Liapunov-Type Inequality for Universal Integral

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The Choquet integral and the Sugeno integral provide a useful tool in many problems in engineering and social choice where the aggregation of data is required. In this paper, previous results of Hong (*Nonlinear Analysis* 2011 74:7296–7303) are improved by relaxing some of their requirements. Carlson's, Sandor's, Bushell–Okrasinski's type inequalities and Fatou's lemma for universal integral are studied in a rather general form, thus generalizing some recent results.

1. INTRODUCTION

The Choquet integral[1–3] and the Sugeno integral[4] provide a useful tool for many problems in engineering and social choice, where the aggregation of data is required. However, their applicability is restricted because of the special operations used in the construction of these integrals. Therefore, Klement et al.[5] provided a universal integral generalizing both the Choquet and the Sugeno cases.

Recently, Hong[7] proposed a Liapunov-type inequality for the Lebesgue measure-based Sugeno integral.

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THEOREM 1.1. *Suppose that f is a nonincreasing concave function on* $[0, 1]$ *and that μ is the Lebesgue measure on $\mathbb{R}$. Then for* $0 < t < s < r$, *the inequality*

$$\left(\frac{t}{t - t\alpha + s\alpha}\right)^{t(r-s)} \mathbf{Su}^{r-t}(\mu, f^s) \leq (\mathbf{Su}^{r-s}(\mu, f^t))(\mathbf{Su}^{s-t}(\mu, f^r)) \tag{1.1}$$

holds where α satisfies the following equation:

$$\left(\frac{t(1-\alpha)}{t - t\alpha + s\alpha}\right)^t \times \frac{\alpha^{\frac{t}{s}}}{(1-\alpha)^t} = \frac{s\alpha}{t - t\alpha + s\alpha}.$$

Unfortunately, the results of Hong[7] are based on concavity of f and some constant $\left(\frac{t}{t-t\alpha+s\alpha}\right)^{t(r-s)}$, but according to probability theory the classical Liapunov inequality[8] is free of these conditions. Our results improve recent results that appeared in Ref. 7.

Also, Caballero and Sadarangani[6] proved a Carlson inequality for the Lebesgue measure–based Sugeno integral, and then Xu and Ouyang[9] further generalized it to comonotone functions and arbitrary nonadditive measure–based Sugeno integrals.

THEOREM 1.2.[9] *Let $(X, \mathcal{F}, \mu)$ be a nonadditive measure space, let $A \in \mathcal{F}$, and let $f_i : X \to \mathbb{R}$, $i = 1, 2, 3$ be measurable functions such that $\mathbf{Su}(\mu, f_i) \leq 1$, $i = 1, 2, 3$. If any two functions of f_i, $i = 1, 2, 3$ are comonotone, then, for any $p, q \geq 1$, we have*

$$\mathbf{Su}(\mu, f_1) \leq \frac{1}{\sqrt{C}} \left(\mathbf{Su}^{\frac{1}{2p}}(\mu, (f_1 f_2)^p)\right) \left(\mathbf{Su}^{\frac{1}{2q}}(\mu, (f_1 f_3)^q)\right) \tag{1.2}$$

where $C = (\mathbf{Su}(\mu, f_2))(\mathbf{Su}(\mu, f_3))$.

Furthermore, the classical Sandor inequality for the Lebesgue measure–based Sugeno integral was proposed in a special form by Caballero and Sadarangani.[10]

The aim of this contribution is to generalize the results of Refs. 6, 9–13 to the frame of the universal integral on monotone measure.

The paper is organized as follows. In the next section, we briefly recall some preliminaries and summarize some previous results. In Section 3, we will focus on some inequalities related to Carlson, Sandor, Liapunov, Bushell–Okrasinski, and Fatou's lemma for universal integral.

2. UNIVERSAL INTEGRAL

In this section, we will review some well-known results from universal integral (see Ref. 5). For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper.

DEFINITION 2.1.[5] *A monotone measure m on a measurable space $(X, \mathcal{A})$ is a function $m : \mathcal{A} \to [0, \infty]$ satisfying*

(i) $m(\phi) = 0$,
(ii) $m(X) > 0$,
(iii) $m(A) \leq m(B)$ *whenever* $A \subseteq B$.

Normed monotone measures on $(X, \mathcal{A})$, i.e., monotone measures satisfying $m(X) = 1$, are also called nonadditive measures,[4,14,15] depending on the context.

For a measurable space $(X, \mathcal{A})$, i.e., a nonempty set X equipped with a σ-algebra $\mathcal{A}$, recall that a function $f : X \to [0, \infty]$ is called $\mathcal{A}$-measurable if, for each $B \in \mathcal{B}([0, \infty])$, the σ-algebra of Borel subsets of $[0, \infty]$, the preimage $f^{-1}(B)$ is an element of $\mathcal{A}$.

DEFINITION 2.2.[5] *Let $(X, \mathcal{A})$ be a measurable space.*

(i) $\mathcal{F}^{(X,\mathcal{A})}$ *denotes the set of all $\mathcal{A}$-measurable functions $f : X \to [0, \infty]$;*
(ii) *For each number $a \in (0, \infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ denotes the set of all monotone measures (in the sense of Definition 2.1) satisfying $m(X) = a$; and we take*

$$\mathcal{M}^{(X,\mathcal{A})} = \bigcup_{a \in (0,\infty]} \mathcal{M}_a^{(X,\mathcal{A})}.$$

Let $\mathcal{S}$ be the class of all measurable spaces, and take

$$\mathcal{D}_{[0,\infty]} = \bigcup_{(X,\mathcal{A}) \in \mathcal{S}} \mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}.$$

The Choquet,[1] Sugeno,[4] and Shilkret[16] integrals (see also Refs. 3, 17–19), respectively, are given, for any measurable space $(X, \mathcal{A})$, for any measurable function $f \in \mathcal{F}^{(X,\mathcal{A})}$ and for any monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$, i.e., for any $(m, f) \in \mathcal{D}_{[0,\infty]}$, by

$$\mathbf{Ch}(m, f) = \int_0^\infty m(\{f \geq t\})dt, \tag{2.1}$$

$$\mathbf{Su}(m, f) = \sup\{\min(t, m(\{f \geq t\})) \mid t \in (0, \infty]\}, \tag{2.2}$$

$$\mathbf{Sh}(m, f) = \sup\{t.m(\{f \geq t\}) \mid t \in (0, \infty]\}, \tag{2.3}$$

where the convention $0.\infty = 0$ is used. All these integrals map $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ into $[0, \infty]$ independently of $(X, \mathcal{A})$. We remark that fixing an arbitrary $m \in \mathcal{M}^{(X,\mathcal{A})}$, they are nondecreasing functions from $\mathcal{F}^{(X,\mathcal{A})}$ into $[0, \infty]$, and fixing an arbitrary $f \in \mathcal{F}^{(X,\mathcal{A})}$, they are nondecreasing functions from $\mathcal{M}^{(X,\mathcal{A})}$ into $[0, \infty]$.

