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General Chebyshev type inequalities for universal integral

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ABSTRACT

A new inequality for the universal integral on abstract spaces is obtained in a rather general form. As two corollaries, Minkowski's and Chebyshev's type inequalities for the universal integral are obtained. The main results of this paper generalize some previous results obtained for special fuzzy integrals, e.g., Choquet and Sugeno integrals. Furthermore, related inequalities for seminormed integral are obtained.

1. Introduction

In the field of fuzzy measure theory, the Sugeno integral [30], see [23,32], has been the first integral proposed to compute an average value of some function with respect to a fuzzy measure, and early applications of fuzzy measures in multicriteria evaluation were based on this integral. Many authors generalized Sugeno and Choquet integrals by using some other operations to replace the special operation(s) $\wedge$ and/or $\vee$ (see, e.g., [16,19,29,31]). In the paper [29], Suárez and Gil presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

The Choquet integral (see [7,9,15,23]) and the Sugeno integral provide a useful tool in many problems in engineering and social choice where the aggregation of data is required. However, their applicability is restricted because of the special operations used in the construction of these integrals. Therefore, Klement et al. [14] provided a universal integral generalizing both the Choquet and the Sugeno case.

The study of inequalities for Sugeno integral was initiated by Román-Flores et al. [11,26], and then followed by many authors (see [1,2,4,11,18,20,21]). In [11], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. [20]. Furthermore, Chebyshev type inequalities for comonotone functions and arbitrary fuzzy measure-based Sugeno integral on an arbitrary measurable space were proposed in a rather general form by Mesiar and

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Ouyang [18]. Recently, Ouyang and Mesiar [22] presented a Chebyshev type inequality for the seminormed fuzzy integrals. There were investigated inequalities also with respect to pseudo-integrals based on a special type of nonadditive measures, so called pseudo-additive measures, see [3,25].

The aim of this contribution is to generalize the Chebyshev type inequalities to the frame of the universal integral on abstract spaces. In general, any integral inequality can be a very strong tool for applications. In particular, when we think of an integral operator as a predictive tool then an integral inequality can be very important in measuring and dimensioning such a process.

The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarize some previous known results. In Section 3, we focus on a new type inequality for the universal integral and then the Minkowski and the Chebyshev type inequalities are deduced as its corollaries. Section 4 includes a reverse inequality for semiconormed fuzzy integrals. Finally, some concluding remarks are given.

2. Universal integral

In this section, we are going to review some well known results on universal integral (see [14]), and related notions and notations used in this paper.

Definition 2.1 ([14,23,30]). A monotone measure m on a measurable space $(X,\mathcal{A})$ is a function $m:\mathcal{A}\to[0,\infty]$ satisfying

(i) $m(\emptyset)=0$,
(ii) $m(X)>0$,
(iii) $m(A)\leqslant m(B)$ whenever $A\subseteq B$.

Normed monotone measures on $(X,\mathcal{A})$, i.e., monotone measures satisfying $m(X)=1$, are also called fuzzy measures (see [12,30,32]).

Let X be a non-empty set and $\mathcal{A}$ a σ-algebra of subsets of X. Then $(X,\mathcal{A})$ is a measurable space and a function $f:X\to[0,\infty]$ is called $\mathcal{A}$-measurable if, for each $B\in\mathcal{B}([0,\infty])$, the σ-algebra of Borel subsets of the interval $[0,\infty]$, the preimage $f^{-1}(B)$ is an element of $\mathcal{A}$.

Definition 2.2 [14]. Let $(X,\mathcal{A})$ be a measurable space.

