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A Generalized Möbius Transform of Games on MV-algebras and Its Application to a Cimmino-type Algorithm for the Core

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This paper is dedicated to the memory of Dan Butnariu, my dear friend and teacher.

Abstract. A generalization of the Möbius transform of games with finitely many players is introduced for games on MV-algebras. The variety of MV-algebras subsumes most coalition models. We characterize the class of games for which the generalized Möbius transform exists. An application of the proposed transform to a Cimmino-type algorithm is shown for the core solution in games with finitely-many players.

1. Introduction

The Möbius transform, which originated in the work of Rota [**Rot64**], was introduced to deal with problems in combinatorics and number theory. The scope of its applications is, however, very broad, so one of the areas in which it is used extensively is coalition game theory or theory of non-additive set functions in general. Since coalition games in the classical setting of games with finitely many players [**PS07**] are just set functions on the algebra of all subsets, we will prefer the shorter term "game" instead of "set function". The Möbius transform of a game is an equivalent representation of the game such that this representation can be viewed as an additive set function defined on a much larger algebra than the original game. This point of view is not so illuminating when processing games with finitely many players, yet its underlying idea enables extensions of the Möbius transform to games on infinite sets [**Sch86, Den97**]. There are numerous fields and results intertwining with the theory of the Möbius transform: the theory of integral representations of Choquet [**Cho54**], Stone's representation theorem for Boolean algebras or random set theory [**Mol05**]. A comprehensive account of these aspects can be found in [**Mol05**, Chapter 1].

The main goal of this paper is to show that the Möbius transform can be naturally generalized to games on MV-algebras [**CDM00**]. Most algebraic structures used for modeling coalitions in coalition game theory, such as Boolean algebras of sets [**PS07, AS74**] or particular families of real-valued functions [**Aub74, BK93**],

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are special cases of MV-algebras ("many-valued"-algebras). It was Dan Butnariu who envisioned the study of games on MV-algebras already in the book [**BK93**], and who further encouraged the author to pursue the study of games in the many-valued settings.

While Boolean algebras give semantics to the classical two-valued logic, MV-algebras are the algebras of Łukasiewicz infinite-valued logic [**CDM00**, Chapter 4]. It was argued in [**Kro09**] that the idea of many-valued coalitions fits within the framework of coalition game theory. We will give the necessary background on MV-algebras in Section 2.1. The essential tool in investigating the generalized Möbius transform and the solution of games is the concept of measure (or state) introduced in [**Mun95**]. As a matter of fact, the notion of measure on an MV-algebra is just an "integral" extension of the notion of Borel measure — see Theorem 2.10 in Section 2.2.

Section 3 contains the main result (Theorem 3.5), which singles out the class of MV-algebras and games for which the generalized Möbius transform exists. This result is preceded by the motivation leading to the introduction of the MV-algebra of all continuous functions over the space of compact subsets on which the Möbius transform should be defined: this construction is in line with the idea of the Möbius transform in the Boolean setting. Results similar to those in Section 3 appear in [**Sch86**] and [**Den97**] but it is rather difficult to compare them directly since there are essential differences in their basic settings. Indeed, the most important point of dissimilarity is that we always work with a particular set of continuous functions over a compact Hausdorff space. The tools and techniques of random set theory and Choquet integration [**Mol05**] lie at the heart of the proof of Theorem 3.5. In particular, the fundamental Choquet-Matheron-Kendall theorem [**Mol05**, Theorem 1.13] is used to establish the assertion of Theorem 3.5(ii).

In Section 4 we will show that the information provided by the generalized Möbius transform can be used for recovering elements of cores of games on MV-algebras by a Cimmino-type algorithm. The core of a game (Definition 4.1) is one of the basic solution concepts in coalition game theory. In [**BK09**], Butnariu and the author designed an iterative procedure to recover the core elements. This procedure is based on the Cimmino-type projection algorithm introduced by Butnariu and Shklyar in [**BS08**]. The so-called coalitional assessment of a given game (Definition 4.2), which is a crucial input parameter of the procedure, can be easily derived from the generalized Möbius transform of a particular class of games (Theorem 4.4).