We stress the following important common property for all three integrals from (2.1), (2.2), to (2.3). Namely, these integrals does not make difference between the pairs $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$ which satisfy, for all for all $t \in (0, \infty]$,

$$m_1(\{f_1 \geq t\}) = m_2(\{f_2 \geq t\}).$$

Therefore, such equivalence relation between pairs of measures and functions was introduced in Ref. 5.

DEFINITION 2.3. *Two pairs* $(m_1, f_1) \in \mathcal{M}^{(X_1,\mathcal{A}_1)} \times \mathcal{F}^{(X_1,\mathcal{A}_1)}$ *and* $(m_2, f_2) \in \mathcal{M}^{(X_2,\mathcal{A}_2)} \times \mathcal{F}^{(X_2,\mathcal{A}_2)}$ *satisfying*

$$m_1(\{f_1 \geq t\}) = m_2(\{f_2 \geq t\}) \quad \textit{for all} \quad t \in (0, \infty],$$

will be called integral equivalent, in symbols

$$(m_1, f_1) \sim (m_2, f_2).$$

To introduce the notion of the universal integral, we shall need instead of the usual plus and product more general real operations.

DEFINITION 2.4.[3,20] *A function* $\otimes: [0, \infty]^2 \to [0, \infty]$ *is called a pseudomultiplication if it satisfies the following properties:*

(i) *it is nondecreasing in each component, i.e., for all* $a_1, a_2, b_1, b_2 \in [0, \infty]$ *with* $a_1 \leq a_2$ *and* $b_1 \leq b_2$ *we have* $a_1 \otimes b_1 \leq a_2 \otimes b_2$*;*
(ii) 0 *is an annihilator of* $\otimes$*, i.e., for all* $a \in [0, \infty]$ *we have* $a \otimes 0 = 0 \otimes a = 0$*;*
(iii) $\otimes$ *has a neutral element different from* 0*, i.e., there exists an element* $e \in (0, \infty]$ *such that, for all* $a \in [0, \infty]$*, we have* $a \otimes e = e \otimes a = a$.

Restricting to the interval [0, 1], a pseudomultiplication and a pseudoaddition with additional properties of associativity and commutativity can be considered as the t-norm T and the t-conorm S (see Ref. 21), respectively.

In some constructions, for a given pseudomultiplication on $[0, \infty]$, we suppose the existence of a pseudoaddition $\oplus: [0, \infty]\ 2 \to [0, \infty]$, which is continuous, associative, non-decreasing and has 0 as neutral element (then the commutativity of $\oplus$ follows, see Ref. 21), and which is left-distributive with respect to $\otimes$, i.e., for all $a, b, c \in [0, \infty]$ we have $(a \oplus b) \otimes c = (a \oplus c) \otimes (b \oplus c)$. The pair $(\oplus, \otimes)$ is then called an integral operation pair; see Refs. 5, 17.

Each of the integrals mentioned in (2.1), (2.2), and (2.3) maps $\mathcal{D}_{[0,\infty]}$ into $[0, \infty]$, and their main properties can be covered by the following common integral given in Ref. 5:

DEFINITION 2.5. *A function* $\mathbf{I}\colon \mathcal{D}_{[0,\infty]} \to [0,\infty]$ *is called a universal integral if the following axioms hold:*

(I1) *For any measurable space* $(X, \mathcal{A})$*, the restriction of the function* $\mathbf{I}$ *to* $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ *is nondecreasing in each coordinate;*

(I2) *there exists a pseudomultiplication* $\otimes\colon [0,\infty]^2 \to [0,\infty]$ *such that for all pairs* $(m, c.\mathbf{1}_A) \in \mathcal{D}_{[0,\infty]}$

$$\mathbf{I}(m, c.\mathbf{1}_A) = c \otimes m(A);$$

(I3) *for all integral equivalent pairs* $(m_1, f_1), (m_2, f_2) \in \mathcal{D}_{[0,\infty]}$*, we have* $\mathbf{I}(m_1, f_1) = \mathbf{I}(m_2, f_2)$.

By Proposition 3.1 from Ref. 5, we have the following important characterization:

THEOREM 2.6. *Let* $\otimes\colon [0,\infty]^2 \to [0,\infty]$ *be a pseudomultiplication on* $[0,\infty]$*. Then the smallest universal integral* $\mathbf{I}$ *and the greatest universal integral* $\mathbf{I}$ *based on* $\otimes$ *are given by*

$$\mathbf{I}_{\otimes}(m, f) = \sup\{t \otimes m(\{f \geq t\}) \mid t \in (0,\infty]\},$$

$$\mathbf{I}^{\otimes}(m, f) = essup_m f \otimes \sup\{m(\{f \geq t\}) \mid t \in (0,\infty]\},$$

where $essup_m f = \sup\{t \in [0,\infty] \mid m(\{f \geq t\}) > 0\}$.

Specially, we have $\mathbf{Su} = \mathbf{I}_{Min}$ and $\mathbf{Sh} = \mathbf{I}_{Prod}$, where the pseudomultiplications Min and $Prod$ are given (as usual) by $Min(a,b) = (a \wedge b)$ and $Prod(a,b) = a.b$. Note that the nonlinearity of the Sugeno integral $\mathbf{Su}$ (see, e.g., Refs. 22, 23) implies that universal integrals are also nonlinear, in general.