(i) $\mathcal{F}^{(X,\mathcal{A})}$ is the set of all $\mathcal{A}$ -measurable functions $f:X\to[0,\infty]$;
(ii) For each number $a\in]0,\infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ is the set of all monotone measures (in the sense of Definition 2.1) satisfying $m(X)=a$; and we take

$$\mathcal{M}^{(X,\mathcal{A})}=\bigcup_{a\in]0,\infty]}\mathcal{M}_a^{(X,\mathcal{A})}.$$

Let $\mathcal{S}$ be the class of all measurable spaces, and take

$$\mathcal{D}_{[0,\infty]}=\bigcup_{(X,\mathcal{A})\in\mathcal{S}}\mathcal{M}^{(X,\mathcal{A})}\times\mathcal{F}^{(X,\mathcal{A})}.$$

The Choquet [7], Sugeno [30] and Shilkret [28] integrals (see also [6,17,23,24]), respectively, are defined, for any measurable space $(X,\mathcal{A})$, for any measurable function $f\in\mathcal{F}^{(X,\mathcal{A})}$ and for any monotone measure $m\in\mathcal{M}^{(X,\mathcal{A})}$,i.e., for any $(m,f)\in\mathcal{D}_{[0,\infty]}$, by

$$\mathbf{Ch}(m,f)=\int_0^\infty m(\{f\geqslant t\})dt, \tag{2.1}$$

$$\mathbf{Su}(m,f)=\sup\{\min(t,m(\{f\geqslant t\}))|t\in]0,\infty]\}, \tag{2.2}$$

$$\mathbf{Sh}(m,f)=\sup\{t\cdot m(\{f\geqslant t\})|t\in]0,\infty]\}, \tag{2.3}$$

where the convention $0\cdot\infty=0$ is used. All these integrals map $\mathcal{M}^{(X,\mathcal{A})}\times\mathcal{F}^{(X,\mathcal{A})}$ into $[0,\infty]$ independently of $(X,\mathcal{A})$. We remark that fixing an arbitrary $m\in\mathcal{M}^{(X,\mathcal{A})}$, they are non-decreasing functions from $\mathcal{F}^{(X,\mathcal{A})}$ into $[0,\infty]$, and fixing an arbitrary $f\in\mathcal{F}^{(X,\mathcal{A})}$, they are non-decreasing functions from $\mathcal{M}^{(X,\mathcal{A})}$ into $[0,\infty]$.

We stress the following important common property for all three integrals from (2.1)–(2.3). Namely, these integrals do not make differences between the pairs $(m_1,f_1),(m_2,f_2)\in\mathcal{D}_{[0,\infty]}$, which satisfy, for all $t\in]0,\infty]$,

$$m_1(\{f_1\geqslant t\})=m_2(\{f_2\geqslant t\}).$$

Therefore, such equivalence relation between pairs of measures and functions was introduced in [14].

Definition 2.3. Two pairs $(m_1,f_1)\in\mathcal{M}^{(X_1,\mathcal{A}_1)}\times\mathcal{F}^{(X_1,\mathcal{A}_1)}$ and $(m_2,f_2)\in\mathcal{M}^{(X_2,\mathcal{A}_2)}\times\mathcal{F}^{(X_2,\mathcal{A}_2)}$ satisfying

$$m_1(\{f_1\geqslant t\})=m_2(\{f_2\geqslant t\})\quad\text{for all } t\in]0,\infty],$$

will be called integral equivalent, in symbols

$$(m_1, f_1) \sim (m_2, f_2).$$

To introduce the notion of the universal integral we shall need instead of the usual plus and product more general real operations.

Definition 2.4 ([23,31]). A function $\otimes : [0,\infty]^2 \to [0,\infty]$ is called a pseudo-multiplication if it satisfies the following properties:

(i) it is non-decreasing in each component, i.e., for all a_1, a_2, b_1, $b_2 \in [0,\infty]$ with $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ we have $a_1 \otimes b_1 \leqslant a_2 \otimes b_2$;
(ii) 0 is an annihilator of, i.e., for all $a \in [0,\infty]$ we have $a \otimes 0 = 0 \otimes a = 0$;
(iii) has a neutral element different from 0, i.e., there exists an $e \in]0,\infty]$ such that, for all $a \in [0,\infty]$, we have $a \otimes e = e \otimes a = a$.

Restricting to the interval $[0,1]$ a pseudo-multiplication and a pseudo-addition with additional properties of associativity and commutativity can be considered as the t-norm T and the t-conorms S (see [13]), respectively.