2. Preliminaries

Basic definitions and results concerning MV-algebras are recalled in Section 2.1. Measures on MV-algebras will be briefly introduced in Section 2.2. The interested reader is referred to the book [**CDM00**] and Chapter 22 in [**BW02**] for further details.

2.1. MV-algebras.

DEFINITION 2.1. An *MV-algebra* is an algebra

$$\langle M, \oplus, \neg, 0 \rangle$$

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with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0 such that $\langle M, \oplus, 0\rangle$ is an abelian monoid and the following equations hold true for every $a, b \in M$:

$$\neg\neg a = a,$$
$$a \oplus \neg 0 = \neg 0,$$
$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$$

On every MV-algebra M, we define

$$1 = \neg 0,$$
$$a \odot b = \neg(\neg a \oplus \neg b).$$

For any two elements $a, b \in M$, we write $a \leq b$ if $\neg a \oplus b = 1$. The relation $\leq$ is in fact a partial order. Further, the operations $\vee, \wedge$ defined by

$$a \vee b = \neg(\neg a \oplus b) \oplus b,$$
$$a \wedge b = \neg(\neg a \vee \neg b),$$

respectively, make the algebraic structure $\langle M, \wedge, \vee, 0, 1\rangle$ into a distributive lattice with bottom element 0 and top element 1.

EXAMPLE 2.2 (Algebra of sets). *Every Boolean algebra $\mathcal{A}$ of subsets of a set X is an MV-algebra in which $\oplus = \vee = \cup$, $\odot = \wedge = \cap$, $\neg$ is the set complement $^{\complement}$, and $0 = \emptyset$, $1 = X$.*

EXAMPLE 2.3 (Standard MV-algebra). *The basic example of an MV-algebra is the* standard MV-algebra, *which is the real unit interval* $[0, 1]$ *equipped with operations*

$$a \oplus b = \min(1, a + b) \quad \textit{and} \quad \neg a = 1 - a.$$

This implies

$$a \odot b = \max(0, a + b - 1)$$

by the definition of the operation $\odot$. The partial order $\leq$ of the standard MV-algebra coincides with the usual order of reals from the unit interval $[0, 1]$. *The operations $\odot, \oplus$ are also called the* Łukasiewicz t-norm *and the* Łukasiewicz t-conorm [**BK93**], *respectively.*

The set $[0, 1]^X$ of all functions $X \to [0, 1]$ becomes an MV-algebra if the operations $\oplus, \neg$ and the element 0 are defined pointwise. The corresponding lattice operations $\vee, \wedge$ are then the pointwise maximum and the pointwise minimum of two functions $X \to [0, 1]$, respectively.

DEFINITION 2.4. Let X be a nonempty set. A *clan* over X is a collection M_X of functions $X \to [0, 1]$ such that the zero function 0 is in M_X and the following conditions are satisfied:

(i) if $a \in M_X$, then $\neg a \in M_X$,
(ii) if $a, b \in M_X$, then $a \oplus b \in M_X$.

In particular, a clan M_X contains the constant function 1 and it is closed with respect to the operation $\odot$ and thus every clan is an MV-algebra. Interestingly, most mathematical structures for modeling coalitions of players are captured by clans. Suppose that X is a set of players. If X is finite, then the algebra $\mathcal{A} = 2^X$ of all subsets of X is the classical framework used in coalition game theory — see, for instance, [**PS07**]. This setting can be further generalized in a few directions.