There is neither a smallest nor a greatest pseudomultiplication on $[0,\infty]$. But, if we fix the neutral element $e \in (0,\infty]$, then the smallest pseudomultiplication $\otimes_e$ and the greatest pseudomultiplication $\otimes^e$ with neutral element e are given by

$$a \otimes_e b = \begin{cases} 0 & if\ (a,b) \in [0,e)^2, \\ \max(a,b) & if\ (a,b) \in [e,\infty]^2, \\ \min(a,b) & \text{otherwise,} \end{cases}$$

and

$$a \otimes^e b = \begin{cases} \min(a,b) & if\ \min(a,b) = 0 \text{ or } (a,b) \in (0,e]^2, \\ \infty & if\ (a,b) \in (e,\infty]^2, \\ \max(a,b) & \text{otherwise.} \end{cases}$$

Then by Proposition 3.2 from Ref. 5, there exists the smallest universal integral $\mathbf{I}_{\otimes_e}$

among all universal integrals satisfying the conditions

(i) for each $m \in \mathcal{M}_e^{(X,\mathcal{A})}$ and each $c \in [0, \infty]$, we have $\mathbf{I}(m, c.\mathbf{1}_X) = c$ and
(ii) for each $m \in \mathcal{M}^{(X,\mathcal{A})}$ and each $A \in \mathcal{A}$, we have $\mathbf{I}(m, e.\mathbf{1}_X) = m(A)$, given by

$$\mathbf{I}_{\otimes_e}(m, f) = \max\{m(\{f \geq e\}), essinf_m f\}$$

where $essinf_m f = \sup\{t \in [0, \infty] \mid m(\{f \geq t\}) = m(X)\}$. Restricting now to the unit interval [0, 1], we shall consider functions $f \in \mathcal{F}^{(X,\mathcal{A})}$ satisfying $Ran(f) \subseteq [0, 1]$ (in which case, we shall write shortly $f \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$). Observe that, in this case, we have the restriction of the pseudomultiplication $\otimes$ with neutral element $e = 1$ to $[0, 1]^2$ (called a semicopula or a conjunctor, i.e., a binary operation $\circledast: [0, 1]^2 \to [0, 1]$, which is nondecreasing in both components, has 1 as neutral element and satisfies $a \circledast b \leq \min(a, b)$ for all $(a, b) \in [0, 1]^2$; see Refs. 24, 25), and universal integrals are restricted to the class $\mathcal{D}_{[0,1]} = \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \mathcal{M}_1^{(X,\mathcal{A})} \times \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$. In a special case, for a fixed strict t-norm T, the corresponding universal integral $\mathbf{I}_T$ is the so-called Sugeno–Weber integral.[26] The smallest universal integral $\mathbf{I}_\circledast$ on the [0, 1] scale related to the semicopula $\circledast$ is given by

$$\mathbf{I}_\circledast(m, f) = \sup\{t \circledast m(\{f \geq t\}) \mid t \in [0, 1]\}.$$

This type of integral was called the seminormed integral in Ref. 27.

Before starting our main results, we need the following definitions:

DEFINITION 2.7. *Functions $f, g: X \to \mathbb{R}$ are said to be comonotone if for all $x, y \in X$,*

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

and f and g are said to be countermonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$, or $f(x) > f(y)$ and $g(x) < g(y)$. Similarly, if f and g are countermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ ($f(x) > f(y)$ and $g(x) > g(y)$) cannot happen. Observe that the concept of comonotonicity was first introduced in Ref. 28.

DEFINITION 2.8. *Let $A, B: [0, \infty]^2 \to [0, \infty]$ be two binary operations. Recall that A dominates B (or B is dominated by A), denoted by $A \gg B$, if*

$$A(B(a, b), B(c, d)) \geq B(A(a, c), A(b, d))$$

holds for any $a, b, c, d \in [0, \infty]$. *For a deeper investigation of complete domination of aggregation functions, the reader is referred to Ref. 29.*

DEFINITION 2.9. *Let* $\star: [0, \infty]^2 \to [0, \infty]$ *be a binary operation and consider* $\Phi : [0, \infty] \to [0, \infty]$. *Then we say that* Φ *is subdistributive over* $\star$ *if*

$$\Phi(x \star y) \leq \Phi(x) \star \Phi(y)$$

for all $x, y \in [0, \infty]$. *Analogously, we say that* Φ *is superdistributive over* $\star$ *if*

$$\Phi(x \star y) \geq \Phi(x) \star \Phi(y)$$

for all $x, y \in [0, \infty]$.

Now, our results can be stated as follows:

3. MAIN RESULTS

This section provides some inequalities related to Carlson, Bushell–Okrasinski, Liapunov, and Fatou's lemma for universal integral, thus generalizing the results of Refs. 6, 9–13.

3.1. Carlson's and Bushell–Okrasinski's Inequality

Before stating Carlson's inequality for universal integral, we need two lemmas.

LEMMA 3.1. *Let a fixed* $s \in [1, \infty)$. *And let* $f \in \mathcal{F}^{(X,\mathcal{A})}$ *be a measurable function and* $\otimes_e: [0, \infty]^2 \to [0, \infty]$ *be the smallest pseudomultiplication on* $[0, \infty]$ *with neutral element* $e \in (0, \infty]$ *and* $m \in \mathcal{M}^{(X,\mathcal{A})}$ *be a monotone measure such that* $\mathbf{I}_{\otimes_e}(m, f) < \infty$ *and* $x \geq x^s$ *for all* $x \in [0, \infty)$. *Then the inequality*

$$\mathbf{I}_{\otimes_e}\left(m, f^s\right) \geq \mathbf{I}^s_{\otimes_e}(m, f) \tag{3.1}$$

holds whenever $(.)^s$ *is subdistributive over* $\otimes_e$.