For a given pseudo-multiplication $\otimes$ on $[0,\infty]$, we suppose the existence of a pseudo-addition $\oplus : [0,\infty]^2 \to [0,\infty]$ such that it is continuous, associative, non-decreasing and has 0 as neutral element (the commutativity of $\oplus$ follows, see [13]), and which is left-distributive with respect to $\otimes$ i.e., for all a,b, $c \in [0,\infty]$ we have $(a \oplus b) \otimes c = (a \oplus c) \otimes (b \oplus c)$. The pair $(\oplus, \otimes)$ is then called an integral operation pair, see [6,14].

Each of the integrals mentioned in (2.1)–(2.3) maps $\mathcal{D}_{[0,\infty]}$ into $[0,\infty]$ and their main properties can be covered by the following common integral given in [14].

Definition 2.5. A function $\mathbf{I} : \mathcal{D}_{[0,\infty]} \to [0,\infty]$ is called a universal integral if the following axioms hold:

(I1) For any measurable space $(X, \mathcal{A})$, the restriction of the function $\mathbf{I}$ to $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ is non-decreasing in each coordinate;
(I2) there exists a pseudo-multiplication $\otimes : [0,\infty]^2 \to [0,\infty]$ such that for all pairs $(m, c \cdot \mathbf{1}_A) \in \mathcal{D}_{[0,\infty]}$ (where $\mathbf{1}_A$ is the characteristic function of the set A)

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A);$$

(I3) for all integral equivalent pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$ we have

$$\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2).$$

By Proposition 3.1 from [14] we have the following important characterization.

Theorem 2.6. *Let $\otimes : [0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication on $[0,\infty]$. Then the smallest universal integral $\mathbf{I}$ and the greatest universal integral $\mathbf{I}$ based on $\otimes$ are given by*

$$\mathbf{I}_{\otimes}(m,f) = \sup\{t \otimes m(\{f \geqslant t\}) | t \in]0,\infty]\},$$
$$\mathbf{I}^{\otimes}(m,f) = \mathrm{essup}_m f \otimes \sup\{m(\{f \geqslant t\}) | t \in]0,\infty]\},$$

where $\mathrm{essup}_m f = \sup\{t \in [0,\infty] | m(\{f \geqslant t\}) > 0\}$.

Specially, we have $\mathbf{Su} = \mathbf{I}_{Min}$ and $\mathbf{Sh} = \mathbf{I}_{Prod}$, where the pseudo-multiplications *Min* and *Prod* are given by $Min(a,b) = \min(a,b)$ and $Prod(a,b) = a \cdot b$. There is neither a smallest nor a greatest pseudo-multiplication on $[0,\infty]$. But, if we fix a neutral element $e \in]0,\infty]$, then the smallest pseudo-multiplication $\otimes_e$ with neutral element e is given by

$$a \otimes_e b = \begin{cases} 0 & \text{if } (a,b) \in [0,e[^2, \\ \max(a,b) & \text{if } (a,b) \in [e,\infty]^2, \\ \min(a,b) & \textit{otherwise}. \end{cases}$$

Then by Proposition 3.2 from [14] there exists the smallest universal integral $\mathbf{I}_{\otimes_e}$ among all universal integrals satisfying the conditions

(i) for each $m \in \mathcal{M}_e^{(X,\mathcal{A})}$ and each $c \in [0,\infty]$ we have $\mathbf{I}(m, c \cdot \mathbf{1}_X) = c$,
(ii) for each $m \in \mathcal{M}^{(X,\mathcal{A})}$ and each $A \in \mathcal{A}$ we have $\mathbf{I}(m, e \cdot \mathbf{1}_X) = m(A)$, given by

$$\mathbf{I}_{\otimes_e}(m,f) = \max(m(\{f \geqslant e\}), \mathrm{essinf}_m f),$$

where $\mathrm{essinf}_m f = \sup\{t \in [0,\infty] | m(\{f \geqslant t\}) = e\}$.