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The coalition structures in the form of the MV-algebra $[0,1]^X$ with X finite were investigated by Aubin [**Aub74**] under the name "games with fuzzy coalitions". Relaxing the finiteness assumption, Aumann and Shapley [**AS74**] investigated games on the Borel measurable subsets of the player set $X = [0,1]$. They also came up with the idea of many-valued coalitions under the name of "ideal coalitions", which they identified with Borel measurable functions $X \to [0,1]$. Butnariu and Klement [**BK93**] focused on games defined on so-called tribes. A *tribe* over X is a clan M_X closed with respect to countable (pointwise) suprema of its elements:

$$\text{if } (a_n) \in M_X^{\mathbb{N}}, \text{ then } \bigvee_{n=1}^{\infty} a_n \in M_X.$$

In this contribution, we will focus on the class of so-called semisimple MV-algebras, which subsumes all the algebraic structures mentioned above for modeling coalitions.

Let M be an MV-algebra. A *filter* in M is a subset F of M such that

(i) $1 \in F$,
(ii) if $a, b \in F$, then $a \odot b \in F$,
(iii) if $a \in F$ and $a \leq b \in M$, then $b \in F$.

A filter F in M is *proper* if $F \neq M$. We say that a proper filter is *maximal* whenever it is not strictly included in any proper filter. Let X_M be the set of all maximal filters in M. It can be shown that $X_M \neq \emptyset$. The set X_M can be endowed with a topology whose family of closed sets is given by all sets $C_F = \{F' \in X_M \mid F' \supseteq F\}$, where F is a filter in M. Then the space X_M becomes compact and Hausdorff.

An MV-algebra M is called *semisimple* (cf. [**CDM00**, Chapter 3.6]) if

$$\bigcap \{ F \mid F \in X_M \} = \{1\}.$$

A clan M_X of functions $X \to [0,1]$ is *separating* whenever for every $x, y \in X$ with $x \neq y$, there exists a function $a \in M_X$ such that $a(x) \neq a(y)$. Every semisimple MV-algebra has a convenient representation by a separating clan of continuous functions over some compact Hausdorff space: this is the assertion of Theorem 2.5 below. See [**CDM00**, Chapter 3] for details and the proof.

If M_1, M_2 are MV-algebras, then a mapping $h : M_1 \to M_2$ is an *isomorphism* provided h is a bijection that preserves the operations $\oplus, \neg$ and the constant 0.

THEOREM 2.5. *The following assertions are equivalent for any MV-algebra M:*

(i) *M is semisimple,*
(ii) *M is isomorphic to a separating clan of continuous $[0,1]$-valued functions over the compact Hausdorff space X_M,*
(iii) *M is isomorphic to a separating clan of continuous $[0,1]$-valued functions over a compact Hausdorff space X,*
(iv) *M is isomorphic to a clan of $[0,1]$-valued functions over a set X.*

Without loss of generality, every semisimple MV-algebra can be thought of as a separating clan M_X of continuous functions over a compact Hausdorff space X and vice versa. Thus semisimple MV-algebras can be viewed as common generalizations of coalition structures provided that the set of players is identified with the compact Hausdorff space X. Equipping the set of players X with a topology is not unusual in game theory. For instance, Aumman and Shapley [**AS74**] assume that X is a Polish space or the real unit interval $[0,1]$. If $a \in M_X$ is a coalition and $x \in X$

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is a player, then the continuity of a means that the degrees of membership $a(y)$ of the players $y \in X$ do not change abruptly when y is sufficiently close to x.

Let M_X be a separating clan of continuous functions over a compact Hausdorff space X. There exists a one-to-one correspondence between certain filters in M_X and closed subsets of X. For every set $A \subseteq X$, the subset of M_X given by

$$F_A = \{ a \in M_X \mid a(x) = 1 \} \tag{2.1}$$

is a filter in M_X. In particular, $F_\emptyset = M_X, F_X = \{1\}$, and the filter $F_{\{x\}}$ is maximal for every $x \in X$. Conversely, a closed subset V_F of X can be assigned to every filter F in M_X by putting

$$V_F = \bigcap \left\{ a^{-1}(1) \;\middle|\; a \in F \right\},$$

since every function $a \in F$ is continuous.