Proof. Let $e \in (0, \infty]$ be the neutral element of $\otimes_e$. If $\mathbf{I}_{\otimes_e}(m, f) = q < \infty$, then for any $\varepsilon > 0$, there exist q_ε such that $M = \mu(A \cap \{f \geq q_\varepsilon\})$, where $(q_\varepsilon \otimes_e M) \geq q - \varepsilon$. Since $(.)^s$ is subdistributive over $\otimes_e$ and $x \geq x^s$ for all $x \in [0, \infty)$ and $s \geq 1$, then

$$\mathbf{I}_{\otimes_e}(m, f^s) = \sup\{t \otimes m(\{f^s \geq t\}) \mid t \in (0, \infty]\} \geq q^s_\varepsilon \otimes_e \mu\big(A \cap \{f^s \geq q^s_\varepsilon\}\big)$$

$$= q^s_\varepsilon \otimes_e M \geq q^s_\varepsilon \otimes_e M^s \geq (q_\varepsilon \otimes_e M)^s \geq (q - \varepsilon)^s.$$

whence $\mathbf{I}_{\otimes_e}(m, f^s) \geq \mathbf{I}^s_{\otimes_e}(m, f)$ follows from the arbitrariness of ε. ■

LEMMA 3.2.[30] *Let $f, g \in \mathcal{F}^{(X,\mathcal{A})}$ be two comonotone measurable functions and $\otimes_e\colon [0,\infty]^2 \to [0,\infty]$ be the smallest pseudomultiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f)$ and $\mathbf{I}_{\otimes_e}(m, g)$ are finite. Let $\star\colon [0,\infty)^2 \to [0,\infty)$ be continuous and nondecreasing in both arguments. If*

$$[(a \star b) \otimes_e c] \geq [(a \otimes_e c) \star b] \vee [a \star (b \otimes_e c)]\,, \tag{3.2}$$

then the inequality

$$\mathbf{I}_{\otimes_e}(m, (f \star g)) \geq \left(\mathbf{I}_{\otimes_e}(m, f)\right) \star \left(\mathbf{I}_{\otimes_e}(m, g)\right) \tag{3.3}$$

holds.

Remark 3.3. If $(x \star e) \vee (e \star x) \leq x$ for any $x \in [0,\infty)$ and $\otimes_e$ dominates $\star$, then (3.2) holds readily. Indeed,

$$[(a \star b) \otimes_e c] \geq (a \star b) \otimes_e (c \star e) \geq [(a \otimes_e c) \star (b \otimes_e e)] = [(a \otimes_e c) \star b]\,,$$

and $[(a \star b) \otimes_e c] \geq [a \star (b \otimes_e c)]$ follows similarly, i.e.,

$$[(a \star b) \otimes_e c] \geq (a \star b) \otimes_e (c \star e) \geq [(a \otimes_e e) \star (b \otimes_e c)] = [a \star (b \otimes_e c)]\,.$$

Lemmas 3.1 and 3.2 help us to reach the following results:

THEOREM 3.4 (Carlson-type inequality for universal integral (I)). *Let a fixed $s \in [1,\infty)$. And let $f_i \in \mathcal{F}^{(X,\mathcal{A})}$, $i = 1, 2, 3$ be measurable functions and $\otimes_e\colon [0,\infty]^2 \to [0,\infty]$ be the smallest pseudomultiplication on $[0,\infty]$ with neutral element $e \in (0,\infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f_i) < \infty$, $i = 1, 2, 3$ and $x \geq x^s$ for all $x \in [0,\infty)$. Let $\star\colon [0,\infty]^2 \to [0,\infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum such that $\otimes_e$ dominates $\star$. If any two functions of f_i, $i = 1, 2, 3$ are comonotone, then, for any $p, q \geq 1$, we have*

$$\begin{aligned}&([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_2)]) \star ([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_3)])\\ &\leq \left[\mathbf{I}^{\frac{1}{p}}_{\otimes_e}(m, (f_1 \star f_2)^p)\right] \star \left[\mathbf{I}^{\frac{1}{q}}_{\otimes_e}(m, (f_1 \star f_3)^q)\right],\end{aligned} \tag{3.4}$$

where $(.)^p$, $(.)^q$ are subdistributive over $\otimes_e$.

Proof. Let $\mathbf{I}_{\otimes_e}(m, f_i) < \infty$, $i = 1, 2, 3$. Since $\star$ is bounded from above by minimum ($\star \leq \min$), it is easy to show that $\mathbf{I}_{\otimes_e}(m, f_1 \star f_2) < \infty$, $\mathbf{I}_{\otimes_e}(m, f_1 \star f_3) < \infty$.

By Lemma 3.1, we have

$$\mathbf{I}^{p}_{\otimes_e}(m,\, f_1 \star f_2) \le \mathbf{I}_{\otimes_e}(m, (f_1 \star f_2)^p),$$

$$\mathbf{I}^{q}_{\otimes_e}(m,\, f_1 \star f_3) \le \mathbf{I}_{\otimes_e}(m, (f_1 \star f_3)^q).$$

Then

$$[\mathbf{I}_{\otimes_e}(m,\, f_1 \star f_2)] \star [\mathbf{I}_{\otimes_e}(m,\, f_1 \star f_3)]$$
$$\le \left[\mathbf{I}^{\frac{1}{p}}_{\otimes_e}(m, (f_1 \star f_2)^p)\right] \star \left[\mathbf{I}^{\frac{1}{q}}_{\otimes_e}(m, (f_1 \star f_3)^q)\right]. \tag{3.5}$$

Since $\otimes_e$ dominates $\star$ and $\star \le \min$, then Lemma 3.2 implies that

$$\mathbf{I}_{\otimes_e}(m,\, f_1 \star f_2) \ge [\mathbf{I}_{\otimes_e}(m,\, f_1)] \star [\mathbf{I}_{\otimes_e}(m,\, f_2)],$$
$$\mathbf{I}_{\otimes_e}(m,\, f_1 \star f_3) \ge [\mathbf{I}_{\otimes_e}(m,\, f_1)] \star [\mathbf{I}_{\otimes_e}(m,\, f_3)]. \tag{3.6}$$

Therefore, (3.5) and (3.6) imply that

$$([\mathbf{I}_{\otimes_e}(m,\, f_1)] \star [\mathbf{I}_{\otimes_e}(m,\, f_2)]) \star ([\mathbf{I}_{\otimes_e}(m,\, f_1)] \star [\mathbf{I}_{\otimes_e}(m,\, f_3)])$$
$$\le \left[\mathbf{I}^{\frac{1}{p}}_{\otimes_e}(m, (f_1 \star f_2)^p)\right] \star \left[\mathbf{I}^{\frac{1}{q}}_{\otimes_e}(m, (f_1 \star f_3)^q)\right],$$

and the proof is completed. ■

Since $\otimes$ is minimum and $\star$ is the usual product $(\cdot)$ such that $\cdot|_{[0,1]^2} \le \min$, then the following result holds (see Refs. 6, 9 for similar results).

