Do We Know How to Integrate?

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Abstract. After a short history of integration on real line, some examples of optimization tasks are given to illustrate the philosophy behind some types of integrals with respect to monotone measures and related to the standard arithmetics on real line. Basic integrals are then described both in discrete case and general case. A general approach to integration known as universal integrals is recalled, and introduced types of integrals as universal integrals are discussed. A special stress is given to copula–based universal integrals. Several types of integrals based on arithmetics different from the standard one are given, too. Finally, some concluding remarks are added.

Keywords: Choquet integral, monotone measure, Sugeno integral, universal integral.

1 Introduction and Historical Remarks

When asking a randomly chosen person whether he/she knowns something about integrals, almost all positively reacting persons have in mind the Riemann integral. This integral is a background of the classical natural sciences, and it acts on (possibly n–dimensional) real line equipped with the standard Lebesgue measure. Obviously, the history of integration, at least till 1925, is related to the Riemann integral and its genuine generalizations. As a first trace of constructive approaches to integration can be considered a formula for the volume of a frustum of a square pyramid proposed in ancient Egypt around 1850 BC (the Moscow Mathematical Papyrus, Problem 14). The first documented systematic technique allowing to determine integrals is the exhaustion method of the ancient Greek astronomer Eudoxus (around 370 BC). This method was further developed by several Greek mathematicians, including Archimedes. Similar methods were independently developed in China (Liu Hui around the third century, father Zu Chogzhi and son Zu Geng in the fifth century describing the volume of a sphere) and in India (Aryabhata in the fifth century). Only more than 1000 years later, several European scientists have done next important steps in the integration area. We recall J. Kepler (his approach to computation of the volume of barrels in now known as Simpson rule), Cavalieri (with his method of indivisibles he was able to integrate polynomials till order 9), J. Wallis (algebraic law

V. Torra et al. (Eds.): MDAI 2012, LNAI 7647, pp. 13–22, 2012.

for integration), P. de Fermat (his was the first to use infinite series in his integration method). Modern notation for (indefinite) integral was introduced by G. Leibniz in 1675. He adapted the integral symbol $\int$ from the letter known as long $\int$, standing for "summa". The modern notation for the definite integral was first used by J. Fourier around 1820. In this period, A. Cauchy developed a method for integration of continuous functions. All the roots and backgrounds for the "integral", including the fundamental work of I. Newton and G. Leibniz, were known in the middle of the 19th century. It was B. Riemann in his Habilitation Thesis at University of Göttingen [15] in 1854 who gave the first indubitable access to integration. This integral, now called the Riemann integral, is the best known integral, taught in each Calculus course. Its limitations (real line, standard Lebesgue measure) were challenging several scholars to generalize it. We recall here H. Lebesgue [7] who in 1904 introduced a rather general integral, acting on an arbitrary measurable space $(X, \mathcal{A})$, and defined for an σ–additive measure $m : \mathcal{A} \to [0, \infty]$. Observe that this integral is a background of the probability theory, among others. The next words bring a quotation from H. Lebesgue lecture held in May 8, 1926, in Copenhagen and entitled "The development of the notion of the integral": "... a generalization mode not for the vain pleasure of generalizing, but rather for the solution of problems previously posed, is always a fruitful generalization. The diverse applications which have already taken the concepts which we have just examined prove this superabundantly" (for the full text see [18]). Note that there is no concept of improper Lebesgue integral as it is the case of Riemann integral. Therefore there is no guarantee that a Riemann integrable function is also Lebesgue integrable. As a typical example consider the function $f : R \to R$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin x}{x} & \text{else.} \end{cases}$$

Then the Lebesgue integral $\int_R f(x)\ d\mu$ with μ being the standard Lebesgue measure on Borel subsets of R does not exist (indeed, $\int_R |f(x)|\ d\mu = +\infty$), but the (improper) Riemann integral can be computed to be finite.

All till now mentioned integrals were additive (as functionals) and defined with respect to an (σ–) additive measure. The first known approach to integration based on a monotone but not necessarily additive measure is due to Vitali [21] from 1925. Approach of G. Vitali (dealing with inner and outer measures) is a predecessor of the fundamental work of G. Choquet [4] yielding the Choquet integral. Another fundamental integral defined for monotone measures is due to M. Sugeno [20] in 1974.