THEOREM 2.6. *Let M_X be a separating clan of continuous functions over a compact Hausdorff space X.*

(i) *The mapping $x \in X \mapsto F_{\{x\}}$ is a one-to-one correspondence between X and the set of all maximal filters in M_X.*
(ii) *If $A \subseteq X$ is closed, then $A = V_{F_A}$.*
(iii) *If F is a proper filter that is an intersection of all maximal filters containing F, then $F = F_{V_F}$.*

See [**CDM00**, Chapter 3.4] for the proof and examples of filters that are not intersections of maximal filters. Theorem 2.6(iii) establishes a one-to-one order-reversing correspondence between the set of all nonempty closed subsets of X and the set of all proper filters in M_X that are intersections of maximal filters. This fact is crucial for developing a faithful generalization of the Möbius transform in Section 3.2.

2.2. Measures on MV-algebras. Throughout this section we assume that M_X is a separating clan of continuous functions over a compact Hausdorff space X. We think of the clan M_X as the set of all possible coalitions arising in a game with the player set X. Plausible solutions to the game are conceived as real functionals on M_X that are additive with respect to the operation $\oplus$. Particular classes of such functionals on MV-algebras are known as measures [**BW02**] and states [**RM02**]. Our terminology is similar to that used in [**BW02**].

DEFINITION 2.7. A *measure* on M_X is a function $m : M_X \to \mathbb{R}$ such that $m(0) = 0$ and

$$m(a \oplus b) = m(a) + m(b) \tag{2.2}$$

for every $a, b \in M_X$ with $a \odot b = 0$. We say that a measure m is *bounded* if

$$\sup \{ |m(a)| \mid a \in M \} < \infty$$

and m is *nonnegative* when $m(a) \geq 0$ for every $a \in M_X$.

In particular, every nonnegative measure m is bounded since there is a nonnegative real α and a nonnegative measure s with $s(1) = 1$ such that $m = \alpha s$. The condition (2.2) is a kind of additivity since

$$a \odot b = 0 \quad \text{if and only if} \quad a + b \leq 1, \tag{2.3}$$

where $+$ is the pointwise sum of real functions. Every distribution of profit in a coalition game with the coalition set M_X is viewed as a bounded measure on M_X

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and vice versa. This is natural since the profit $m(a \oplus b)$ is split into the sum of terms $m(a)$ and $m(b)$ corresponding to the incompatible coalitions $a, b \in M_X$ in the sense of (2.3).

Although a bounded measure on M_X quantifies rather a potential profit of coalitions than that of players, Theorems 2.8 and 2.10 below guarantee that every distribution of profit among all the coalitions induces a unique distribution of wealth among all the players. Moreover, the profit distributed in this way to each coalition $a \in M_X$ is precisely the "mean value" of the profits assigned to the individual players with weights given by the membership degrees of all the players participating in a. Let $\mathfrak{B}(X)$ be the σ-algebra of all Borel subsets of X. The term "measure on $\mathfrak{B}(X)$" stands for "σ-additive real-valued function on $\mathfrak{B}(X)$". We will need the following integral representation theorem arising from [**Kro06**, Theorem 28] or [**Pan08**, Proposition 1.1].

THEOREM 2.8. *The set of all nonnegative measures on M_X is in a one-to-one correspondence with the set of all nonnegative regular Borel measures on $\mathfrak{B}(X)$ via the mapping $\mu \mapsto m_\mu$, where μ is a nonnegative regular Borel measure and*

$$m_\mu(a) = \int_X a \, \mathrm{d}\mu, \quad a \in M_X.$$

Bounded measures on M_X possess the following Jordan decomposition property — see [**BW02**, Theorem 3.1.3].