COROLLARY 3.5. *Let $f_i \in \mathcal{F}^{(X,\mathcal{A})}$, $i = 1, 2, 3$ be measurable functions and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{Su}(m,\, f_i) \le 1$, $i = 1, 2, 3$. If any two functions of f_i, $i = 1, 2, 3$ are comonotone, then, for any $p, q \ge 1$, we have*

$$\mathbf{Su}(m,\, f_1) \le \frac{1}{\sqrt{C}} \left(\mathbf{Su}^{\frac{1}{2p}}(m, (f_1 f_2)^p)\right) \left(\mathbf{Su}^{\frac{1}{2q}}(m, (f_1 f_3)^q)\right) \tag{3.7}$$

where $C = (\mathbf{Su}(m,\, f_2))(\mathbf{Su}(m,\, f_3))$.

Remark 3.6. We can use an example in Ref. 9 to show that the condition $\mathbf{Su}(m,\, f_i) \le 1$, $i = 1, 2, 3$ in Corollary 3.5 (and thus in Theorem 3.4) cannot be abandoned, and so we omit it here.

Remark 3.7. If $\otimes$ is the standard product in Theorem 3.4, then we have the Carlson-type inequality for the Shilkret integral.

Notice that when working on [0, 1] in Theorem 3.4, we mostly deal with $e = 1$, then $\otimes = \circledast$ is semicopula (t-seminorm) and the following result holds:

COROLLARY 3.8. *Let $f_i \in \mathcal{F}^{(X,\mathcal{A})}_{[0,1]}, i = 1, 2, 3$ be measurable functions and $m \in \mathcal{M}^{(X,\mathcal{A})}_1$ be a monotone measure. Let $\star$: $[0, 1]^2 \to [0, 1]$ be continuous and nondecreasing in both arguments and bounded from above by minimum such that the semicopula $\circledast$ dominates $\star$. If any two functions of $f_i, i = 1, 2, 3$ are comonotone, then, for any $p, q \geq 1$, we have*

$$([\mathbf{I}_{\circledast}(m, f_1)] \star [\mathbf{I}_{\circledast}(m, f_2)]) \star ([\mathbf{I}_{\circledast}(m, f_1)] \star [\mathbf{I}_{\circledast}(m, f_3)])$$

$$\leq \Big[\mathbf{I}^{\frac{1}{p}}_{\circledast}(m, (f_1 \star f_2)^p)\Big] \star \Big[\mathbf{I}^{\frac{1}{q}}_{\circledast}(m, (f_1 \star f_3)^q)\Big], \tag{3.8}$$

where $(.)^p, (.)^q$ are subdistributive over $\circledast$.

We get the following theorems with an analogous proof as the proof of Theorem 3.4:

THEOREM 3.9 (Carlson-type inequality for universal integral (II)). *Let a fixed $s \in [1, \infty)$. And let $f_i \in \mathcal{F}^{(X,\mathcal{A})}, i = 1, 2, 3$ be measurable functions and $\otimes_e$: $[0, \infty]^2 \to [0, \infty]$ be the smallest pseudomultiplication on $[0, \infty]$ with neutral element $e \in (0, \infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f_i) < \infty, i = 1, 2, 3$ and $x \geq x^s$ for all $x \in [0, \infty)$. Let $\star$: $[0, \infty]^2 \to [0, \infty]$ be continuous and nondecreasing in both arguments and bounded from above by minimum such that $\otimes_e$ dominates $\star$. If any two functions of $f_i, i = 1, 2, 3$ are comonotone, then, for any $p, q \geq 1$, we have*

$$([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_2)])^p \star ([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_3)])^q$$

$$\leq [\mathbf{I}_{\otimes_e}(m, (f_1 \star f_2)^p)] \star [\mathbf{I}_{\otimes_e}(m, (f_1 \star f_3)^q)], \tag{3.9}$$

where $(.)^p, (.)^q$ are subdistributive over $\otimes_e$.

Remark 3.10. If $\otimes$ is minimum and $\star$ is the usual product $(\cdot)$ such that $\cdot|_{[0,1]^2} \leq \min$ in Theorem 3.9 (similar to Corollary 3.5), then we have the Carlson-type inequality for the Sugeno integrals, which were obtained by Wang and Bai.[13]

THEOREM 3.11 (Carlson-type inequality for universal integral (III)). *Let $\varphi_i : [0, \infty) \to [0, \infty)$, $i = 1, 2$ be continuous and strictly increasing functions. And let $f_i \in \mathcal{F}^{(X,\mathcal{A})}, i = 1, 2, 3$ be measurable functions and $\otimes_e$: $[0, \infty]^2 \to [0, \infty]$ be the smallest pseudomultiplication on $[0, \infty]$ with neutral element $e \in (0, \infty]$ and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that $\mathbf{I}_{\otimes_e}(m, f_i) < \infty, i = 1, 2, 3$ and $\varphi_i(x) \leq x, i = 1, 2$ for all $x \in [0, \infty)$. Let $\star$: $[0, \infty]^2 \to [0, \infty]$ be continuous and nondecreasing* *in both arguments and bounded from above by minimum such that* $\otimes_e$ *dominates* $\star$. *If any two functions of* f_i, $i = 1, 2, 3$ *are comonotone, then we have*

$$([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_2)]) \star ([\mathbf{I}_{\otimes_e}(m, f_1)] \star [\mathbf{I}_{\otimes_e}(m, f_3)]) \\ \leq \left[\varphi_1^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi_1(f_1 \star f_2)))\right] \star \left[\varphi_2^{-1}(\mathbf{I}_{\otimes_e}(m, \varphi_2(f_1 \star f_3)))\right], \tag{3.10}$$

where φ_i, $i = 1, 2$ *are subdistributive over* $\otimes_e$.

Also, it is easy to prove the following theorems.