All mentioned integrals consider the (non-negative) real values of both functions and measures. There are numerous kinds of integrals defined on more general structures. In this contribution, we consider only the framework of already mentioned integrals, i.e., we will deal with measurable spaces $(X, \mathcal{A})$ from the class $\mathcal{S}$ of all measurable spaces (X is a non–empty set, universe, and $\mathcal{A} \subseteq 2^X$ is a σ–algebra of subsets of X), with $\mathcal{A}$–measurable functions

$f : X \to [0, \infty]$ from the class $\mathcal{F}_{(X,\mathcal{A})}$ of all such functions, and with monotone $m : \mathcal{A} \to [0, \infty]$, $m(\emptyset) = 0, m(X) > 0$, from the class $\mathcal{M}_{(X,\mathcal{A})}$ of all such measures. Integral is then a mapping

$$I : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0, \infty]$$

with some special properties we will discuss in the next sections.

The aim of this paper is to discuss some approaches to integration with respect to monotone measures. In the next section, we bring an optimisation problem under different constraints, illustrating the philosophy of several such integrals linked to the standard summation and multiplication of reals. These integrals are then properly defined and further discussed, including the study of their relationship. Section 3 is devoted to the introduction of a framework of universal integrals recently proposed in [5]. We introduce here some universal integrals, including copula based integrals (here we restrict our considerations to the unit interval $[0, 1]$). In Section 4, pseudo-arithmetical operations based integrals are discussed. Finally, several concluding remarks are added.

2 Optimisation of a Global Performance and Integrals

Consider a group $X = \{a, b, c\}$ of three workers with working capacity $f : X \to [0, \infty]$ given in hours by $f(a) = 5,\ f(b) = 4,\ f(c) = 3$. A performance per hour of a group of our workers is given by a set function $m : 2^X \to [0, \infty]$,

$m(\emptyset) = 0,\ m(\{a\}) = 2,\ m(\{b\}) = 3,\ m(\{c\}) = 4,$

$m(\{a, b\}) = 7,\ m(\{b, c\}) = 5,\ m(\{a, c\}) = 4,\ m(\{a, b, c\}) = 8.$

Our aim is to find a strategy to reach the optimal total performance of our workers under given work constraints:

(1) only one group can work for a fixed time period;
(2) several disjoint groups can work (fixed working time in each group may differ);
(3) one group starts to work, once a worker stops to work, he cannot start to work again;
(4) several disjoint groups can start to work, in each group after some working time we can split a working group into smaller groups, and a worker after stopping to work cannot start again;
(5) there are no constraints.

We formalize the optimal total performances under these five constraints settings and give the solution for our example. Hence the optimal total performance T_i under constraints (**i**) is:

$$T_1 = \max\{k \cdot m(A) \mid k \cdot 1_A \le f\} = \min\{f(a), f(b)\} \cdot m(\{a, b\}) = 4 \cdot 7 = 28;$$

$$T_2 = \max\left\{\sum k_i \cdot m(A_i) \mid \sum k_i \cdot 1_{A_i} \le f, (A_i)_i \text{ is disjoint system }\right\} =$$
$$= \min\{f(a), f(b)\} \cdot m(\{a, b\}) + f(c) \cdot m(\{c\}) = 4 \cdot 7 + 3 \cdot 4 = 40;$$

$$T_3 = \max\left\{\sum k_i \cdot m(A_i) \mid \sum k_i \cdot 1_{A_i} \leq f, (A_i)_i \text{ is a chain }\right\} =$$

$$= \min\{f(a), f(b), f(c)\}\cdot m(\{a,b,c\}) + \min\{f(a)-f(c), f(b)-f(c)\}\cdot m(\{a,b\}) +$$

$$+(f(a)-f(b))\cdot m(\{a\}) = 3\cdot 8 + 1\cdot 7 + 1\cdot 2 = 33;$$

$$T_4 = \max\left\{\sum k_i \cdot m(A_i) \mid \sum k_i \cdot 1_{A_i} \leq f, A_i \cap A_j \in \{\emptyset, A_i, A_j\} \text{ for each i,j}\right\} =$$

$$= \min\{f(a), f(b)\}\cdot m(\{a,b\}) + (f(a)-f(b))\cdot m(\{a\}) + f(c)\cdot m(\{c\}) =$$

$$= 4\cdot 7 + 1\cdot 2 + 3\cdot 4 = 42;$$

$$T_5 = \max\left\{\sum k_i \cdot m(A_i) \mid \sum k_i \cdot 1_{A_i} \leq f\right\} = T_4 = 42.$$