Restricting now to the unit interval [0,1] we shall consider functions $f \in \mathcal{F}^{(X,\mathcal{A})}$ satisfying Ran(f) $\subseteq$ [0,1], and then we shall write $f \in \mathcal{F}^{(X,\mathcal{A})}_{[0,1]}$. In this case we have the restriction of pseudo-multiplications $\otimes$ to $[0,1]^2$ (called a semicopula or a conjunctor, i.e., a binary operation $\circledast : [0,1]^2 \to [0,1]$ which is non-decreasing in both components, has 1 as neutral element and satisfies $a \circledast b \leqslant \min(a,b)$ for all $(a,b) \in [0,1]^2$, see [5,10]), and universal integrals are restricted to the class $\mathcal{D}_{[0,1]} = \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \mathcal{M}_1^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}_{[0,1]}$. In a special case, for a fixed strict t-norm T, the corresponding universal integral $\mathbf{I}_T$ is the so-called Sugeno–Weber integral [33]. The smallest universal integral $\mathbf{I}_\circledast$ on the [0,1] scale related to the semicopula $\circledast$ is given by ([14])

$$\mathbf{I}_\circledast(m,f) = \sup\{t \circledast m(\{f \geqslant t\}) | t \in [0,1])\}.$$

This type of integral was called seminormed integral in [29].

3. Chebyshev's inequality for universal integral

In this section we will prove first the main theorem, and then the Minkowski and Chebyshev type inequalities appear as its corollaries.

We shall need the following important property of a pair of functions, see [8,9,23].

Definition 3.1. Functions $f, g : X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0$$

and f and g are said to be countermonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \leqslant 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$. Similarly, if f and g are countermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ cannot happen.

Theorem 3.2. *Let $\star : [0,\infty[^2 \to [0,\infty[$ be continuous and nondecreasing in both arguments and $\varphi : [0,\infty[\to [0,\infty[$ be continuous and strictly increasing function. Let $f, g \in \mathcal{F}^{(X,\mathcal{A})}$ be two comonotone measurable functions and $\otimes_e : [0,\infty]^2 \to [0,\infty]$ be the smallest pseudo-multiplication on $[0,\infty]$ with neutral element $e \in]0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, \varphi(f))$ and $\mathbf{I}_{\otimes_e}(m, \varphi(g))$ are finite. If*

$$\varphi^{-1}((\varphi(a \star b) \otimes_e c)) \geqslant (\varphi^{-1}((\varphi(a) \otimes_e c)) \star b) \vee (a \star \varphi^{-1}((\varphi(b) \otimes_e c))), \tag{3.1}$$

then the inequality

$$\varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(f \star g))) \geqslant \varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(f))) \star \varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(g))) \tag{3.2}$$

holds.

Proof. . Let $e \in]0,\infty]$ be the neutral element of $\otimes_e$. If $\mathbf{I}_{\otimes_e}(m, \varphi(f)) = \varphi(a) < \infty$ and $\mathbf{I}_{\otimes_e}(m, \varphi(g)) = \varphi(b) < \infty$, then $t \otimes_e m(\{\varphi(f) \geqslant t\}) \leqslant \varphi(a)$ and $t \otimes_e m(\{\varphi(g) \geqslant t\}) \leqslant \varphi(b)$ for all $t \in]0,\infty]$. Thus, for any $\varepsilon > 0$, there exist $\varphi(a_\varepsilon)$ and $\varphi(b_\varepsilon)$ such that

$$m(\{\varphi(f) \geqslant \varphi(a_\varepsilon)\}) = m(\{f \geqslant a_\varepsilon\}) = a_1,$$
$$m(\{\varphi(g) \geqslant \varphi(b_\varepsilon)\}) = m(\{g \geqslant b_\varepsilon\}) = b_1,$$

where $\varphi(a_\varepsilon) \otimes_e a_1 \geqslant \varphi(a - \varepsilon)$ and $\varphi(b_\varepsilon) \otimes_e b_1 \geqslant \varphi(b - \varepsilon)$. The fact of $\{f \geqslant a_\varepsilon\} \cap \{g \geqslant b_\varepsilon\} \subset \{f \star g \geqslant a_\varepsilon \star b_\varepsilon\}$ and the comonotonicity of f, g imply that $m(\{f \star g \geqslant a_\varepsilon \star b_\varepsilon\}) \geqslant a_1 \wedge b_1$. Hence by (3.1)

