of the fusion for each $j \in \mathbb{N}_n$, any of the attributes belonging to $\mathbf{d}_j$ is a parameter with the precise task to reveal an additional information about a particular aspect of the j-th argument. We could say that the presence of the attributes somehow enhances the reliability of the final output value, because the overall information coming from the data, compared with the case of no attributes, is richer. Obviously, all the attributes $d_j^1, \dots, d_j^p$ are independent of the j-th argument and also each other, in order to avoid any problem of superfluity.

Remark 3.2. In the literature, to the best of our knowledge, the only explicit precedent of aggregation operator where any input value is accompanied by a sort of parameter is given by the *Induced Ordered Weighted Averaging* (*IOWA* for short, see [8]). An *IOWA* operator of dimension n essentially aggregates objects which are pairs (u_j, a_j), for $j \in \mathbb{N}_n$, where a_j is the j-th argument, while u_j is the j-th order inducing variable, related to the weight w_j of a given vector W. Actually, the most important difference with our case is that $u_1, \dots, u_n$ play the role of *dummy* variables, because they influence the output value without directly entering the process of fusion.

Definition 3.2. We say that $\boldsymbol{F}$: $\bigcup_{n=1}^{\infty} \mathbb{I}^n \times D^n(p) \to \mathbb{I}$ is a *multi-attribute aggregation operator* (briefly, *M-AgOp*) in $\mathbb{I} \times D(p)$ if every $F_n = F_n(\mathbf{x}, \mathbf{d})$ is non-decreasing in each component of $\mathbf{x}$ and if, fixed any $\mu = \{\mathbf{d}_{(n)}\} \in \mathcal{F}$, then

$$\liminf_{n \to \infty} \inf_{\mathbf{x} \in \mathbb{I}^n} F_n(\mathbf{x}, \mathbf{d}_{(n)}) = a \quad \textit{and}$$

$$\limsup_{n \to \infty} \sup_{\mathbf{x} \in \mathbb{I}^n} F_n(\mathbf{x}, \mathbf{d}_{(n)}) = b. \quad (3.1)$$

It is clear that eq. (3.1) generalizes eq. (2.2), given in Definition 2.3: we stress the fact that the operator $\boldsymbol{F}$ is demanded to reach the boundaries of its range, when the components of the vector $\mathbf{x} \in \mathbb{I}^n$ tend to the border of the domain as n increases, independently of the values of the attributes. This requirement confirms the fact that, at any dimension n, the input values given by the vector $\mathbf{x}$ prevail over the attributes in the process of aggregation. Moreover, eq. (3.1) allows a M-AgOp $\boldsymbol{F}$ in $\mathbb{I} \times D(p)$ to induce infinitely many AgOps $\boldsymbol{G}^{\mu}$ in $\mathbb{I}$ defined as $G^{\mu}_n(\mathbf{x}) = F_n(\mathbf{x}, \mathbf{d}_{(n)})$, for any fixed $\mu = \{\mathbf{d}_{(n)}\} \in \mathcal{F}$.

A special attention is paid to the case of monotonicity of a M-AgOp $\mathbf{F}$ with respect to the attributes.

Definition 3.3. Given any $k \in N_p$, we say that $\mathbb{I}_k$ is *positive* with respect to $\boldsymbol{F}$ if every F_n is non-decreasing in each component of $\mathbf{d}^k \in \mathbb{I}^n_k$, for all $n \in \mathbb{N}$. On the other hand, $\mathbb{I}_k$ is called *negative* if every F_n is non-increasing in each component of $\mathbf{d}^k \in \mathbb{I}^n_k$, for all $n \in \mathbb{N}$.

In the following, with abuse of language, we will say that a M-AgOp $\mathbf{F}$ is *positive* (*negative*) if all the sets of attributes are positive (negative).