From the constraints settings it is obvious that the following inequalities always hold, independently of f and m:

$$T_1 \leq T_i, \quad i \in \{1,2,3,4,5\};$$

$$T_5 \geq T_i, \quad i \in \{1,2,3,4,5\};$$

$$T_4 \geq T_i, \quad i \in \{1,2,3,4\},$$

i.e., we have the following Hasse diagram (see Figure 1).

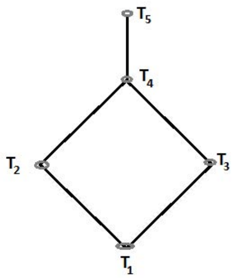

Fig. 1. Hasse diagram for relationships between functionals $T_1 - T_5$

As another example, consider $f : X \to [0,\infty]$ given by $f(a) = 8$, $f(b) = 3$, $f(c) = 6$, and $m : 2^X \to [0,\infty]$ given by,
$m(\emptyset) = 0$, $m(\{a\}) = 2$, $m(\{b\}) = 3$, $m(\{c\}) = 4$ and $m(A) = 10$ in all other cases. Then:

$$T_1 = 60, \quad T_2 = 69, \quad T_3 = 64, \quad T_4 = 73, \quad T_5 = 84,$$

(for more details see [19]).

All introduced functionals can be seen as special instances of decomposition integral proposed recently by Event and Lehrer [2], and some of them are, in fact, famous integrals introduced in past decades. We recall them now in a general setting, considering an arbitrary measurable space $(X, \mathcal{A}) \in \mathcal{S}$, as a mappings

$$I_i : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty],\ i \in \{1,2,3,4,5\}.$$

The first optimal performance T_1 is linked to the Shilkret integral [17],

$$I_1(m,f) = \sup\{k \cdot m(A) \,|\, k \cdot 1_A \le f\}.$$

Note that all sets considered in this paper are supposed to be measurable, $A \in \mathcal{A}$. Evidently, our $T_1 = I_1(m,f)$ for m, f given on $X = \{a,b,c\}$ and $\mathcal{A} = 2^X$.

Concerning the second optimal performance, T_2 is linked to the PAN–integral introduced by Yang in [23],

$$I_2(m,f) =$$

$$= \sup\left\{\sum_{i=1}^{n} k_i \cdot m(A_i) | n \in N, \sum_{i=1}^{n} k_i \cdot 1_{A_i} \le f, (A_i)_{i=1}^{n} \text{ is a disjoint system}\right\}.$$

The third optimization task describes the philosophy of the Choquet integral [4],

$$I_3(m,f) = \sup\left\{\sum_{i=1}^{n} k_i \cdot m(A_i) \,|\, n \in N, \sum_{i=1}^{n} k_i \cdot 1_{A_i} \le f, (A_i)_{i=1}^{n} \text{ is a chain}\right\}.$$

Note that due to the definition of the classical Riemann integral it holds

$$I_3(m,f) = \int_0^\infty m(\{f \ge t\})\,dt.$$

The fourth approach to optimization constraints brings a new integral I_4 proposed recently by Stupňanová [19],

$$I_4(m,f) = \sup\left\{\sum_{i=1}^{n} k_i \cdot m(A_i) | n \in N, \sum_{i=1}^{n} k_i \cdot 1_{A_i} \le f, A_i \cap A_j \in \{\emptyset, A_i, A_j\}\right.$$
$$\left.\text{for any } i,j \in \{1,\cdots,n\}\right\}.$$

Finally, another recent integral is linked to T_5, namely the concave integral introduced by Lehrer [8],

$$I_5(m,f) = \sup\left\{\sum_{i=1}^{n} k_i \cdot m(A_i) \,|\, n \in N, \sum_{i=1}^{n} k_i \cdot 1_{A_i} \le f\right\}.$$

