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On distance distribution functions-valued submeasures related to aggregation functions

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Abstract

Probabilistic submeasures generalizing the classical (numerical) submeasures are introduced and discussed in connection with some classes of aggregation functions. A special attention is paid to triangular norm-based probabilistic submeasures and more general semi-copula-based probabilistic submeasures. Some algebraic properties of classes of such submeasures are also studied.

Keywords: Submeasure; Distribution function; Probabilistic metric space; Triangular norm; Aggregation function; Semi-copula; Copula; Lattice

1. Introduction and motivations

In recent years non-additive set functions have attracted much attention in pure mathematics as well as in various applications: in the decision theory, mathematical economy, social choice problems, artificial intelligence, just to name a few of them. This class includes the well-known set functions such as submeasures, Dobrakov submeasures [3] and semi-measures [4] (their further extensions are described in [14,15]), fuzzy measures, null additive set functions, etc. As a larger overview of non-additive set functions we recommend monographs [25,27], or several chapters in the handbook [24].

1.1. Numerical submeasures

In general, the study of submeasures was initiated in the second half of the last century. In fact, many classical objects of measure theory, as e.g., variations and semi-variations of vector measures, are submeasures. Let us recall

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the definition of this classical object of measure theory: if Σ is a ring of subsets of a fixed (non-empty) set Ω and $\overline{\mathbb{R}}_+ := [0, +\infty]$ is the extended non-negative real half-line, then a mapping $\eta : \Sigma \to \overline{\mathbb{R}}_+$ satisfying the conditions

(i) $\eta(\emptyset) = 0$;
(ii) $\eta(E) \leq \eta(F)$ for $E, F \in \Sigma$ such that $E \subset F$;
(iii) $\eta(E \cup F) \leq \eta(E) + \eta(F)$ whenever $E, F \in \Sigma$;

is said to be a *numerical submeasure* on Σ. Since it takes its values in $\overline{\mathbb{R}}_+$, we add the adjective "numerical" to avoid possible misunderstanding when using other type of submeasures which follows.

1.2. Submeasures in probabilistic metric spaces

In this paper we restrict our attention to submeasures in a specific situation when the exact numerical values of submeasures may not be provided but at least some probabilistic assignment could be done (observe a similar motivation in the case of probabilistic metric spaces introduction, cf. [19,23]). Namely, in our previous papers [16,13] we have investigated a submeasure notion related to probabilistic metric spaces (PM-spaces, for short), see the book [23]. Our considerations of a submeasure notion in paper [16] were closely related to the Menger PM-space $(\Omega, \mathscr{F}, \tau_T)$, see below for further explanation, where τ_T is the triangle function in the form

$$\tau_T(G, H)(x) := \sup_{u+v=x} T(G(u), H(v)) \tag{1}$$

with T being a left-continuous t-norm, and $G, H \in \Delta^+$ (the set of all distance distribution functions=distribution functions of non-negative random variables). The associated submeasure notion was then defined as follows, see [16, Definition 3].

Definition 1.1. Let $T : [0, 1]^2 \to [0, 1]$ be a t-norm, and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ (where $\gamma(E)$ is denoted by γ_E) such that

(a) if $E = \emptyset$, then $\gamma_E(x) = \varepsilon_0(x)$;
(b) if $E \subset F$, then $\gamma_E(x) \geq \gamma_F(x)$;
(c) $\gamma_{E \cup F}(x + y) \geq T(\gamma_E(x), \gamma_F(y))$, $E, F \in \Sigma$

is said to be a τ_T*-submeasure*.

From Definition 1.1 it is obvious that γ is a certain (non-additive) set function taking values in the set of distribution functions of non-negative random variables. In particular, in condition (a) ε_0 is a distance distribution function defined, for all $x \in [-\infty, +\infty]$, by

$$\varepsilon_0(x) := \begin{cases} 0 & \text{for } x \leq 0, \\ 1 & \text{otherwise.} \end{cases}$$

The attribute "submeasure" reflects the property (c) which is a "version" of the classical subadditivity rewritten in context of PM-spaces. Indeed, a PM-space is a set $\Omega \neq \emptyset$ together with a family $\mathscr{F}$ of probability functions $F_{p,q}(x)$ (interpreted as the probability that distance between elements p, q of a non-empty set is less than x) with $F_{p,q}(0) = 0$ satisfying

$$p = q \Rightarrow F_{p,q}(x) = 1, \quad x > 0,$$
$$p \neq q \Rightarrow F_{p,q}(x) < 1 \quad \text{for some } x > 0,$$
$$F_{p,q}(x) = F_{q,p}(x),$$

and the "probabilistic analogue" of the triangle inequality expressed by

$$F_{p,r}(x + y) \geq T(F_{p,q}(x), F_{q,r}(y)), \tag{2}$$

which holds for all $p, q, r \in \Omega$ and all real x, y with a suitable function $T : [0, 1]^2 \to [0, 1]$. In particular, the triple $(\Omega, \mathscr{F}, \tau_T)$ with τ_T given by (1) is called a *Menger PM-space* (under a t-norm T). Thus, one can see the natural connection among the triangular inequality (2), the condition (c) of Definition 1.1 and the "classical" subadditivity property (iii) of numerical submeasure. The connection with the triangle function (1) also explains usage of the name τ_T-submeasure for such a case.

Furthermore, since γ refers to a notion of submeasure in a probabilistic context as explained above for the particular case of Menger PM-spaces, we will also use the adjective "probabilistic" submeasure when speaking of a submeasure γ in general (without any specification of t-norm and/or other aggregation functions).

1.3. Origin and interpretation of probabilistic submeasures

Immediately, a legitimate question may arise: Why to use such a concept, what it is good for to consider probabilistic submeasures? Now, we would like to answer this question at least shortly.

Firstly, a natural origin of notion of probabilistic submeasure comes from the fact that it works in such situations in which we have only a *probabilistic information* about measure of a set (recall a similar situation in the framework of information measures as discussed in [17]). For example, if rounding of reals is considered, then the uniform distributions over intervals describe our information about the measure of a set. Another such probabilistic information occurs in biometric decision-making where the information is often obtained from biometric sensors as well as from the analysis of historical data. Thus, the probabilistic submeasures represent the concept of submeasure probabilistically rather than deterministically.

Secondly, τ_T-submeasures can be seen as fuzzy number-valued submeasures. In this case the value γ_E can be seen as a non-negative LT-fuzzy number, see [6], where $\tau_T(\gamma_E, \gamma_F)$ corresponds to the T-sum of fuzzy numbers γ_E and γ_F. This interpretation in fact resembles the original idea of Menger of PM-spaces, see [19], where the replacement of a positive number by a distance distribution function was motivated by thinking of situations where the exact distance between two objects may not be provided, but at least some probability assignment could be done. Thus, the importance/diameter/measure of a set E might be represented by a distance distribution function, or simply a non-negative LT-fuzzy number γ_E.

Furthermore, having the above interpretation in mind and certain knowledge from fuzzy sets theory, then each τ_M-submeasure γ with the minimum t-norm $M(x, y) = \min\{x, y\}$ (in [16] we call it a *universal* τ_T*-submeasure*) can be represented by means of a non-decreasing system $(\eta_\alpha)_{\alpha\in[0,1]}$ of numerical submeasures as follows:

$$\gamma_E(x) = \sup\{\alpha \in [0, 1]; \eta_\alpha(E) \le x\}, \quad E \in \Sigma.$$

Compare this representation with the horizontal representation $(S_\alpha)_{\alpha\in[0,1]}$ of a fuzzy subset S.