Example 3.1. Consider a scenario in which a central authority must collect the votes of anonymous peers of a network about the *trust value* they express with respect to the behavior of a certain peer. Initially, we suppose that all the votes have the same weight and the central authority combines the input values, belonging to the real unit interval, to obtain a single fused estimate through a *root-mean-power* $\mathbf{G}$ of the kind

$$G_n(\mathbf{x}) = \Big(\frac{1}{n}\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}},$$

where the exponent p is some fixed real number. Suppose now that the central authority intends to distinguish the data according to the *network distance* of any single voting peer to the subject matter of the judgement, in the sense that the bigger is such distance, the smaller is the reliability of the input value. Assume that our basic philosophy is not to overturn the choice of the previous operator: motivated by this idea, we could use a *weighted root-mean-power* $\mathbf{G}^{\mathbf{w}}$ of the kind

$$G^{\mathbf{w}}_n(\mathbf{x}) = \Big(\sum_{i=1}^{n} w_i x_i^p\Big)^{\frac{1}{p}},$$

where the weight vector $\mathbf{w} \in [0,1]^n$ is such that $\sum_{i=1}^{n} w_i = 1$ (see [5] and the references therein). Obviously, the greater is the distance between the j-th voter and the judged peer, the smaller is the j-th component of the weight vector. Actually, it seems more clear and straightforward the introduction of a *single-attribute aggregation operator* (briefly, *S-AgOp*), where the unique attribute here considered, i.e. the network distance, must have a negative influence. Along the lines of the previous choice, we can exactly maintain the same structure of the root-mean-power, but with the crucial innovation that the exponent p dynamically depends upon the attributes. This can be accomplished defining a *negative* operator $\mathbf{F}$ as

$$F_n(\mathbf{x}, \mathbf{d}) = \Big(\frac{1}{n}\sum_{i=1}^{n} x_i^p\Big)^{\frac{1}{p}}, \qquad (3.2)$$

where $p = -\sum_{i=1}^{n} d_i$. It is not difficult to show that such $\mathbf{F}$ is a *M-AgOp* in $[0,1]\times\,]0,\infty[$, with $p = 1$ and $D(1) = \,]0,\infty[$. The only non trivial statement is that $\mathbf{F}$ is *negative*: the proof rests on a general result regarding the quasi-arithmetic means (see [5] and the references therein). Applying Proposition 4.6 of [5], with $f(x) = x^{p'}$ and $g(x) = x^p$, where $p' < p < 0$, directly leads to the claim.

Remark 3.3. Any weighted aggregation can be seen as a multi-attribute aggregation too, with a one-dimensional set of attributes corresponding to considered weights.

4 PROPERTIES FOR MULTI-ATTRIBUTE AGOPS

The main purpose of this section is to rewrite some classical, relevant properties of an aggregation operator for a multi-attribute one. This task is quite elementary for a static property: in this case, we can state a general, natural principle.

Definition 4.1. Let $\mathcal{P}$ be a static property for an arbitrary aggregation operator in $\mathbb{I}$. We say that a *M-AgOp* $\mathbf{F}$ in $\mathbb{I} \times D(p)$ satisfies $\mathcal{P}$ if every *AgOp* $\boldsymbol{G}^{\mu}$ fulfills $\mathcal{P}$ for any $\mu \in \mathcal{F}$.

Note that, for instance, the operator $\mathbf{F}$ defined as in (3.2) is a continuous, idempotent S-AgOp.

In some cases, when the n-ary function F_n of a M-AgOp $\mathbf{F}$ shares a common static property with respect to both the variable $\mathbf{x} \in \mathbb{I}^n$ and the multi-vector of attributes $\mathbf{d} \in D^n(p)$ for all $n \in \mathbb{N}$, then we can speak of *global* property. The next definition provides a clear example.

Definition 4.2. We say that a *M-AgOp* $\mathbf{F}$ in $\mathbb{I} \times D(p)$ is *globally continuous* when every mapping $F_n : \mathbb{I}^n \times D^n(p) \to \mathbb{I}$ is continuous for all $n \in \mathbb{N}$.

Again, the operator $\mathbf{F}$ defined as in (3.2) is globally continuous. Obviously, the class of not globally continuous M-AgOps is not empty.

Example 4.1. Consider the *negative M-AgOp* $\mathbf{F}$ on $[0,1] \times [0,\infty[^2$ given by

$$F_n(\mathbf{x},\mathbf{d}) = \max_{i=1,\ldots,n}\{x_i^{f(d_i^1)+g(d_i^2)}\},$$

where $f, g : [0,\infty[\to [0,\infty[$ are two arbitrary, non-decreasing mappings. This operator is continuous, but generally not globally, if at least one between f and g is not continuous.