The Hasse diagram in Figure 2 depicts the relationships between these integrals. Each of these integrals is linked to the standard arithmetical operations on real line, and for each of them it holds

$$I_i(k \cdot 1_{\{x\}}) = k \cdot m(\{x\}), k \in [0,\infty[,$$

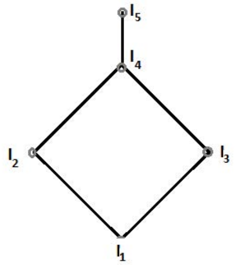

Fig. 2. Hasse diagram for relationships between integrals $I_1 - I_5$

if the singleton $\{x\} \in \mathcal{A}$. However, considering a general $A \in \mathcal{A}$, $I_i(k \cdot 1_A) = k \cdot m(A)$ holds only for $i \in \{1, 3\}$, in general. Consequently, integrals I_2, I_4 and I_5 have a failure admitting the existence of $m_1 \neq m_2$ such that $I(m_1, f) = I(m_2, f)$ for each $f \in \mathcal{F}_{(X,\mathcal{A})}$. Note also that each introduced integral is homogeneous, i.e., $I_i(k \cdot f) = k \cdot I_i(f)$ for each $k \in [0, \infty[$ and $i \in \{1, 2, 3, 4, 5\}$.

Considering special types of monotone measures, we have the next equalities valid for any measurable function $f \in \mathcal{F}_{(X,\mathcal{A})}$:

- if $m \in \mathcal{M}_{(X,\mathcal{A})}$ is supermodular then
$$I_5(m, \cdot) = I_3(m, \cdot),$$
i.e., then the concave integral coincide with the Choquet integral, see [8];
- if m is subadditive then
$$I_5(m, \cdot) = I_2(m, \cdot),$$
i.e., then the concave integral coincide with the PAN–integral, see [19];
- if m is an unaminity measure, i.e., there is $A \in \mathcal{A}, A \neq \emptyset$, so that $m(B) = \begin{cases} 1 \text{ if } A \subseteq B, \\ 0 \ \text{ else} \end{cases}$, then all introduced integrals coincide, and then
$$I_i(m, f) = \inf \{f(x) | x \in A\}, \ i = 1, \cdots, 5,$$
see [19].

Obviously, if X is finite and m is additive then

$$I_i(m, f) = \sum_{x \in X} f(x) \cdot m(\{x\}), \ i \in \{2, 3, 4, 5\}.$$

Moreover, if m is a σ–additive measure, then integrals $I_i(m, \cdot), i \in \{2, 3, 4, 5\}$ coincide with the standard Lebesgue integral,

$$I_i(m, f) = \int_X f \, dm.$$

3 Universal Integrals

To capture the idea of the majority of integrals proposed as functionals on abstract measurable spaces, Klement et al. [5] have recently proposed the concept of universal integrals.

Definition 1. *A mapping* $I : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$ *is called a universal integral whenever it satisfies the next properties:*

(UI1) *I is nondecreasing in both components, i.e.,* $I(m_1, f_1) \le I(m_2, f_2)$ *whenever there is* $(X,\mathcal{A}) \in \mathcal{S}$ *such that* $m_1, m_2 \in \mathcal{M}_{(X,\mathcal{A})}$ *and* $m_1 \le m_2,\ f_1, f_2 \in \mathcal{F}_{(X,\mathcal{A})}$ *and* $f_1 \le f_2$*;*

(UI2) *there is an operation* $\otimes : [0,\infty]^2 \to [0,\infty]$ *(called a pseudo–multiplication) with annihilator* 0 *(i.e.,* $a \otimes 0 = 0 \otimes a = 0$ *for each* $a \in [0,\infty]$*) and a neutral element* $e \in]0,\infty]$ *(i.e.,* $a \otimes e = e \otimes a = a$ *for each* $a \in [0,\infty]$*) so that*

$$I(m, k \cdot 1_A) = k \otimes m(A)$$

for any $(X,\mathcal{A}) \in \mathcal{S}, m \in \mathcal{M}_{(X,\mathcal{A})}, A \in \mathcal{A}$ *and* $k \in [0,\infty]$*;*