Example 1.2. Let η be a numerical submeasure on Σ. Then for each $E \in \Sigma$ the mapping

$$\gamma_E(x) = 1 - \exp\left(-\left[\frac{x}{\lambda\eta(E)}\right]^k\right), \quad \lambda > 0,\ k > 0$$

corresponds to a cumulative distribution function of the Weibull distribution $W(\lambda, k)$ with parameters λ, k. Especially, for $k = 1$ we get the (universal) τ_T-submeasure corresponding to a distribution function of exponential distribution $E(\lambda)$ with parameter λ. Note that the standard conventions for the arithmetic operations on $\overline{\mathbb{R}}_+$ are considered, such as $0 \cdot (+\infty) = 0/0 = 0$.

1.4. Extensions of probabilistic submeasures

Naturally, we may ask about possibility to extend our considerations from Menger PM-spaces to wider spaces with different triangular functions instead of (1). Note that similar considerations were introduced and discussed in the framework of PM-spaces, see for example the monograph [10]. For such reasons in paper [13] we have suggested a generalization of τ_T-submeasures which involves suitable operations L replacing the standard addition $+$ on $\overline{\mathbb{R}}_+$, such that the underlying function

$$\tau_{L,T}(G, H)(x) := \sup_{L(u,v)=x} T(G(u), H(v))$$

is a triangle function, and thus the triple $(\Omega, \mathscr{F}, \tau_{L,T})$ is the so-called L-Menger PM-space under a t-norm T (as a direct generalization of Menger PM-space involving binary operation L).

Furthermore, since t-norms are rather special operations on the unit interval [0, 1], we have also mentioned a number of possible generalizations of a submeasure notion based on aggregation functions, or the convolution of distance distribution functions, i.e., an operation $*$ on Δ^+ given by

$$(G * H)(x) = \begin{cases} 0, & x = 0, \\ \int_0^x G(x-t)\,\mathrm{d}H(t), & x \in]0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

for each $G, H \in \Delta^+$, where the integral is meant in the sense of Lebesgue–Stieltjes. Then the corresponding property (c) of probabilistic submeasure $\gamma : \Sigma \to \Delta^+$ has the form

$$\gamma_{E\cup F}(x) \geq (\gamma_E * \gamma_F)(x), \quad E, F \in \Sigma,$$

providing thus an extension of a notion of submeasure to submeasures which can be used in non-Menger PM-spaces (e.g., in the Wald spaces—those involving the convolution $*$ of distance distribution functions), but also in a wider class of PM-spaces.

1.5. Aim and organization of this paper

Our aim is a further generalization of the concept of probabilistic submeasures considering aggregation functions. In particular, triangular norms applied in (c) of Definition 1.1 are used as binary functions only, and thus their associativity is a superfluous constraint. Therefore, a more general aggregation function can be used here (compare, e.g., the case of fuzzy logics, where the usual triangular norms can be replaced by (quasi-)copulas as discussed in [12]).

The paper is organized as follows: in Section 2 we recall some basic and necessary notions which will be used in this paper. Then in Section 3 we investigate further properties of triangular norm-based probabilistic submeasures which generalize some results obtained in our previous papers. Passing from triangular norms to their natural extension/modification in the form of copulas, quasi-copulas and semi-copulas, we study in Section 4 a notion of submeasure related to these aggregation functions. In the whole paper a number of examples is presented. A lattice structure of spaces of semi-copula and quasi-copula-based submeasures is also investigated.

2. Basic notions and definitions

In order to make the exposition self-contained, here we remind the reader the basic notions and constructions used in this paper.

2.1. Distribution functions

Let Δ be the family of all distribution functions on the extended real line $\overline{\mathbb{R}} := [-\infty, +\infty]$, i.e., $F : \overline{\mathbb{R}} \to [0, 1]$ is non-decreasing, left continuous on the real line $\mathbb{R}$ with $F(-\infty) = 0$ and $F(+\infty) = 1$. A *distance distribution function* is a distribution function whose support is a subset of $\overline{\mathbb{R}}_+$, i.e., a distribution function $F : \overline{\mathbb{R}} \to [0, 1]$ with $F(0) = 0$. The class of all distance distribution functions will be denoted by Δ^+.

A *triangle function* is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ which is symmetric, associative, non-decreasing in each variable and has ε_0 as the identity, where ε_0 is the distribution function of Dirac random variable concentrated in point 0. More precisely, for $a \in [-\infty, +\infty[$ we put

$$\varepsilon_a(x) := \begin{cases} 1 & \text{for } x > a, \\ 0 & \text{otherwise,} \end{cases}$$

whereas for $a = +\infty$ we put

$$\varepsilon_\infty(x) := \begin{cases} 0 & \text{for } x < +\infty, \\ 1 & \text{for } x = +\infty. \end{cases}$$

Clearly, (Δ^+, τ) is an Abelian semi-group with the identity ε_0. For more details on triangle functions we recommend an overview paper [22].

2.2. Triangular norms and binary aggregation functions

A *triangular norm*, shortly a t-norm, is a commutative lattice ordered semi-group on [0, 1] with identity 1. The most important are the minimum t-norm $M(x, y) := \min\{x, y\}$, the product t-norm $\Pi(x, y) := xy$, the Łukasiewicz t-norm $W(x, y) := \max\{x + y - 1, 0\}$, and the drastic product t-norm

$$D(x, y) := \begin{cases} \min\{x, y\} & \text{for } \max\{x, y\} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For more information about t-norms and their properties we refer to books [18,23]. Throughout this paper $\mathscr{T}$ denotes the class of all t-norms.

Triangular norms are a rather special case of aggregation functions on [0, 1]. Under a binary *aggregation function* $A : [0, 1]^2 \to [0, 1]$ we understand a non-decreasing function in both components with the boundary conditions $A(0, 0) = 0$ and $A(1, 1) = 1$. The class of all binary aggregation functions will be denoted by $\mathscr{A}$. For more details on aggregation functions we recommend the recent monograph [11].

Furthermore, let us denote by $\mathscr{L}$ the set of all binary operations L on $\overline{\mathbb{R}}_+$ such that

(i) L is commutative and associative;
(ii) L is jointly strictly increasing, i.e., for all $u_1, u_2, v_1, v_2 \in \overline{\mathbb{R}}_+$ with $u_1 < u_2, v_1 < v_2$ holds $L(u_1, v_1) < L(u_2, v_2)$;
(iii) L is continuous on $\overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+$;
(iv) L has 0 as its neutral element.

Observe that $L \in \mathscr{L}$ is a jointly increasing pseudo-addition on $\overline{\mathbb{R}}_+$ in the sense of [26]. The usual (class of) examples of operations in $\mathscr{L}$ are

$$K_\alpha(x, y) := (x^\alpha + y^\alpha)^{1/\alpha}, \quad \alpha > 0,$$
$$K_\infty(x, y) := \max\{x, y\}.$$

Note that although $\max\{x, y\} \in \mathscr{L}$, its "counterpart" $\min\{x, y\}$ is not a member of $\mathscr{L}$, because $\min\{x, y\}$ does not have 0 as its neutral element.

With $(L, A) \in \mathscr{L} \times \mathscr{A}$ the general form of (1) is as follows:

$$\tau_{L,A}(G, H)(x) := \sup_{L(u,v)=x} A(G(u), H(v)), \quad G, H \in \Delta^+.$$

As it is shown in [22], the appropriate choice for A is a semi-copula S, see Section 4. Indeed, the left-continuity of S guarantees that $\tau_{L,S}$ is a binary operation on Δ^+, cf. [22, Lemma 7.1]. Also, for any semi-copula S

$$\tau_{\max,S}(G, H)(x) = S(G(x), H(x))$$

is the operation pointwise induced by S on Δ^+. However, $\tau_{L,A}$ need not be associative in general, but it has good properties on Δ^+. For more information about (triangular) functions in connection with various aggregation functions and their properties we refer to [22].