THEOREM 3.12 (the Bushell–Okrasinski-type inequality for universal integral (I)). *Let* $g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ *be nonincreasing measurable function and* $\circledast$ *the semicopula on* $[0, 1]$ *and* $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ *be a monotone measure. Let* $\star \colon [0, 1]^2 \to [0, 1]$ *be continuous and nondecreasing in both arguments and bounded from above by minimum. For any* $s \geq 2$, *we have*

$$\mathbf{I}_{\circledast}(m, (1-t)^{s-1} \star g^s(t)) \\ \geq \left[\frac{1}{s} \circledast m\left([0, 1] \cap \left\{t \mid t \leq 1 - \left(\frac{1}{s}\right)^{\frac{1}{s-1}}\right\}\right)\right] \star \mathbf{I}_{\circledast}^s(m, g(t)). \tag{3.11}$$

THEOREM 3.13 (the Bushell–Okrasinski-type inequality for the universal integral (II)). *Let* $g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ *be a nondecreasing measurable function and* $\circledast$ *the semicopula on* $[0, 1]$ *and* $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ *be a monotone measure. Let* $\star \colon [0, 1]^2 \to [0, 1]$ *be continuous and nondecreasing in both arguments and bounded from above by minimum. For any* $s \geq 2$, *we have*

$$\mathbf{I}_{\circledast}(m, t^{s-1} \star g^s(t)) \geq \left[\frac{1}{s} \circledast m\left([0, 1] \cap \left\{t \mid t \geq \left(\frac{1}{s}\right)^{\frac{1}{s-1}}\right\}\right)\right] \star \mathbf{I}_{\circledast}^s(m, g(t)). \tag{3.12}$$

Remark 3.14. If $\circledast = \wedge$ in Theorems 3.12, and 3.13, and $\star$ is the usual product $(\cdot)$ such that $\cdot|_{[0,1]^2} \leq \min$, then we obtain the Bushell–Okrasinski-type inequality for the Sugeno integral (specially, when m is the Lebesgue measure on $\mathbb{R}$ and $s \geq 2$, then the fact of

$$m\left([0, 1] \cap \left\{t \mid t \leq 1 - \left(\frac{1}{s}\right)^{\frac{1}{s-1}}\right\}\right) = m\left([0, 1] \cap \left\{t \mid t \geq \left(\frac{1}{s}\right)^{\frac{1}{s-1}}\right\}\right) \\ = 1 - \left(\frac{1}{s}\right)^{\frac{1}{s-1}} \geq \frac{1}{s},$$

implies the Bushell–Okrasinski-type inequality for a Lebesgue measure-based Sugeno integral, which was obtained by Román-Flores *et al.* 12).

3.2. Liapunov's Inequality

LEMMA 3.15.[30] *Let $f, g \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be two comonotone measurable functions. Let $\star : [0,1]\ 2 \to [0,1]$ be continuous and nondecreasing in both arguments. If the t-semiconorm S satisfies*

$$S((a \star b), c) \leq [(S(a,c)) \star b] \wedge [a \star (S(b,c))], \tag{3.13}$$

then the inequality

$$\mathbf{I}_S(m, (f \star g)) \leq (\mathbf{I}_S(m, f)) \star (\mathbf{I}_S(m, g)) \tag{3.14}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$.

Lemmas 3.1 and 3.15 help us to reach the following result:

THEOREM 3.16 (Liapunov's inequality for universal integral). *Let $s, r, t \in (0, \infty)$, $\beta \in (0,1)$ and $f \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be a measurable functions. If the t-semiconorm S dominates the usual product $(\cdot)$, then the inequality*

$$\mathbf{I}_S^{r-t}(m, f^s) \leq (\mathbf{I}_S(m, f^t))^{r-s} \cdot (\mathbf{I}_S(m, f^r))^{s-t} \tag{3.15}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$ where $\frac{1}{s} = \frac{\beta}{t} + \frac{1-\beta}{r}$.

Proof. Let $r_1 = \frac{r}{s(1-\beta)}$ and $t_1 = \frac{t}{s\beta}$. Then $\frac{1}{r_1} + \frac{1}{t_1} = 1$, $t_1 > 1$. Thus, by Lemmas 3.1 and 3.15

$$\begin{aligned}
\mathbf{I}_S^{r-t}(m, f^s) &= \mathbf{I}_S^{r-t}(m, f^{s\beta + s(1-\beta)}) \leq [(\mathbf{I}_S(m, f^{s\beta})) \cdot (\mathbf{I}_S(m, f^{s(1-\beta)}))]^{r-t} \\
&\leq \left[(\mathbf{I}_S(m, f^{s\beta t_1}))^{\frac{1}{t_1}} \cdot (\mathbf{I}_S(m, f^{s(1-\beta) r_1}))^{\frac{1}{r_1}}\right]^{r-t} \\
&\leq \left[(\mathbf{I}_S(m, f^t))^{\frac{s\beta}{t}} \cdot (\mathbf{I}_S(m, f^r))^{\frac{s(1-\beta)}{r}}\right]^{r-t} \\
&= \left[(\mathbf{I}_S(m, f^t))^{\frac{s\beta(r-s)}{t(r-s)}} \cdot (\mathbf{I}_S(m, f^r))^{\frac{s(1-\beta)(s-t)}{r(s-t)}}\right]^{r-t} \\
&\leq (\mathbf{I}_S(m, f^t))^{\frac{(r-s)s\beta(r-t)}{t(r-s)}} \cdot (\mathbf{I}_S(m, f^r))^{\frac{(s-t)s(1-\beta)(r-t)}{r(s-t)}}.
\end{aligned} \tag{3.16}$$

Since $\frac{1}{s}=\frac{\beta}{t}+\frac{1-\beta}{r}$, then we have

$$\frac{s\beta(r-t)}{t(r-s)}=\frac{s(1-\beta)(r-t)}{r(s-t)}=1. \tag{3.17}$$

Therefore, (3.16) and (3.17) imply that

$$\mathbf{I}_S^{r-t}\left(m, f^s\right) \leq \left(\mathbf{I}_S\left(m, f^t\right)\right)^{(r-s)} \cdot \left(\mathbf{I}_S\left(m, f^r\right)\right)^{(s-t)},$$

and the proof is completed. ∎

Notice that if the semiconorm S is maximum (i.e., for the Sugeno integral), then the following result holds:

COROLLARY 3.17. *Let $s, r, t \in (0,\infty)$, $\beta \in (0,1)$ and $f \in \mathcal{F}_{[0,1]}^{(X,\mathcal{A})}$ be a measurable functions. Then the inequality*

$$\mathbf{Su}^{r-t}\left(m, f^s\right) \leq \left(\mathbf{Su}\left(m, f^t\right)\right)^{r-s} \cdot \left(\mathbf{Su}\left(m, f^r\right)\right)^{s-t} \tag{3.18}$$

holds for any $m \in \mathcal{M}_1^{(X,\mathcal{A})}$, where $\frac{1}{s}=\frac{\beta}{t}+\frac{1-\beta}{r}$ and the usual product $(\cdot)$ dominates min.