$$\begin{aligned}\varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(f \star g))) &= \varphi^{-1}(\sup\{t \otimes_e m(\{\varphi(f \star g) \geqslant t\}) | t \in]0,\infty]\})\}) \geqslant \varphi^{-1}(\varphi(a_\varepsilon \star b_\varepsilon) \otimes_e (a_1 \wedge b_1)) \\ &= \varphi^{-1}((\varphi(a_\varepsilon \star b_\varepsilon) \otimes_e a_1)) \wedge \varphi^{-1}(\varphi(a_\varepsilon \star b_\varepsilon) \otimes_e b_1) \geqslant (\varphi^{-1}(\varphi(a_\varepsilon) \otimes_e a_1) \star b_\varepsilon) \wedge (a_\varepsilon \star \varphi^{-1}(\varphi(b_\varepsilon) \otimes_e b_1)) \\ &\geqslant ((a - \varepsilon) \star b_\varepsilon) \wedge (a_\varepsilon \star (b - \varepsilon)) \geqslant (a - \varepsilon) \star (b - \varepsilon),\end{aligned}$$

whence $\varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(f \star g))) \geqslant a \star b = \varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(f))) \star \varphi^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi(g)))$ follows from the continuity of $\star$ and the arbitrariness of ε. □

Let $\varphi(x) = x^s$ for all $s > 0$. Then we obtain an inequality related to Minkowski type for universal integral.

Corollary 3.3. *Let $f, g \in \mathcal{F}^{(X,\mathcal{A})}$ be two comonotone measurable functions and $\otimes_e : [0,\infty]^2 \to [0,\infty]$ be the smallest pseudo-multiplication on $[0,\infty]$ with neutral element $e \in]0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f^s)$ and $\mathbf{I}_{\otimes_e}(m, g^s)$ are finite. Let $\star : [0,\infty[^2 \to [0,\infty[$ be continuous and nondecreasing in both arguments. If*

$$((a\bigstar b)^s\otimes_e c)^{\frac{1}{s}} \geqslant \left((a^s\otimes_e c)^{\frac{1}{s}}\bigstar b\right) \vee \left(a\bigstar(b^s\otimes_e c)^{\frac{1}{s}}\right),$$

then the inequality

$$(\mathbf{I}_{\otimes_e}(m,(f\bigstar g)^s))^{\frac{1}{s}} \geqslant (\mathbf{I}_{\otimes_e}(m,f^s))^{\frac{1}{s}}\bigstar(\mathbf{I}_{\otimes_e}(m,g^s))^{\frac{1}{s}}$$

holds for all $s > 0$.

Again, we get the Chebyshev type inequality whenever $s = 1$.

Corollary 3.4. *Let $f,g \in \mathcal{F}^{(X,\mathcal{A})}$ be two comonotone measurable functions and $\otimes_e : [0,\infty]^2 \to [0,\infty]$ be the smallest pseudo-multiplication on $[0,\infty]$ with neutral element $e \in]0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m,f)$ and $\mathbf{I}_{\otimes_e}(m,g)$ are finite. Let $\bigstar : [0,\infty[^2 \to [0,\infty[$ be continuous and nondecreasing in both arguments. If*

$$(a\bigstar b)\otimes_e c) \geqslant [(a\otimes_e c)\bigstar b] \vee [a\bigstar(b\otimes_e c)],$$

then the inequality

$$\mathbf{I}_{\otimes_e}(m,(f\bigstar g)) \geqslant \mathbf{I}_{\otimes_e}(m,f)\bigstar\mathbf{I}_{\otimes_e}(m,g)$$

holds.

Notice that when working on the unit interval $[0,1]$ in Theorem 3.2, we mostly deal with $e = 1$, and then $\otimes = \circledast$ is a semi-copula (t-seminorm), implying the following result.