2.3. Probabilistic submeasures in general setting

Now we introduce the following probabilistic submeasure notion in its general form (note that neither the left-continuity of a t-norm T nor of an aggregation function A is required in what follows).

Definition 2.1. Let $(L, A) \in \mathscr{L} \times \mathscr{A}$ and Σ be a ring of subsets of $\Omega \neq \emptyset$. A mapping $\gamma : \Sigma \to \Delta^+$ such that

(a′) $\gamma_\emptyset(x) = \varepsilon_0(x)$;
(b′) $\gamma_E(x) \geq \gamma_F(x)$, whenever $E \subset F$;

(c′) $\gamma_{E\cup F}(L(x, y)) \geq A(\gamma_E(x), \gamma_F(y))$, $E, F \in \Sigma$,

is said to be a $\tau_{L,A}$-*submeasure*.

Let us mention that for better readability we use the following convention: since Δ^+ is the set of all distribution functions with support $\overline{\mathbb{R}}_+$, we state the expression for a $\tau_{L,A}$-submeasure $\gamma : \Sigma \to \Delta^+$ with $(L, A) \in \mathscr{L} \times \mathscr{A}$ just for positive values of x. In case $x \leq 0$ we always suppose $\gamma_{.}(x) = 0$. For that reason the information "for $x > 0$" is omitted from Definition 2.1 as well as from its particular case Definition 1.1.

If $L = K_1$ (the usual sum), then its index is usually omitted, and we simply speak about τ_A-submeasure. Clearly, for $A = T$ (a left-continuous t-norm), and $L = K_1$ the $\tau_{L,A}$-submeasure reduces to τ_T-submeasure from [16], see also Definition 1.1. Further, for $L = K_\infty$ we get a $\tau_{\max,T}$-submeasure related to a non-Archimedean Menger PM-space $(\Omega, \mathscr{F}, \tau_{\max,T})$ with $T \in \mathscr{T}$. It is worth to note that in this case condition (c′) reads as follows:

$$\gamma_{E\cup F}(z) \geq T(\gamma_E(z), \gamma_F(z)), \quad E, F \in \Sigma.$$

Remark 2.2. In general, $L \in \mathscr{L}$ if and only if there is a (possibly empty) system $(]a_k, b_k[)_{k\in K}$ of pairwise disjoint open subintervals of $]0, +\infty[$, and a system $(\ell_k)_{k\in K}$ of increasing bijections $\ell_k : [a_k, b_k] \to \overline{\mathbb{R}}_+$ so that

$$L(x, y) = \begin{cases} \ell_k^{-1}(\ell_k(x) + \ell_k(y)) & \text{if } (x, y) \in]a_k, b_k[^2, \\ \max\{x, y\} & \text{otherwise.} \end{cases}$$

For more details see [18]. For $L = K_\alpha \in \mathscr{L}$ and $A = T \in \mathscr{T}$ we have

$$\tau_{K_\alpha,T}(G, H)(x) = \tau_T(G, H)(x^\alpha),$$

which motivates us to say that for $L \in \mathscr{L}$ generated by a strictly increasing bijection $\ell : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$, we denote $L_\ell(x, y) = \ell^{-1}(\ell(x) + \ell(y))$, we have

$$\tau_{L_\ell,T}(G, H)(x) = \tau_T(G, H)(\ell(x)).$$

In this light, in general, we have

$$\gamma_{E\cup F}(L(x, y)) = \gamma_{F\cup E}(L(y, x)) \geq \max\{A(\gamma_E(x), \gamma_F(y)), A(\gamma_F(y), \gamma_E(x))\},$$

for $E, F \in \Sigma$. So, we may (equivalently) take the symmetrization

$$A_{\text{sym}}(u, v) = \max\{A(u, v), A(v, u)\}$$

instead of $A \in \mathscr{A}$.

Easily, by standard methods of measure theory it is possible to extend a $\tau_{L,A}$-submeasure γ from a ring $\Sigma \subset \mathfrak{P}(\Omega)$ of subsets of $\Omega \neq \emptyset$ to a set function $\gamma^* : \mathfrak{P}(\Omega) \to \Delta^+$ as follows:

$$\gamma^*_E(x) := \sup\{\gamma_F(x); E \subseteq F \in \Sigma\}, \quad E \in \Omega.$$

Problem 1. Fix $(L, A) \in \mathscr{L} \times \mathscr{A}$, and let γ be a $\tau_{L,A}$-submeasure on a ring Σ of subsets of $\Omega \neq \emptyset$. Is the (Jordan) extension γ^* of γ also a $\tau_{L,A}$-submeasure on $\mathfrak{P}(\Omega)$? If not, under which conditions will it be?

2.4. Spaces of probabilistic submeasures

Throughout the paper $\Theta_{L,A}$ denotes the set of all $\tau_{L,A}$-submeasures on Σ for a fixed $(L, A) \in \mathscr{L} \times \mathscr{A}$, and

$$\Theta_{\mathscr{L},\mathscr{A}} := \{\Theta_{L,A}; (L, A) \in \mathscr{L} \times \mathscr{A}\}$$

the set of all $\tau_{L,A}$-submeasures on Σ for all possible pairs $(L, A) \in \mathscr{L} \times \mathscr{A}$ (or, the "superset" of all sets of $\tau_{L,A}$-submeasures on Σ). Here also, as a convention, we omit the index $L = K_1$ and write Θ_A instead of $\Theta_{K_1,A}$.

Example 2.3. For the set $\Omega = \{\omega_1, \omega_2\}$ and positive constants a, b, c such that $c \le \min\{a, b\}$, put

$$\gamma_{\omega_1}(x) := \max\{0, 1 - e^{-ax}\},$$
$$\gamma_{\omega_2}(x) := \max\{0, 1 - e^{-bx}\},$$
$$\gamma_{\Omega}(x) := \max\{0, 1 - e^{-cx}\}.$$

Then $\gamma \in \Theta_A$, $A \in \mathscr{A}$, if and only if

$$A(u, v) \le (1-u)^{c/a} \cdot (1-v)^{c/b}.$$

Hence, for $a = b = \frac{3}{2}c$, we have $\gamma \in \Theta_\Pi$, but $\gamma \notin \Theta_M$. Observe also that γ is not related to any numerical submeasure, see [16] for more details on relation of numerical submeasures and probabilistic submeasures.

Example 2.4. For a positive real number p consider the class $\mathbf{M}_p \subset \mathscr{A}$ which is usually called the *p-mean* (or, the Hölder mean), and is defined as

$$\mathbf{M}_p(x, y) := \left(\frac{x^p + y^p}{2}\right)^{1/p}, \quad x, y \ge 0.$$

If η is a numerical submeasure on Σ, then $\gamma \in \Theta_{\mathbf{M}_p}$, where

$$\gamma_E(x) = 2^{-1/p}(1 + (\max\{\min\{\sqrt[p]{\max\{1 + p(x - \eta(E)), 0\}}, 1\}, 0\})^p)^{1/p}$$

for $E \in \Sigma$. Since $\lim_{p\to 0} \mathbf{M}_p = \mathbf{G}$, the *geometric mean*, then $\gamma \in \Theta_{\mathbf{G}}$ has the form

$$\gamma_E(x) = \sqrt{\min\{e^{x-\eta(E)}, 1\}}, \quad E \in \Sigma.$$

Also, for $p = 1$, resp. $p = 2$, the p-mean is nothing but the *arithmetic mean* $\mathbf{A}$, resp. the *quadratic mean* $\mathbf{Q}$, and therefore we easily get the corresponding $\tau_{\mathbf{A}}$-, resp. $\tau_{\mathbf{Q}}$-submeasure.