(UI3) *for any two pairs* $(m_1, f_1) \in (X_1, \mathcal{A}_1), (m_2, f_2) \in (X_2, \mathcal{A}_2)$ *such that* $m_1(\{f_1 \ge t\}) = m_2(\{f_2 \ge t\})$ *for each* $t \in]0,\infty]$ *(such pairs are called integral equivalent) it holds*

$$I(m_1, f_1) = I(m_2, f_2).$$

Similarly we can introduce the concept of universal integrals on the unit interval $[0,1]$ (compare the concepts of measure theory and probability theory). In such a case, we deal with normed monotone measures, $m(X) = 1$ (these measures are also called fuzzy measures or capacities), measurable functions $f : X \to [0,1]$, and the considered pseudo-multiplication $\otimes$ is defined on $[0,1]^2$, $\otimes : [0,1]^2 \to [0,1]$, with neutral element $e = 1$ (then $\otimes$ is called a semicopula, or conjuctor, or weak t–norm, depending on the literature). For more details we recommend [5]. Here we recall only two distinguished classes of universal integrals.

Proposition 1. *Let* $\otimes : [0,1]^2 \to [0,1]$ *be a fixed pseudo-multiplication. Then the mapping* $I_\otimes : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}_{(X,\mathcal{A})} \times \mathcal{F}_{(X,\mathcal{A})}\right) \to [0,\infty]$ *given by*

$$I_\otimes(m, f) = \sup\left\{t \otimes m\left(\{f \ge t\}\right) | t \in [0,\infty]\right\}$$

is a universal integral which is the smallest one linked to $\otimes$ *through the axiom* ***(UI2)****.*

Note that the Shilkret integral I_1 is related to the standard product $\cdot$, $I_1 = I_\cdot$, while $\otimes = \wedge$ (min) yields the famous Sugeno integral [20],

$$I_1(m, f) = Su(m, f) = \sup\left\{t \wedge m\left(\{f \ge t\}\right) | t \in [0,\infty]\right\}.$$

The second type of universal integrals we recall is defined on $[0,1]$ and it is linked to copulas.

Observe that a copula $C : [0,1]^2 \to [0,1]$ is a pseudo-multiplication on $[0,1]$ which is supermodular, i.e., for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$ it holds $C(\mathbf{x} \wedge \mathbf{y}) + C(\mathbf{x} \vee \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y})$. Copulas are in a one-to-one correspondence with probability measures on the Borel subsets of $[0,1]^2$ with uniformly distributed margins. This link is fully characterized by the equality

$$P_C\left([0,u] \times [0,v]\right) = C\left(u,v\right)$$

valid for all $u, v \in [0,1]$. For more details we recommend [14].

Proposition 2. *Let* $C : [0,1]^2 \to [0,1]$ *be a fixed copula. Then the mapping* $I_{(C)} : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}^1_{(X,\mathcal{A})} \otimes \mathcal{F}^1_{(X,\mathcal{A})}\right) \to [0,1]$*, where* $\mathcal{M}^1_{(X,\mathcal{A})} = \left\{m \in \mathcal{M}_{(X,\mathcal{A})} | m(X) = 1\right\}$, $\mathcal{F}^1_{(X,\mathcal{A})} = \left\{f \in \mathcal{F}_{(X,\mathcal{A})} | \ \mathit{Ran}\ f \subseteq [0,1]\right\}$, *given by*

$$I_{(C)}(m,f) = P_C\left(\left\{(u,v) \in [0,1]^2 | v \leq m(\{f \geq u\})\right\}\right)$$

is a universal integral on $[0,1]$.

Observe that for the product copula Π one have $I_{\Pi} = T_3$ (restricted to $[0,1]$), i.e., the Choquet integral is obtained. Similarly, for the greatest copula Min (i.e., for $\wedge$), the Sugeno integral on $[0,1]$ is obtained, $I_{(Min)} = Su$.

Finally note that integrals I_2 (PAN-integral), I_4 (Stupňanová integral) and I_5 (concave integral of Lehrer) introduced in the previous section are not universal integrals.