Remark 3.18. Let $r=2, t=\frac{1}{3}, \beta=\frac{1}{2}, s=\frac{4}{7}$ and m be the Lebesgue measure on $\mathbb{R}$. Then we can use a same example presented in Ref.[7] (Example 1) to show that the condition of "the usual product $(\cdot)$ dominates min" in Corollary 3.17 (and thus in Theorem 3.16) cannot be abandoned, and so we omit it here.

Remark 3.19. Let $r=2, s=1, \beta=t=\frac{1}{2}$ and m be the Lebesgue measure on $\mathbb{R}$. Then we can use the same example as in Ref.[7] (Example 1) to show that the condition of $\frac{1}{s}=\frac{\beta}{t}+\frac{1-\beta}{r}$ in Corollary 3.17 (and thus in Theorem 3.16) cannot be abandoned, and so we omit it here.

Also, in a similar way, we can prove the following theorem, and then the Liapunov-type inequalities for the Sugeno integral appear as its corollaries (thus improving the results of Ref.[7]).

THEOREM 3.20. *Let $f \in \mathcal{F}^{(X,\mathcal{A})}$ be a measurable function and $m \in \mathcal{M}^{(X,\mathcal{A})}$ be a monotone measure such that* $\mathbf{Su}(m, f) \leq 1$. *For any $s, r, t \in (0,\infty)$, $\beta \in (0,1)$, we have*

$$\mathbf{Su}^{r-t}\left(m, f^s\right) \leq \left(\mathbf{Su}\left(m, f^t\right)\right)^{r-s} \cdot \left(\mathbf{Su}\left(m, f^r\right)\right)^{s-t} \tag{3.19}$$

where $\frac{1}{s} = \frac{\beta}{t} + \frac{1-\beta}{r}$ *and the usual p roduct* $(\cdot)$ *dominates* min.

Remark 3.21. In Ref. 7, it requires that f be a concave function. But, in Theorem 3.20, this condition can be abandoned.

3.3. Sandor's Inequality and Fatou's Lemma

THEOREM 3.22 (Sandor's inequality for universal integral). *Let* $f \in \mathcal{F}^{([a,b],\mathcal{A})}$ *be a convex function. If* $\otimes_e$ *is a smallest pseudomultiplication and* $\otimes^e$ *is a greatest pseudomultiplication on* $[0, \infty]$ *with neutral element* $e \in (0, \infty]$*, then for any monotone measure* $m \in \mathcal{M}^{(X,\mathcal{A})}$*, we have*

(a) *if* $f(a) < f(b)$, *then*

$$\mathbf{I}_{\otimes_e}(m, f) \leq \left[\begin{array}{c} \left(\frac{\left(2Af(b)+A^2\right)-\sqrt{4A^3 f(b)+A^4}}{2} \right) \otimes^e \\ m\left([a,b] \cap \left\{ x \mid x \geq b + \frac{(b-a)}{f(b)-f(a)} \left(\sqrt{\frac{\left(2Af(b)+A^2\right)-\sqrt{4A^3 f(b)+A^4}}{2}} - f(b) \right) \right\} \right) \end{array} \right]$$

where $A = \frac{b-a}{f(b)-f(a)}$.

(b) *if* $f(a) = f(b)$, *then*

$$\mathbf{I}_{\otimes_e}(m, f) \leq [f(a) \otimes_e m([a,b])] .$$

(c) *if* $f(a) > f(b)$, *then*

$$\left[\begin{array}{c} \left(\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2} \right) \otimes^e \\ m\left([a,b] \cap \left\{ x \mid x \leq b + \frac{b-a}{f(b)-f(a)} \left(\sqrt{\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}} - f(b) \right) \right\} \right) \end{array} \right]$$

where $A = \frac{b-a}{f(b)-f(a)}$.

Proof. Let $f : [a,b] \to [0,\infty)$ be a convex function. Suppose that $g\colon [a,b] \to [0,\infty)$ is a linear function satisfying $g(a) = f(a)$ and $g(b) = f(b)$. Therefore,

$$g(x) = f(b) + \frac{f(b) - f(a)}{b-a}(x-b).$$

(a) If $f(a) < f(b)$, then

$$\begin{aligned}
&\mathbf{I}_{\otimes_e}(m, f^2) \leq \mathbf{I}_{\otimes_e}(m, g^2)\\
&\leq \bigvee_{\alpha>0}\left(\alpha \otimes_e m\left([a,b] \cap \left\{f(b) + \frac{f(b)-f(a)}{b-a}(x-b) \geq \alpha^{\frac{1}{2}}\right\}\right)\right)\\
&= \bigvee_{\alpha>0}\left(\alpha \otimes_e m\left([a,b] \cap \left\{x \mid x \geq b + \frac{b-a}{f(b)-f(a)}\left(\sqrt{\alpha} - f(b)\right)\right\}\right)\right)\\
&\leq \left[\begin{array}{l} \left(\frac{\left(2Af(b)+A^2\right)-\sqrt{4A^3f(b)+A^4}}{2}\right) \otimes^e \\ m\left([a,b] \cap \left\{x \mid x \geq b + \frac{b-a}{f(b)-f(a)}\right.\right. \\ \left.\left.\quad \times\left(\sqrt{\frac{(2Af(b)+A^2)-\sqrt{4A^3f(b)+A^4}}{2}} - f(b)\right)\right\}\right) \end{array}\right]
\end{aligned}$$

where $A = \frac{b-a}{f(b)-f(a)}$.