3. Triangular norm-based submeasures

In what follows we use the usual point-wise order $\le$ between real-valued functions. Since $\gamma \in \Delta^+$ is non-decreasing, then for a fixed $T \in \mathscr{T}$ each $\tau_{L_1,T}$-submeasure γ is a $\tau_{L_2,T}$-submeasure whenever $L_1 \le L_2$. Moreover, if $T_2 \le T_1$ (it is usually said that T_2 is a *weaker* t-norm than T_1, or (equivalently) T_1 is *stronger* than T_2, see [18]), then each τ_{L_1,T_1}-submeasure γ is a τ_{L_2,T_2}-submeasure. In accordance with this motivation introduce the order $\preceq$ on $\Theta_{\mathscr{L},\mathscr{T}}$ as follows:

$$\Theta_{L_1,T_1} \preceq \Theta_{L_2,T_2} \quad \text{if and only if} \quad L_1 \le L_2 \text{ and } T_2 \le T_1.$$

Then $(\Theta_{\mathscr{L},\mathscr{T}}, \preceq)$ is a partially ordered set and for each $(L, T) \in \mathscr{L} \times \mathscr{T}$ we have

$$\Theta_{L,M} \preceq \Theta_{L,T} \preceq \Theta_{L,D}.$$

Note that for $L = K_1$ the order $\preceq$ on $\Theta_{\mathscr{T}}$ is nothing but order-inverted image of the point-wise order $\le$ of t-norms.

Remark 3.1. Observe that the partial order $\preceq$ is a coarsening of the standard inclusion ordering, i.e.,

$$\Theta_{L_1,T_1} \preceq \Theta_{L_2,T_2} \Longrightarrow \Theta_{L_1,T_1} \subset \Theta_{L_2,T_2}.$$

On the other hand, consider for example $L = K_\infty$. Then $\Theta_{K_\infty,T}$ does not depend on T (in fact, it consists of probabilistic submeasures γ satisfying $\gamma_E = \gamma_\Omega$ for any non-empty $E \subset \Omega$), although there are incomparable t-norms T_1 and T_2, i.e., Θ_{K_∞,T_1} and Θ_{K_∞,T_2} are $\preceq$-incomparable.

Table 1
Some well-known families of t-norms and their corresponding parameterized families of τ_T-submeasures.

Family of t-norms	**Corresponding family of τ_T-submeasures**
Aczél-Alsina t-norms	$\gamma_E^{AA,0}(x) = \varepsilon_{\eta(E)}(x)$
T_λ^{AA}, $\lambda \in [0, +\infty[$	$\gamma_E^{AA,\lambda}(x) = \exp(-[\max\{\eta(E) - x, 0\}]^{1/\lambda})$
Dombi t-norms	$\gamma_E^{D,0}(x) = \gamma_E^{AA,0}(x)$
T_λ^{D}, $\lambda \in [0, +\infty[$	$\gamma_E^{D,\lambda}(x) = (1 + [\max\{\eta(E) - x, 0\}]^{1/\lambda})^{-1}$
Frank t-norms	$\gamma_E^{F,1}(x) = \min\{\exp(x - \eta(E)), 1\}$
T_λ^{F}, $\lambda \in]0, +\infty]$	$\gamma_E^{F,+\infty}(x) = \max\{\min\{1 + x - \eta(E), 1\}, 0\}$
	$\gamma_E^{F,\lambda}(x) = \min\{\log_\lambda(1 + (\lambda - 1)\exp(x - \eta(E))), 1\}$
Hamacher t-norms	$\gamma_E^{H,+\infty}(x) = \gamma_E^{AA,0}(x)$
T_λ^{H}, $\lambda \in [0, +\infty]$	$\gamma_E^{H,0}(x) = \min\{(1 + \eta(E) - x)^{-1}, 1\}$
	$\gamma_E^{H,\lambda}(x) = \min\{\lambda(\exp(\eta(E) - x) + \lambda - 1)^{-1}, 1\}$
Yager t-norms	$\gamma_E^{Y,0}(x) = \gamma_E^{AA,0}(x)$
T_λ^{Y}, $\lambda \in [0, +\infty[$	$\gamma_E^{Y,\lambda}(x) = \max\{\min\{1 - [\max\{\eta(E) - x, 0\}]^{1/\lambda}, 1\}, 0\}$
Sugeno–Weber t-norms	$\gamma_E^{SW,-1}(x) = \gamma_E^{AA,0}(x)$
T_λ^{SW}, $\lambda \in [-1, +\infty]$	$\gamma_E^{SW,0}(x) = \gamma_E^{F,+\infty}(x)$
	$\gamma_E^{SW,+\infty}(x) = \gamma_E^{F,1}(x)$
	$\gamma_E^{SW,\lambda}(x) = \max\{\min\{\lambda^{-1}((1 + \lambda)^{1+x-\eta(E)} - 1), 1\}, 0\}$

Example 3.2. Let $\eta : \Sigma \to \overline{\mathbb{R}}_+$ be a numerical submeasure on a ring Σ of a non-empty set Ω, and $E \in \Sigma$. Then

(i) $\gamma \in \Theta_{L,M}$, where $L \in \mathscr{L}$, $K_1 \leq L$, and

$$\gamma_E(x) = \begin{cases} 0 & \text{for } x \leq 0, \\ 1/2 & \text{for } x \in]0, \eta(E)]; \\ 1 & \text{for } x > \eta(E), \end{cases}$$

(ii) $\gamma \in \Theta_{L,D}$, where $L \in \mathscr{L}$ is arbitrary, and

$$\gamma_E(x) = \frac{x}{x + \eta(E)};$$

(iii) $\gamma \in \Theta_{L,W}$, where $L \in \mathscr{L}$, $K_1 \leq L$, and

$$\gamma_E(x) = \max\{\min\{1 + x - \eta(E), 1\}, 0\};$$

(iv) for

$$\gamma_E(x) = \min\left\{\frac{1 + x}{1 + \eta(E)}, 1\right\}$$

we have that $\gamma \in \Theta_D$, however it is not an element of Θ_W, neither Θ_Π nor Θ_M.

More examples of τ_T-submeasures related to some well-known parameterized families of t-norms T, see [18], are summarized in Table 1. Note that in all cases we omit the minimum t-norm $M = T_{+\infty}^{AA} = T_{+\infty}^{D} = T_0^{F} = T_{-\infty}^{SS} = T_{+\infty}^{Y}$ (one example of such a universal submeasure is given in Example 3.2(i)).

3.1. Probabilistic versus numerical submeasures

As it is already known, see [16, Theorem 1], to each numerical submeasure η on Σ corresponds $\gamma \in \Theta_{L,M}$, $K_1 \le L$, in the form

$$\gamma_E(x) = \varepsilon_0(x - \eta(E)), \quad E \in \Sigma,$$

where the number $\gamma_E(x)$ may be interpreted as the probability that the value of submeasure η of a set $E \in \Sigma$ is less than x. To underline the interesting relationship between the probabilistic $\tau_{L,T}$-submeasure γ and the numerical submeasure η on Σ we give the following result which improves and generalizes [16, Theorem 4]. For the sake of completeness we give its short direct proof here.