Unlike the static case, the extension of a dynamic property from an aggregation operator to a M-AgOp $\mathbf{F}$ seems to be really problematic, essentially because the arbitrariness in the choice of the sequence $\mu \in \mathcal{F}$ suffers the big drawback of imposing extreme rigidity to the property, when passing from a generic n-ary function F_n to the following ones, up to a substantial loss of its significance. Consider, for instance, the case of self-identity: adopting the principle as in Definition 4.1, we should say that a M-AgOp $\mathbf{F}$ in $\mathbb{I} \times D(p)$ is self-identical when

$$F_n(\mathbf{x},\mathbf{d}_{(n)}) = \\ = F_{n+1}((x_1,\mathbf{d}_{1\,(n+1)}),\ldots,(x_n,\mathbf{d}_{n\,(n+1)}),\mathbf{u})$$

for all $\mathbf{x} \in \mathbb{I}^n$, for any $\mu = \{\mathbf{d}_{(n)}\} \in \mathcal{F}$ and for any $n \in \mathbb{N}$, where $\mathbf{u} = (F_n(\mathbf{x},\mathbf{d}_{(n)}),\mathbf{d}_{n+1\,(n+1)})$. As argued above, the absolute arbitrariness of $\mathbf{d}_{(n)}$ and $\mathbf{d}_{(n+1)}$ makes practically impossible the fulfillment of this property by any M-AgOp: even if we imposed a natural condition as $\mathbf{d}_{j\,(n+1)} = \mathbf{d}_{j\,(n)}$ as $j \in \mathbb{N}_n$, the problem of a rational choice for $\mathbf{d}_{n+1\,(n+1)}$ would remain open. The problem of how defining a dynamic property for a M-AgOp has to be considered case by case: we propose now the concept of dynamic neutral element.

Definition 4.3. We say that a *M-AgOp* $\mathbf{F}$ in $\mathbb{I}\times D(p)$ admits a *dynamic neutral element* $e \in \mathbb{I}$ if, for every $i \in \mathbb{N}_n$ and for all $\mathbf{x} \in \mathbb{I}^n$ such that $x_i = e$, then

$$F_n(\mathbf{x},\mathbf{d}_{(n)}) = F_{n-1}(\times_{j\neq i}(x_j,\mathbf{d}_{j\,(n)})) \tag{4.1}$$

for any $\mu = \{\mathbf{d}_{(n)}\} \in \mathcal{F}$ and for all $n > 1$.

We can provide examples which show that, even for M-AgOps, static and dynamic "neutrality" are each other independent.

A dynamic property whose extension to a M-AgOp is particularly interesting is the asymptotic idempotency (see [3] and [4]). We recall that an AgOp $\mathbf{G}$ in $\mathbb{I}$ is *asymptotically idempotent* if

$$\lim_{n\to\infty} \delta_{G_n}(x) = x \quad \text{for all } x \in \mathbb{I}. \tag{4.2}$$

Definition 4.4. We say that a *M-AgOp* $\mathbf{F}$ in $\mathbb{I}\times D(p)$ is *asymptotically idempotent* if there exists a sequence $\mu = \{\mathbf{d}_{(n)}\} \in \mathcal{F}$ such that

$$\lim_{n\to\infty} \delta_{G_n^\mu}(x) = x \quad \textit{for all } x \in \mathbb{I}. \tag{4.3}$$

Example 4.2. Let us consider two cases of *negative S-AgOps* in $[0,1]\times[0,1]$. In the first, $\mathbf{F}$ is defined as

$$F_n(\mathbf{x},\mathbf{d}) = \begin{cases} \max\limits_{i=1,\ldots,n}\{x_i{}^{s(n)}\}\cdot\left(1-\frac{1}{2n}\right)+\frac{1}{2n}, & n \textit{ even};\\ \max\limits_{i=1,\ldots,n}\{x_i{}^{s(n)}\}\cdot\left(1-\frac{1}{2n}\right), & \textit{otherwise}, \end{cases}$$

where $s(n) := \sum_{i=1}^n d_i$. Observe that condition (3.1) is satisfied, because, fixed any $\mathbf{d} \in [0,1]^n$, we have

$$\inf_{\mathbf{x}\in\mathbb{I}^n} F_n(\mathbf{x},\mathbf{d}) = \begin{cases} \frac{1}{2n}, & \text{if } n \textit{ is even};\\ 0, & \textit{otherwise}, \end{cases}$$

and

$$\sup_{\mathbf{x}\in\mathbb{I}^n} F_n(\mathbf{x},\mathbf{d}) = \begin{cases} 1, & \text{if } n \textit{ is even};\\ 1-\frac{1}{2n}, & \textit{otherwise}. \end{cases}$$