Corollary 3.5. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\bigstar : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and $\varphi : [0,1] \to [0,1]$ be continuous and strictly increasing function. If the semicopula $\circledast$ satisfies*

$$\varphi^{-1}((\varphi(a\bigstar b)\circledast c)) \geqslant (\varphi^{-1}((\varphi(a)\circledast c))\bigstar b) \vee (a\bigstar\varphi^{-1}((\varphi(b)\circledast c))),$$

then the inequality

$$\varphi^{-1}(\mathbf{I}_{\circledast}(m,\varphi(f\bigstar g))) \geqslant \varphi^{-1}(\mathbf{I}_{\circledast}(m,\varphi(f)))\bigstar\varphi^{-1}(\mathbf{I}_{\circledast}(m,\varphi(g)))$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Let $\varphi(x) = x^s$ for all $0 < s < \infty$. Then we get the reverse Minkowski type inequality for seminormed fuzzy integrals.

Corollary 3.6. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\bigstar : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If semicopula $\circledast$ satisfies*

$$((a\bigstar b)^s\circledast c)^{\frac{1}{s}} \geqslant \left((a^s\circledast c)^{\frac{1}{s}}\bigstar b\right) \vee \left(a\bigstar(b^s\circledast c)^{\frac{1}{s}}\right),$$

then the inequality

$$(\mathbf{I}_{\circledast}(m,(f\bigstar g)^s))^{\frac{1}{s}} \geqslant (\mathbf{I}_{\circledast}(m,f^s))^{\frac{1}{s}}\bigstar(\mathbf{I}_{\circledast}(m,g^s))^{\frac{1}{s}}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ and for all $0 < s < \infty$.

Again, we get the Chebyshev type inequality for t-seminormed fuzzy integrals whenever $s = 1$ [22].

Corollary 3.7. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\bigstar : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If semicopula $\circledast$ satisfies*

$$(a\bigstar b)\circledast c \geqslant ((a\circledast c)\bigstar b) \vee (a\bigstar(b\circledast c)),$$

then the inequality

$$\mathbf{I}_{\circledast}(m,(f\bigstar g)) \geqslant \mathbf{I}_{\circledast}(m,f)\bigstar\mathbf{I}_{\circledast}(m,g)$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Remark 3.8. We can use an example from [22] to show that the condition

$$(a\bigstar b)\circledast c \geqslant ((a\circledast c)\bigstar b) \vee (a\bigstar(b\circledast c))$$

in Corollary 3.7 (and thus in Theorem 3.2) cannot be abandoned, and so we omit it here.

Let $V, U : [0,1]^2 \to [0,1]$ be two binary operations. Recall that V dominates U (or U is dominated by V), denoted by $V \gg U$, if

$$V(U(a,b),U(c,d)) \geqslant U(V(a,c),V(b,d))$$

holds for any a, b, c, $d \in [0,1]$. For a deeper investigation of complete domination of aggregation functions the reader is referred to [27].

Notice that if the semicopula (t-seminorm) $\circledast$ is minimum (i.e., for Sugeno integral) and $\star$ is bounded from above by minimum and φ is a continuous and strictly increasing function, then $\star$ is dominated by minimum. Therefore the following results hold.

Corollary 3.9 [2]. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum and $\varphi : [0,1] \to [0,1]$ be continuous and strictly increasing function. Then the inequality*

$$\varphi^{-1}(\mathbf{Su}(m,\varphi(f\star g))) \geqslant \varphi^{-1}(\mathbf{Su}(m,\varphi(f)))\star\varphi^{-1}(\mathbf{Su}(m,\varphi(g)))$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Corollary 3.10 [21]. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. Then the inequality*

$$(\mathbf{Su}(m,(f\star g)^s))^{\frac{1}{s}} \geqslant (\mathbf{Su}(m,f^s))^{\frac{1}{s}}\star(\mathbf{Su}(m,g^s))^{\frac{1}{s}}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ and for all $0 < s < \infty$.