Theorem 3.3. *Let $L \le K_1$ and $\gamma \in \Theta_{L,T_1}$. If t is an additive generator of a continuous Archimedean t-norm T such that $T \le T_1$, then a mapping $\eta_{\gamma,t} : \Sigma \to \mathbb{R}_+$ given by*

$$\eta_{\gamma,t}(E) := \sup\{z \in \mathbb{R}_+; t(\gamma_E(z)) \ge z\}$$

is a numerical submeasure.

Proof. The equality $\eta_{\gamma,t}(\emptyset) = 0$ and the monotonicity of $\eta_{\gamma,t}$ are obvious. Moreover, it is evident that γ is an element of Θ_T, and hence for $E, F \in \Sigma$ we have

$$\begin{aligned}\eta_{\gamma,t}(E \cup F) &= \sup\{z \in \mathbb{R}_+; t(\gamma_{E\cup F}(z)) \ge z\}\\ &\le \sup\{z \in \mathbb{R}_+; t(T(\gamma_E(x), \gamma_F(z-x))) \ge x + z - x \text{ for some } x \in [0, z]\}\\ &= \sup\{z \in \mathbb{R}_+; \min\{t(0), t(\gamma_E(x)) + t(\gamma_F(z-x))\} \ge z \text{ for some } x \in [0, z]\}\\ &\le \eta_{\gamma,t}(E) + \eta_{\gamma,t}(F),\end{aligned}$$

which proves that $\eta_{\gamma,t}$ is a numerical submeasure on Σ. □

3.2. Transformations and aggregations of probabilistic submeasures

In the context of t-norms (but not limited to this case, as we will use later) it is very natural to consider the following simple transformations which often manifest in different applied fields. Consider the *group $\mathcal{H}$ of automorphisms* (strictly increasing bijections) of the unit interval [0, 1] acting on the class $\mathcal{X}$ of all functions G from $[0, 1]^2$ to [0, 1] as follows:

$$(\Psi_h G)(x, y) = h^{-1}(G(h(x), h(y))), \quad h \in \mathcal{H},$$

for all $x, y \in [0, 1]$. We shall denote by $\Psi_{\mathcal{H}}$ this class of transformations (an element of $\Psi_{\mathcal{H}}$ is determined by a function $h \in \mathcal{H}$). Clearly, $\Psi_{\mathcal{H}}$ is a group under the composition with the inverse $\Psi_h^{-1} = \Psi_{h^{-1}}$ and the identity $\Psi_{\mathrm{id}_{[0,1]}}$. The mapping $\Psi : \mathcal{X} \times \mathcal{H} \to \mathcal{X}$ is the action of the group $\mathcal{H}$ on $\mathcal{X}$. Since $\Psi_h M = M$ for each $h \in \mathcal{H}$, then each Ψ_h-transform of a universal submeasure is a universal submeasure as well. Moreover, $\Theta_{L,M} = \Theta_{L,\Psi_h M}$ for each $(L, h) \in \mathcal{L} \times \mathcal{H}$. Also it is known, see [8, Proposition 2.6], that the class $\mathcal{T}$ of t-norms is closed under Ψ.[1]

Proposition 3.4. *Let $h \in \mathcal{H}$. Then*

(i) *if h is supermultiplicative, then for each $L_1, L_2 \in \mathcal{L}$ such that $L_1 \le L_2$ holds $\Theta_{L_1,\Pi} \preceq \Theta_{L_2,\Psi_h \Pi}$;*
(ii) *if the function $1 - h(1-x)$ is subadditive, then for each $L_1, L_2 \in \mathcal{L}$ such that $L_1 \le L_2$ holds $\Theta_{L_1,W} \preceq \Theta_{L_2,\Psi_h W}$;*
(iii) *if $(L_1, T_1), (L_2, T_2) \in \mathcal{L} \times \mathcal{T}$ such that $\Theta_{L_1,T_1} \preceq \Theta_{L_2,T_2}$, then $\Theta_{L_1,\Psi_h T_1} \preceq \Theta_{L_2,\Psi_h T_2}$;*
(iv) *for each $(L, T, h) \in \mathcal{L} \times \mathcal{T} \times \mathcal{H}$ holds $\Theta_{L,M} \preceq \Theta_{L,\Psi_h T} \preceq \Theta_{L,\Psi_h D}$;*
(v) *for each $(L, T) \in \mathcal{L} \times \mathcal{T}$ and each involution h on* [0, 1] *holds: $\gamma \in \Theta_{L,T}$ if and only if $h \circ \gamma \in \Theta_{L,\Psi_h T}$.*

[1] A class $\mathcal{X}$ is *closed under* Ψ, if $\Psi_h(\mathcal{X}) \subset \mathcal{X}$ for each $h \in \mathcal{H}$.

Example 3.5. Let η be a numerical submeasure on Σ. If $h(x) = \tan(\pi/4)x$ for $x \in [0, 1]$, then for $K_1 \leq L$ we get $\gamma \in \Theta_{L,\Psi_h W}$, where

$$\gamma_E(x) = \max\left\{\min\left\{\frac{4}{\pi}\ \arctan(1-\eta(E)+x), 1\right\}, 0\right\}, \quad E \in \Sigma.$$

It is easy to verify that the convex combination of numerical submeasures is again a numerical submeasure. If we consider the pseudo-convex combination in the spirit of weighted quasi-arithmetic mean, the result for probabilistic submeasures will be the same, i.e., for an arbitrary $L \in \mathscr{L}$ the weighted quasi-arithmetic mean

$$\mathbf{A}_t^w(x_1, \ldots, x_n) := t^{(-1)}\left(\sum_{i=1}^n w_i t(x_i)\right)$$

generated by an additive generator t of a continuous Archimedean t-norm $T \in \mathscr{T}$ preserves the class $\Theta_{L,T}$ of probabilistic $\tau_{L,T}$-submeasures. Here for $i = 1, \ldots, n$ we consider $x_i \in [0, 1]$, w_i are non-negative weights with $\sum_{i=1}^n w_i = 1$ and $t^{(-1)}$ is the pseudo-inverse function to t, see [18] for more details. Recall that $t : [0, 1] \to \overline{\mathbb{R}}_+$ is an *additive generator* of a continuous Archimedean t-norm T if and only if it is continuous, strictly decreasing and satisfying $t(1) = 0$. Moreover, its pseudo-inverse $t^{(-1)} : \overline{\mathbb{R}}_+ \to [0, 1]$ is given by

$$t^{(-1)}(x) := t^{-1}(\min\{t(0), x\}).$$

Proposition 3.6. *Let $L \in \mathscr{L}$ and t be an additive generator of a continuous Archimedean t-norm $T \in \mathscr{T}$. If $\gamma^{(i)} \in \Theta_{L,T}$ for $i = 1, 2, \ldots, n$, then*

$$\gamma := \mathbf{A}_t^w(\gamma^{(1)}, \ldots, \gamma^{(n)}) \in \Theta_{L,T}.$$

Proof. The first two properties (a′) and (b′) of Definition 2.1 are easy to verify, therefore we show only the triangle inequality (c′).