This operator is asymptotically idempotent, because we can find infinitely many sequences of attributes such that eq. (4.3) holds: for instance, we can choose $\mathbf{d}_{(n)} = (0,0,\ldots,0,1)$ or $(0,0,\ldots,0,1/2,1/2)$ or $(1/n,\ldots,1/n)$, with $s(n) = 1$. On the contrary, if we pick $\mu^* := \{\mathbf{d}^*_{(n)}\}_{n\in\mathbb{N}}$, with $\mathbf{d}^*_{(n)} = (1,\ldots,1)$ and $s(n) = n$, we get

$$\lim_{n\to\infty} \delta_{G_n^{\mu^*}}(x) = 0 \quad \textit{for all } x \in \mathbb{I}.$$

In the second case, let $\mathbf{F}$ be given by

$$F_n(\mathbf{x},\mathbf{d}) = \max(\mathbf{x})\cdot\frac{\sum_{i=1}^n x_i{}^{d_i}}{1+\sum_{i=1}^n x_i{}^{d_i}}.$$

Note that condition (3.1) holds, because, for any $\mathbf{d} \in [0,1]^n$, we have

$$\sup_{\mathbf{x}\in\mathbb{I}^n} F_n(\mathbf{x},\mathbf{d}) = \frac{n}{1+n}.$$

We claim that such operator fulfills eq. (4.3) for any sequence $\mu \in \mathcal{F}$. First of all, note that, for any $x \in \mathbb{I}$, we have

$$\delta_{G_n^\mu}(x) \le x \tag{4.4}$$

for every sequence $\mu \in \mathcal{F}$. Indeed, for any $x \in \mathbb{I}$, we easily get

$$\delta_{G_n^{\mu^*}}(x) = x\cdot\frac{nx}{1+nx}. \tag{4.5}$$

Now, it is immediate to verify that an arbitrary $\mu \in \mathcal{F}$ is pointwise majorized by μ^*, hence, combining the negativity of the operator with eq. (4.4) and (4.5), we obtain that

$$\delta_{G_n^{\mu^*}}(x) \le \delta_{G_n^\mu}(x) \le x,$$

and the claim follows from $\lim_{n\to\infty} \delta_{G_n^{\mu^*}}(x) = x$ for all $x \in \mathbb{I}$.

Remark 4.1. Observe that Definition 2.9 may be extended without any change also to *M-AgOps*: particularly, we now present a generalization for a *S-AgOp* of Examples 2.2 and 2.3.

Example 4.3. Let $q \in \mathbb{N}$ and consider the *positive S-AgOp* $\mathbf{F}$ in $[0,1]\times\,]0,\infty[$ defined as

$$F_n(\mathbf{x},\mathbf{d}) = \begin{cases} \left(\frac{1}{q}\sum_{i=1}^{n} x_i^{s(n)}\right)^{\frac{1}{s(n)}}, & \textit{if } n \le q; \\ \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{s(n)}\right)^{\frac{1}{s(n)}}, & \textit{otherwise}. \end{cases}$$

It is easy to check that such $\mathbf{F}$ is a q-idempotent *S-AgOp*. Note that the positivity is based upon the result recalled in Example 3.1 and the elementary inequality $z^{\frac{1}{p}} < z^{\frac{1}{p'}}$ for all $z \in\,]0,1[$, with $0 < p < p'$.

Example 4.4. Consider the *negative S-AgOp* $\mathbf{F}$ in $[0,1]\times\,]0,\infty[$ given by

$$F_n(\mathbf{x},\mathbf{d}) = \begin{cases} \max_{i=1,\ldots,n} \{\alpha(x_i)^{d_i}\}, & \textit{if } n > 1; \\ id, & \textit{otherwise}, \end{cases}$$

where α is defined as in Example 2.3. It is not difficult to see that such $\mathbf{F}$, according to Definition 4.3, admits $e = 0$ as 3-dynamic (but not static) neutral element.

5 CONCLUSIONS

In this work, firstly we have introduced a new idea of aggregation, motivated by the fact that the usual boundary conditions which characterize a classical aggregation operator do not cover all the possible functions which play the role of aggregators. Secondly, we have unified all the important cases, like weighted means, in which one has to fuse both proper arguments and collateral parameters into a single output value, under an organic theory, called multi-attributes aggregation. We have provided several examples, both from a theoretical and practical point of view, in which such theory is shown to be more general and significant than the classical one.

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