(b) If $f(a) = f(b)$, then $g(x) = f(a)$. Thus

$$\mathbf{I}_{\otimes_e}(m, f^2) \leq \mathbf{I}_{\otimes_e}(m, g^2) = \mathbf{I}_{\otimes_e}\left(m, f^2(a)\right) = \left[f^2(a) \otimes_e m([a,b])\right].$$

(c) If $f(a) > f(b)$, then

$$\begin{aligned}
&\mathbf{I}_{\otimes_e}(m, f^2) \leq \mathbf{I}_{\otimes_e}(m, g^2)\\
&= \bigvee_{\alpha>0}\left(\alpha \otimes_e m\left([a,b] \cap \left\{f(b) + \frac{f(b)-f(a)}{b-a}(x-b) \geq \alpha^{\frac{1}{2}}\right\}\right)\right)\\
&= \bigvee_{\alpha>0}\left(\alpha \otimes_e m\left([a,b] \cap \left\{x \mid x \leq b + \frac{b-a}{f(b)-f(a)}\left(\sqrt{\alpha} - f(b)\right)\right\}\right)\right)\\
&\leq \left[\begin{array}{l} \left(\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}\right) \otimes^e \\ m\left([a,b] \cap \left\{x \mid x \leq b + \frac{b-a}{f(b)-f(a)}\right.\right. \\ \left.\left.\quad \times\left(\sqrt{\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}} - f(b)\right)\right\}\right) \end{array}\right]
\end{aligned}$$

where $A = \frac{b-a}{f(b)-f(a)}$. And the proof is completed. ■

Remark 3.23.

(I) If $\otimes_e = \wedge$ in Theorem 3.22, then we obtain the Sandor-type inequality for the Sugeno integral (specially, when m is the Lebesgue measure on $\mathbb{R}$, then the fact of $\mathbf{Su}(m, f) \leq m([a, b])$,

$$\left[\begin{array}{l}\left(\frac{(2Af(b)+A^2)-\sqrt{4A^3 f(b)+A^4}}{2}\right) \\ = m\left([a, b] \cap \left\{x \mid x \geq b + \frac{(b-a)}{f(b)-f(a)}\left(\sqrt{\frac{(2Af(b)+A^2)-\sqrt{4A^3 f(b)+A^4}}{2}} - f(b)\right)\right\}\right)\end{array}\right],$$

$$\left[\begin{array}{l}\left(\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}\right) \\ = m\left([a, b] \cap \left\{x \mid x \leq b + \frac{b-a}{f(b)-f(a)}\left(\sqrt{\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}} - f(b)\right)\right\}\right)\end{array}\right],$$

and $\otimes^e = \vee$ imply the Sandor-type inequality for a Lebesgue measure-based Sugeno integral, which was obtained by Caballero and Sadarangani[10]).

(II) If $\otimes_e$ is the standard product in Theorem 3.22, then we have the Sandor type inequality for Shilkret integral.

(III) When working on [0, 1] in Theorem 3.22, then we mostly deal with $e = 1$, then $\otimes_e = \circledast$ is semicopula (t-seminorm). Then we have a Sandor type inequality for seminormed non-additive integrals.

We get the following theorem with an analogous proof as the proof of Theorem 3.22.

THEOREM 3.24 (Reverse Sandor-type inequality for universal integral). *Let $f \in \mathcal{F}^{([a,b],\mathcal{A})}$ be a concave function. If $\otimes_e\colon [0, \infty]^n \to [0, \infty]$ is a smallest pseudo-multiplication on $[0, \infty]$ with neutral element $e \in (0, \infty]$, then for any monotone measure $m \in \mathcal{M}^{(X,\mathcal{A})}$, we have*

(a) *if $f(a) < f(b)$, then*

$$\mathbf{I}_{\otimes_e}(m, f) \geq \left[\begin{array}{c}\left(\frac{(2Af(b)+A^2)-\sqrt{4A^3 f(b)+A^4}}{2}\right) \otimes_e \\ m\left([a, b] \cap \left\{x \mid x \geq b + \frac{(b-a)}{f(b)-f(a)}\left(\sqrt{\frac{(2Af(b)+A^2)-\sqrt{4A^3 f(b)+A^4}}{2}} - f(b)\right)\right\}\right)\end{array}\right],$$

where $A = \frac{b-a}{f(b)-f(a)}$.

(b) *if $f(a) = f(b)$, then*

$$\mathbf{I}_{\otimes_e}(m, f) \geq [f(a) \otimes_e m([a, b])].$$

(c) *if* $f(a) > f(b)$, *then*

$$\mathbf{I}_{\otimes_e}(m, f) \geq \left[\begin{array}{c} \left(\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}\right) \otimes_e \\ m\left([a,b] \cap \left\{ x \mid x \leq b + \frac{(b-a)\left(\sqrt{\frac{2[(b-a)-Af(b)]+A^2-\sqrt{4A^2(b-a-Af(b))+A^4}}{2}}-f(b)\right)}{f(b)-f(a)} \right\}\right) \end{array} \right],$$

where $A = \frac{b-a}{f(b)-f(a)}$.

Also, it is easy to prove the following theorem:

THEOREM 3.25 (Fatou's lemma for universal integral). *Let* $\{f_n\}$ *be a nondecreasing (nonincreasing) sequence of nonnegative and measurable functions. If* $\otimes_e\colon [0,\infty]\ n \to [0,\infty]$ *is a smallest pseudomultiplication on* $[0,\infty]$ *with neutral element* $e \in (0,\infty]$*, then for any finite monotone measure* $m \in \mathcal{M}^{(X,\mathcal{A})}$*, we have*

$$\mathbf{I}_{\otimes_e}\left(m, \lim_{n\to\infty} \inf f_n\right) \leq \lim_{n\to\infty} \inf \left(\mathbf{I}_{\otimes_e}(m, f_n)\right).$$

Remark 3.26.

(I) If $\otimes_e = \wedge$ in Theorem 3.25, then we obtain the Fatou's lemma for the Sugeno integral, which was obtained by Agahi et al.[11]).

(II) When working on [0, 1] in Theorem 3.25, then we mostly deal with $e = 1$, then $\otimes_e = \circledast$ is semicopula (t-seminorm). Then we have the Fatou's lemma for seminormed nonadditive integrals.

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