Let $E, F \in \Sigma$. Since $\gamma^{(i)} \in \Theta_{L,T}$, then

$$\gamma^{(i)}_{E\cup F}(L(x, y)) \geq t^{-1}(\min\{t(0), t(\gamma^{(i)}_E(x)) + t(\gamma^{(i)}_F(y))\}),$$

and we have

$$\begin{aligned}\gamma_{E\cup F}(L(x, y)) &= t^{(-1)}\left(\sum_{i=1}^n w_i t(\gamma^{(i)}_{E\cup F}(L(x, y)))\right)\\ &\geq t^{(-1)}\left(\sum_{i=1}^n w_i\ \min\{t(0), t(\gamma^{(i)}_E(x)) + t(\gamma^{(i)}_F(y))\}\right)\\ &\geq t^{(-1)}(t(\gamma_E(x)) + t(\gamma_F(y)))\\ &= T(\gamma_E(x), \gamma_F(y)),\end{aligned}$$

thus γ is a $\tau_{L,T}$-submeasure on Σ. □

Corollary 3.7. *Let $(L, h) \in \mathscr{L} \times \mathscr{H}$ and t be an additive generator of a continuous Archimedean t-norm $T \in \mathscr{T}$. If $\gamma^{(i)} \in \Theta_{L,\Psi_h T}$ for $i = 1, 2, \ldots, n$, then $\gamma = \mathbf{A}_t^w(\gamma^{(1)}, \ldots, \gamma^{(n)}) \in \Theta_{L,\Psi_h T}$.*

From these observations we state the following open problem:

Problem 2. Characterize the class of mappings (aggregation functions) $\mathscr{M}$ which preserves the class $\Theta_{L,A}$ of probabilistic submeasures for a fixed $(L, A) \in \mathscr{L} \times \mathscr{A}$, i.e., $\mathscr{M}(\Theta_{L,A}) \subseteq \Theta_{L,A}$.

3.3. Topological rings of probabilistic submeasures

Now we will consider the Fréchet–Nikodym topology $\Gamma(\gamma)$ generated by probabilistic submeasure γ on Σ. This notion was introduced and studied by Drewnowski in [5] for numerical submeasures on a ring of sets. Recall that a topology σ on a ring Σ is said to be a *ring topology* if the mappings $(E,F)\to E\triangle F$ and $(E,F)\to E\cap F$ of $\Sigma\times\Sigma\to\Sigma$ are continuous (with respect to the product topology on $\Sigma\times\Sigma$). A ring topology σ is said to be a *Fréchet–Nikodym topology* on Σ if for each σ-neighborhood U of $\emptyset$ in Σ there is a σ-neighborhood V of $\emptyset$ in Σ such that $F\subset U$ for all $F\subseteq E\in V$, $F\in\Sigma$. In particular, a family $\{\eta_i; i\in I\}$ of numerical submeasures on Σ defines a Fréchet–Nikodym topology $\Gamma(\eta_i; i\in I)$ and conversely, for each Fréchet–Nikodym topology Γ on Σ there is a family $\{\zeta_j; j\in J\}$ of numerical submeasures on Σ such that $\Gamma=\Gamma(\zeta_j; j\in J)$.

Define the set function $\rho:\Sigma\times\Sigma\to\Delta^+$ by $\rho(E,F):=\gamma_{E\triangle F}$ where $\gamma\in\Theta_{L,T}$. Then

$$\begin{aligned}\rho_{E,F}(L(x,y))&=\gamma_{E\triangle F}(L(x,y))\geq\gamma_{(E\triangle G)\cup(G\triangle F)}(L(x,y))\\&\geq T(\gamma_{E\triangle G}(x),\gamma_{G\triangle F}(y))=T(\rho_{E,G}(x),\rho_{G,F}(y)),\end{aligned}$$

which means, in the other words, that ρ is an L-Menger pseudo-metric on Σ. Moreover, ρ is translation invariant, i.e.,

$$\rho_{E,F}=\gamma_{E\triangle F}=\gamma_{(E\triangle G)\triangle(G\triangle F)}=\rho_{E\triangle G,G\triangle F}.$$

Thus, the triple $(\Sigma,\rho,\tau_{L,T})$ is an L-Menger probabilistic pseudo-metric space, see [13, Theorem 3.2]. Since for $E_1,E_2,F_1,F_2\in\Sigma$ holds

$$(E_1\cap F_1)\triangle(E_2\cap F_2)\subset(E_1\triangle E_2)\cup(F_1\triangle F_2),$$

then we get

$$\begin{aligned}\rho_{E_1\cap F_1,E_2\cap F_2}(x)&=\gamma_{(E_1\cap F_1)\triangle(E_2\cap F_2)}(x)\geq\gamma_{(E_1\triangle E_2)\cup(F_1\triangle F_2)}(L(z,z))\\&\geq T(\gamma_{E_1\triangle E_2}(z),\gamma_{F_1\triangle F_2}(z))=T(\rho_{E_1,E_2}(z),\rho_{F_1,F_2}(z)),\end{aligned}\tag{3}$$

where $L(z,z)<x$. If $(E_n,F_n)\to(E,F)$ in topology $\Gamma(\gamma)$, then $E_n\to E$ and $F_n\to F$. Thus, $\rho_{E_n,E}(z)\to1$ and $\rho_{F_n,F}(z)\to1$. Moreover, if we consider a continuous t-norm T, then from (3) we get $\rho_{E_n\cap E,F_n\cap F}(z)\to1$ for each $x>0$. In fact, it proves continuity of $\cap$ in the product topology $\Sigma\times\Sigma$. These observations lead to the following result.

Proposition 3.8. *Let $(L,T)\in\mathscr{L}\times\mathscr{T}$, where T is a continuous t-norm and $\gamma\in\Theta_{L,T}$. For $\varepsilon>0$ and $\delta>0$ put*

$$\mathscr{B}(\varepsilon,\delta):=\{E\in\Sigma;\gamma_E(\varepsilon)>1-\delta\}.$$

Then

(i) $\mathfrak{B}:=\{\mathscr{B}(\varepsilon,\delta);\varepsilon>0,\delta>0\}$ *is a normal base of neighborhoods of* $\emptyset$ *for the Fréchet–Nikodym topology* $\Gamma(\gamma)$;
(ii) $(\Sigma,\triangle,\cap,\Gamma(\gamma))$ *is a topological ring of sets.*

4. Semi-copula-based submeasures

In what follows we consider the natural extension/modification of t-norms: copulas, quasi-copulas and semi-copulas, see [7]. Recall that a *semi-copula* is an aggregation function $S:[0,1]^2\to[0,1]$ with 1 as its neutral element. Denote by $\mathscr{S}$ the set of all semi-copulas and $\mathscr{S}_c$ the set of all continuous semi-copulas. A *quasi-copula* Q is a 1-Lipschitz semi-copula, i.e., a semi-copula Q satisfying

$$|Q(x,y)-Q(x',y')|\leq|x-x'|+|y-y'|$$

for all $x,x',y,y'\in[0,1]$. The set of all quasi-copulas will be denoted by $\mathscr{Q}$. A semi-copula C which is 2-increasing, i.e., for each $x,y,x',y'\in]0,1]$ such that $x\leq x'$ and $y\leq y'$ holds

$$C(x',y')-C(x,y')-C(x',y)+C(x,y)\geq0$$

is called a *copula*. Denote by $\mathscr{C}$ the set of all copulas. Then $\mathscr{C} \subset \mathscr{Q} \subset \mathscr{S}$. All these sets are partially ordered equipped with the usual point-wise order $\leq$ between real functions. Clearly, for each $h \in \mathscr{H}$ the mapping Ψ_h is order-preserving on $\mathscr{S}$ and for a given $h \in \mathscr{H}$ the partially ordered set

$$\mathscr{K} := (\{S \in \mathscr{S};\ \Psi_h S = S\}, \leq)$$

is a complete lattice (by Knaster–Tarski theorem). As we have already mentioned, $M \in \mathscr{K}$, but also $D \in \mathscr{K}$.