Corollary 3.11 [18]. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum. Then the inequality*

$$\mathbf{Su}(m,f\star g) \geqslant \mathbf{Su}(m,f)\star\mathbf{Su}(m,g)$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

4. A reverse inequality for semiconormed fuzzy integrals

If we take T a t-seminorm and S its dual t-semiconorm, $S(x,y) = 1 - T(1-x, 1-y)$ and m a normed fuzzy measure, i.e., satisfying $m(X) = 1$ (see [12,30,32]), then

$$(\mathbf{I}_T(f,m))^d = 1 - \mathbf{I}_T(1-f,m) = \mathbf{I}_S(f,m^d),$$

where the dual fuzzy measure m^d is given by

$$m^d(A) = 1 - m(X - A)$$

and thus by the duality, all results for t-seminormed integrals can be transformed into results for t-semiconormed integrals. We get the following theorem with an analogous proof as the proof of Theorem 3.2.

Theorem 4.1. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and $\varphi : [0,1] \to [0,1]$ be continuous and strictly increasing function. If the t-semiconorm S satisfies*

$$\varphi^{-1}(S(\varphi(a\star b),c)) \leqslant (\varphi^{-1}(S(\varphi(a),c))\star b) \wedge (a\star\varphi^{-1}(S(\varphi(b),c))),$$

then the inequality

$$\varphi^{-1}(\mathbf{I}_S(m,\varphi(f\star g))) \leqslant \varphi^{-1}(\mathbf{I}_S(m,\varphi(f)))\star\varphi^{-1}(\mathbf{I}_S(m,\varphi(g)))$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Let $\varphi(x) = x^k$ for all $0 < k < \infty$, then we get the Minkowski inequality for semiconormed fuzzy integrals (if $k = 1$, then we have the reverse Chebyshev inequality for semiconormed fuzzy integrals [22]).

Corollary 4.2. *Let $f,g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the t-semiconorm S satisfies*

$$(S((a\star b)^k,c))^{\frac{1}{k}} \leqslant \left((S(a^k,c))^{\frac{1}{k}}\star b\right) \wedge \left(a\star(S(b^k,c))^{\frac{1}{k}}\right),$$

then the inequality

$$(\mathbf{I}_S(m,(f\star g)^k))^{\frac{1}{k}} \leqslant (\mathbf{I}_S(m,f^k))^{\frac{1}{k}}\star(\mathbf{I}_S(m,g^k))^{\frac{1}{k}}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ and for all $0 < k < \infty$.

Notice that if the t-semiconorm S is maximum (i.e., for Sugeno integral) and $\star$ is bounded from below by maximum and φ be a continuous and strictly increasing function, then S is dominated by $\star$. Thus the following results hold.

Corollary 4.3 [2]. *Let $f, g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum and $\varphi : [0,1] \to [0,1]$ be a continuous and strictly increasing function. Then the inequality*

$$\varphi^{-1}(\mathbf{Su}(m, \varphi(f \star g))) \leqslant \varphi^{-1}(\mathbf{Su}(m, \varphi(f))) \star \varphi^{-1}(\mathbf{Su}(m, \varphi(g)))$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Corollary 4.4 [1]. *Let $f, g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality*

$$(\mathbf{Su}(m, (f \star g)^k))^{\frac{1}{k}} \leqslant (\mathbf{Su}(m, f^k))^{\frac{1}{k}} \star (\mathbf{Su}(m, g^k))^{\frac{1}{k}}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ and for all $0 < k < \infty$.

Corollary 4.5 [22]. *Let $f, g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]^2 \to [0,1]$ be continuous and nondecreasing in both arguments and bounded from below by maximum. Then the inequality*

$$\mathbf{Su}(m, (f \star g)) \leqslant \mathbf{Su}(m, f) \star \mathbf{Su}(m, g)$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

5. Conclusion

We have proved a new general inequality for the universal integral. As two corollaries, Minkowski's and Chebyshev's type inequalities for the universal integral are obtained, covering in special cases many inequalities for Choquet, Sugeno and seminormed integrals, some of them proved in recent papers. For further investigation, it would be a challenging problem to determine the conditions under which (3.2) becomes an equality.

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