4 Integrals and Pseudo-Arithmetical Operations

In Section 2, we have tried to answer the question how to integrate under different constraint settings, utilizing as a basic tool for our processing the standard arithmetical operations on the real line. There are possible several modifications of these operations, yielding new types of integrals. First of all, we can rescale our original scale $[0,\infty]$ by means of some automorphism $\varphi : [0,\infty] \to [0,\infty]$ (i.e., φ is an increasing bijection). Then the standard addition $+$ becomes a pseudo-addition $\oplus : [0,\infty]^2 \to [0,\infty]$ given by

$$u \oplus v = \varphi^{-1}\left(\varphi(u) + \varphi(v)\right).$$

Similarly, pseudo-multiplication $\otimes : [0,\infty]^2 \to [0,\infty]$ is given by

$$u \otimes v = \varphi^{-1}\left(\varphi(u) \cdot \varphi(v)\right).$$

Modifying I_1 (Shilkret integral) into

$$I_{1,\varphi}(m,f) = \sup\left\{k \otimes m(A) \,|\, k \cdot 1_A \leq f\right\}$$

one gets the universal integral $I_{\otimes}$. However, $I_{\otimes}(m,f) = \varphi^{-1}\left(I_1(\varphi \cdot m, \varphi \cdot f)\right)$, i.e., we have a φ-transform of I_1 only. Similarly, the remaining integrals $I_i, i =$

2, 3, 4, 5, can be transformed. Note that the transformed Choquet integral $I_{3,\varphi}$ is a special instance of Choquet-like integrals introduced by Mesiar [9].

Pseudo-addition $\oplus$ and pseudo-multiplication $\otimes$ can be introduced axiomatically, see e.g. [1]. Not going more deeply into details, recall only that for $\oplus = \vee$ (max, supremum), and any pseudo-multiplication $\otimes$ as given in Definition 1, when replacing $+$ by $\vee$ and $\cdot$ by $\otimes$, in the definition of integrals $I_i, i = 1, \cdots, 5$, all of them collapse into the universal integral $I_\otimes$ characterized in Proposition 1. For a deeper overview of integrals based on pseudo-arithmetical operations (on finite spaces) we recommend [11].

5 Concluding Remarks

We have discussed the integrals, first from historical point of view, and then as optimization procedures when considering different constraints settings. The concept of universal integrals on $[0, \infty]$ and on $[0, 1]$ was also given, and several positive and negative examples were added.

Note that the axiomatic approach to several of introduced integrals was introduced several years after their constructive introduction. This is not the case of Riemann integral only, but for example the Choquet integral was axiomatized by Schmeidler in 1986 [16]. For an overview of axiomatic approaches to integrals we recommend [6].

Adding some constraints on monotone measures, one can get some distinguished aggregation functions. So, for example, when considering universal integrals on $[0, 1]$ and symmetric monotone measure on finite space X (i.e., $m(A)$ depends on the cardinality of A only), then the Choquet integral becomes OWA operator [22], [3], and copula-based integral $I_{(C)}(m, \cdot)$ becomes OMA operator [10] (i.e., ordered modular average).

Integrals can be also combined. So, for example, any convex combination $I = \lambda\, I_{(1)} + (1 - \lambda)\, I_{(2)}$, of two universal integrals related to pseudo-multiplications $\otimes_1$ and $\otimes_2$ with the same neutral element e is a universal integral related to the pseudo-multiplication $\otimes = \lambda \otimes_1 + (1 - \lambda) \otimes_2$, independently of $\lambda \in [0, 1]$. For two copulas C_1, C_2, also $C = \lambda\, C_1 + (1 - \lambda)\, C_2$ is a copula, and then $I_{(C)} = \lambda\, I_{C_1} + (1 - \lambda)\, I_{C_2}$. Another approach to combine integrals was proposed by Narukawa and Torra [12], and multidimensional integrals were introduced and discussed by the same authors in [13].

As we see, though we have touched the problem how to integrate, this area is an expanding field attracting an intensive research and we believe to see not only many new theoretical results soon, but first of all numerous applications in several engineering and human reasoning connected branches.

Acknowledgement. The work on this contribution was supposed by grants VEGA 1/0171/12 and GAČR P–402–11–0378.

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