4.1. Generated probabilistic submeasures

For the class of Archimedean copulas, i.e., copulas of the form

$$C(x, y) := \varphi^{[-1]}(\varphi(x) + \varphi(y))$$

for all $x, y \in [0, 1]$, where $\varphi : [0, 1] \to \overline{\mathbb{R}}$ is a continuous, strictly decreasing convex function with $\varphi(1) = 0$ and the pseudo-inverse $\varphi^{[-1]}$ (such a function is called an *additive generator* of C, cf. [20]), we immediately have the following characterization.

Proposition 4.1. *Let η be a numerical submeasure on Σ. If φ is an additive generator of $C \in \mathscr{C}$, then $\gamma \in \Theta_C$, where*

$$\gamma_E(x) = \varphi^{[-1]}(\eta(E) - x), \quad E \in \Sigma.$$

Moreover, for each $h \in \mathscr{H}$ holds $\gamma \in \Theta_{\Psi_h C}$, where

$$\gamma_E(x) = (\varphi \circ h)^{[-1]}(\eta(E) - x), \quad E \in \Sigma.$$

Easily it is possible to state the analogous result for the multiplicative generator of $C \in \mathscr{C}$.

Example 4.2. Let η be a numerical submeasure on Σ and $E \in \Sigma$. Then
(i) $\gamma \in \Theta_{C_\lambda^{GH}}$, where

$$\gamma_E(x) = \exp(-[\max\{\eta(E) - x, 0\}]^{1/\lambda})$$

corresponds to the *Gumbel–Hougaard family* of (strict) copulas C_λ^{GH} given by

$$C_\lambda^{GH}(u, v) := \exp(-[(-\ln u)^\lambda + (-\ln v)^\lambda]^{1/\lambda}),$$

with $\lambda \in [1, +\infty[$, see [20]; for $\lambda = 1$ we have the independence copula Π (and the corresponding $\gamma \in \Theta_\Pi$)—clearly, $\Theta_\Pi \in \Theta_{\mathscr{C}}$ and for each $h \in \mathscr{H}$ we have $\Theta_{\Psi_h \Pi} \in \Theta_{\mathscr{T}}$;
(ii) $\gamma \in \Theta_{C_\lambda}$, where

$$\gamma_E(x) = \max\left\{\min\left\{\frac{1 - \eta(E) + x}{1 + (\lambda - 1)(\eta(E) - x)}, 1\right\}, 0\right\}$$

corresponds to the family of (non-strict) Archimedean copulas

$$C_\lambda(u, v) := \max\left\{\frac{\lambda^2 uv - (1 - u)(1 - v)}{\lambda^2 - (\lambda - 1)^2(1 - u)(1 - v)}, 0\right\}, \quad \lambda \in [1, +\infty[.$$

Observe that similarly as in the case of continuous Archimedean t-norms, the weighted quasi-arithmetic mean

$$\mathbf{A}_\varphi^w(x_1, \ldots, x_n) := \varphi^{[-1]}\left(\sum_{i=1}^n w_i \varphi(x_i)\right)$$

generated by an additive generator φ of an Archimedean copula $C \in \mathscr{C}$ preserves the class Θ_C of probabilistic copula-based submeasures (even their generalization involving an arbitrary $L \in \mathscr{L}$).

Remark 4.3. Especially, in connection with copulas the following possibility to extend probabilistic submeasures is provided in [13, Section 4.2]: for $(L, C) \in \mathscr{L} \times \mathscr{C}$ define the function $\sigma_{L,C} : \Delta^+ \times \Delta^+ \to \Delta^+$ for $x \in \overline{\mathbb{R}}_+$ by

$$\sigma_{L,C}(G,H)(x) = \begin{cases} 0, & x = 0, \\ \int_{L(x)} \mathrm{d}C(G(u), H(v)), & x \in]0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

for each $G, H \in \Delta^+$, where $L(x) = \{(u, v); u, v \in \mathbb{R}_+, L(u, v) < x\}$, and the integral is of Lebesgue–Stieltjes type. Then a mapping $\gamma : \Sigma \to \Delta^+$ such that

(a$''$) $\gamma_\emptyset(x) = \varepsilon_0(x)$;
(b$''$) $\gamma_E(x) \geq \gamma_F(x)$, whenever $E \subset F$;
(c$''$) $\gamma_{E\cup F}(x) \geq \int_{L(x)} \mathrm{d}C(\gamma_E(u), \gamma_F(v)),\ E, F \in \Sigma$,

is said to be a $\sigma_{L,C}$-*submeasure*.

Given a class $\mathscr{X}$ of functions from $[0, 1]^2$ to $[0, 1]$, we shall denote by $\Psi_{\mathscr{H}}(\mathscr{X})$ the class of operators obtained by transforming all elements of $\mathscr{X}$ by all elements of $\Psi_{\mathscr{H}}$. Since the classes $\mathscr{S}$ and $\mathscr{S}_c$ are closed under Ψ, cf. [8], we have the following relations

$$\Psi_{\mathscr{H}}(\mathscr{C}) \subset \Psi_{\mathscr{H}}(\mathscr{Q}) \subset \Psi_{\mathscr{H}}(\mathscr{S}_c) = \mathscr{S}_c \subset \mathscr{S} = \Psi_{\mathscr{H}}(\mathscr{S}).$$

Recall that the identity $I \in \Psi_{\mathscr{H}}$. Moreover, $\mathscr{C} \subset \Psi_{\mathscr{H}}(\mathscr{C})$ and $\mathscr{Q} \subset \Psi_{\mathscr{H}}(\mathscr{Q})$, see [1].

Proposition 4.4. *If $S_1, S_2 \in \mathscr{S}$ such that $\Theta_{S_1} \preceq \Theta_{S_2}$, then for each $h \in \mathscr{H}$ it holds $\Theta_{\Psi_h S_1} \preceq \Theta_{\Psi_h S_2}$. Moreover, $\Theta_{\mathscr{S}} = \Theta_{\Psi_{\mathscr{H}}(\mathscr{S})}$ and $\Theta_{\mathscr{S}_c} = \Theta_{\Psi_{\mathscr{H}}(\mathscr{S}_c)}$.*

4.2. Lattice structure of spaces of probabilistic submeasures

In what follows we are interested in lattice structure of submeasure spaces in $\Theta_{\mathscr{S}}$. As shown in [9], the class $\mathscr{S}$ of semi-copulas constitutes the lattice completion of the class $\mathscr{T}$ of t-norms, in the sense that every semi-copula may be represented as the point-wise supremum and infimum of a suitable subset of t-norms. Let $\vee$ and $\wedge$ denote the point-wise supremum and infimum, respectively. Observe that if γ is a τ_{S_1}- and τ_{S_2}-submeasure for some $S_1, S_2 \in \mathscr{S}$, then γ is a $\tau_{S_1 \vee S_2}$- as well as $\tau_{S_1 \wedge S_2}$-submeasure. Thus, for $S_1, S_2 \in \mathscr{S}$ put

$$\Theta_{S_1} \sqcup \Theta_{S_2} := \Theta_{S_1 \wedge S_2} \quad \text{and} \quad \Theta_{S_1} \sqcap \Theta_{S_2} := \Theta_{S_1 \vee S_2}.$$

It is easy to see that $\sqcup$ and $\sqcap$ are *lattice operations*. Since $(\mathscr{S}, \leq, \vee, \wedge)$ is a complete lattice, see [9], then we have the following observation.

Proposition 4.5. *The family $\Theta_{\mathscr{S}}$ of all probabilistic submeasure spaces is a distributive lattice.*

Proof. Indeed, for $S_1, S_2, S_3 \in \mathscr{S}$, we have

$$\begin{aligned} \Theta_{S_1} \sqcup (\Theta_{S_2} \sqcap \Theta_{S_3}) &= \Theta_{S_1} \sqcup \Theta_{S_2 \vee S_3} = \Theta_{S_1 \wedge (S_2 \vee S_3)} = \Theta_{(S_1 \wedge S_2) \vee (S_1 \wedge S_3)} \\ &= \Theta_{S_1 \wedge S_2} \sqcap \Theta_{S_1 \wedge S_3} = (\Theta_{S_1} \sqcup \Theta_{S_2}) \sqcap (\Theta_{S_1} \sqcup \Theta_{S_3}). \end{aligned}$$

Analogously for $\Theta_{S_1} \sqcap (\Theta_{S_2} \sqcup \Theta_{S_3})$. By [2, Theorem 2.2] $\Theta_{\mathscr{S}}$ is a distributive lattice. □

Since for each $S \in \mathscr{S}$ holds $\Theta_M \preceq \Theta_S \preceq \Theta_D$, then Θ_M is bottom and Θ_D is top in the lattice $\Theta_{\mathscr{S}}$, thus $\Theta_{\mathscr{S}}$ is a *bounded distributive lattice*.

In what follows recall [2] that an *ideal* $\mathfrak{I}$ of a lattice $(\mathfrak{L}, \leq, \vee, \wedge)$ is a subset such that

(i) if $a \in \mathfrak{I}$ and $b \leq a$, then $b \in \mathfrak{I}$;
(ii) if $a, b \in \mathfrak{I}$, then $a \vee b \in \mathfrak{I}$.

Dually, a *filter* $\mathfrak{F}$ of a lattice $(\mathfrak{L}, \leq, \vee, \wedge)$ is a subset such that

(i) if $a \in \mathfrak{F}$ and $a \leq b$, then $b \in \mathfrak{F}$;
(ii) if $a, b \in \mathfrak{F}$, then $a \wedge b \in \mathfrak{F}$.

Proposition 4.6. *For every $S_1 \in \mathscr{S}$ the set*

$$\mathfrak{I}_{S_1} := \{\Theta_S \in \Theta_{\mathscr{S}}; \Theta_S \preceq \Theta_{S_1}, S \in \mathscr{S}\}$$

is an ideal in $\Theta_{\mathscr{S}}$.

Proof. First, observe that for $S_1 \in \mathscr{S}$ the set $\mathfrak{I}_{S_1}$ is the set of all τ_S-submeasures related to a semi-copula S which are also τ_{S_1}-submeasures.

Let $\Theta_{S_2} \in \mathfrak{I}_{S_1}$, i.e., $\Theta_{S_2} \preceq \Theta_{S_1}$ and let $\Theta_{S_3} \preceq \Theta_{S_2}$. From it follows that $S_1 \leq S_2$ and $S_2 \leq S_3$. Thus, $S_1 \leq S_3$ which shows that $\Theta_{S_3} \preceq \Theta_{S_1}$, i.e., $\Theta_{S_3} \in \mathfrak{I}_{S_1}$.

Let $\Theta_{S_2}, \Theta_{S_3} \in \mathfrak{I}_{S_1}$, i.e., $S_1 \leq S_2$ and $S_1 \leq S_3$. Since $S_1 \leq S_2 \wedge S_3$, then $\Theta_{S_2} \sqcup \Theta_{S_3} = \Theta_{S_2 \wedge S_3} \preceq \Theta_{S_1}$, i.e., $\Theta_{S_2} \sqcup \Theta_{S_3} \in \mathfrak{I}_{S_1}$. Therefore, $\mathfrak{I}_{S_1}$ is an ideal in $\Theta_{\mathscr{S}}$. □

Dually to Theorem 4.6, we obtain the following corollary.

Corollary 4.7. *For every $S_2 \in \mathscr{S}$, the set*

$$\mathfrak{F}_{S_2} := \{\Theta_S \in \Theta_{\mathscr{S}}; \Theta_{S_2} \preceq \Theta_S, S \in \mathscr{S}\}$$

is a filter in $\Theta_{\mathscr{S}}$.

From it follows that for $S_1, S_2 \in \mathscr{S}$ such that $S_1 \leq S_2$ the set

$$[\Theta_{S_2}, \Theta_{S_1}] = \mathfrak{I}_{S_1} \cap \mathfrak{F}_{S_2}$$

is an order interval in $\Theta_{\mathscr{S}}$.

Theorem 4.8. *$(\Theta_{\mathscr{S}}, \preceq, \sqcap, \sqcup, \Theta_D, \Theta_M)$ is a complete lattice.*

Proof. Let $\Theta_{\mathscr{P}}$ be any subset of $\Theta_{\mathscr{S}}$ and put $\sqcup\Theta_{\mathscr{P}} = \Theta_{\wedge\mathscr{P}}$, $\sqcap\Theta_{\mathscr{P}} = \Theta_{\vee\mathscr{P}}$, where

$$\vee\mathscr{P}(x, y) := \sup\{P(x, y); P \in \mathscr{P}\}, \quad \wedge\mathscr{P}(x, y) := \inf\{P(x, y); P \in \mathscr{P}\}$$

for each $(x, y) \in [0, 1]^2$. Since $(\mathscr{S}, \leq, \vee, \wedge)$ is complete, then for each $\mathscr{P} \subseteq \mathscr{S}$ holds $\vee\mathscr{P} \in \mathscr{S}$ and $\wedge\mathscr{P} \in \mathscr{S}$. Thus $\Theta_{\vee\mathscr{P}} \in \Theta_{\mathscr{S}}$ and $\Theta_{\wedge\mathscr{P}} \in \Theta_{\mathscr{S}}$. □

Remark 4.9. Since $(\mathscr{Q}, \leq, \vee, \wedge)$ is a complete lattice, see [21], all the above assertions hold also for $\Theta_{\mathscr{Q}}$, i.e., $(\Theta_{\mathscr{Q}}, \preceq, \sqcap, \sqcup, \Theta_W, \Theta_M)$ is a complete sublattice of $(\Theta_{\mathscr{S}}, \preceq, \sqcap, \sqcup, \Theta_D, \Theta_M)$, where for each $Q_1 \in \mathscr{Q}$ the set

$$\mathfrak{I}_{Q_1} := \{\Theta_Q \in \Theta_{\mathscr{Q}}; \Theta_Q \preceq \Theta_{Q_1}, Q \in \mathscr{Q}\}$$

is an ideal in $\Theta_{\mathcal{Q}}$, and for each $Q_2 \in \mathcal{Q}$ the set

$$\mathfrak{F}_{Q_2} := \{\Theta_Q \in \Theta_{\mathcal{Q}}; \Theta_{Q_2} \preceq \Theta_Q, Q \in \mathcal{Q}\}$$

is a filter in $\Theta_{\mathcal{Q}}$.

5. Concluding remarks

We have discussed probabilistic submeasures generalizing the classical submeasures in some different ways. First of all, the generalization of the exact numerical values of submeasures by means of probabilistic information in the form of distribution functions was considered. Next, the standard addition was generalized into a jointly strictly monotone pseudo-addition. Finally, some special generalizations of triangle functions aggregating distribution functions were introduced and applied. Moreover, we have studied the structure of spaces of introduced probabilistic submeasure spaces.

The area of generalized non-additive measure theory has a great potential not only in the pure mathematics but also in decision theory, game theory and some other applied fields. Up to the two open problems explicitly formulated in this paper, there are several problems and directions for the future research, such as the set-valued submeasures and their different possible generalizations, dealing with Banach spaces in general, or some more specific spaces.

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