THEOREM 2.9. *Let m be a bounded measure on M_X. For every $a \in M_X$, put*

$$m^+(a) = \sup \{ m(b) \mid b \le a, \, b \in M_X \},$$
$$m^-(a) = m^+(a) - m(a).$$

Then m^+, m^- are nonnegative measures on M_X such that

$$m = m^+ - m^-.$$

Let μ be a regular Borel measure. Then the classical Jordan decomposition of the Borel measure μ makes it possible to write $\mu = \mu^1 - \mu^2$ for two nonnegative regular Borel measures μ^1, μ^2. The integral

$$\int_X a \, \mathrm{d}\mu = \int_X a \, \mathrm{d}\mu^1 - \int_X a \, \mathrm{d}\mu^2, \quad a \in M_X$$

is denoted by $m_\mu(a)$.

THEOREM 2.10. *The set of all bounded measures on M_X is in a one-to-one correspondence with the set of all regular Borel measures on $\mathfrak{B}(X)$.*

PROOF. If μ is a regular Borel measure on $\mathfrak{B}(X)$, then m_μ is a measure on M_X. It is also bounded, since m_μ can be extended to a bounded linear functional on the Banach space of all continuous functions over X with the supremum norm. Conversely, let m be a bounded measure on M_X. Then Theorem 2.9 gives a pair of nonnegative measures m^+, m^- with $m = m^+ - m^-$ and Theorem 2.8 says that $m^+ = m_{\mu^1}, m^- = m_{\mu^2}$ for the unique nonnegative regular Borel measures μ^1, μ^2. Setting $\mu = \mu^1 - \mu^2$ shows that $m = m_\mu$. So we need only to check that μ is the unique measure with this property. Let ν be a regular Borel measure with $m_\nu = m$ and ν^1, ν^2 be nonnegative regular Borel measures such that $\nu = \nu^1 - \nu^2$. Then the equality $m_{\nu^1 - \nu^2} = m_{\mu^1 - \mu^2}$ can be expressed as $m_{\nu^1 + \mu^2} = m_{\mu^1 + \nu^2}$. Since both

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$\nu^1+\mu^2$ and $\mu^1+\nu^2$ are nonnegative, it follows by Theorem 2.8 that $\nu^1+\mu^2=\mu^1+\nu^2$. Hence $\nu=\nu^1-\nu^2=\mu^1-\mu^2=\mu$. □

3. Games on MV-algebras

The set of all plausible coalitions is represented by a separating clan M_X of continuous functions over X. It was emphasized in Section 2.1 that this assumption captures most structures used for modeling coalitions of players. Each coalition $a \in M_X$ in a game is assigned a real number, the worth of a. This assignment then determines the coalition game.

Definition 3.1. A *game* on M_X is a function

$$v : M_X \to \mathbb{R}$$

satisfying $v(0)=0$ and $\sup\{\, |v(a)| \mid a \in M_X \,\} < \infty$.

The number $v(a)$ is the total worth generated by the players in the coalition $a \in M_X$ as a result of their cooperation. The main objective in coalition game theory is to find a final distribution of the profit among the players, which depends only on the results of cooperation a priori captured by the function v. In Section 2.2, we identified each such profit distribution with a bounded measure on M_X.

Hence we can formalize the task of "solving" a coalition game as follows. Let Γ be a class of games on M_X. A *solution* on Γ is a mapping σ sending every $v \in \Gamma$ to a set $\sigma(v)$ of bounded measures on M_X. Various assumptions of economic and behavioral rationality lead to different solutions σ. The core is one of the most important solution concepts in coalition game theory [**PS07**, Section 3]. This solution will be further discussed in Section 4 (Definition 4.1), in which we apply the generalized Möbius transform, developed in the next section, to a class of games on the clan over a finite player set. The underlying idea is that an alternative representation of the game v is convenient for understanding the structure of the solution set $\sigma(v)$ or for enhancing the computations with the solution set. Namely, Möbius transform is frequently used for the representation of games on the set of all subsets of a finite set [**PS07**, Section 8.1]. In the next section, we are going to generalize the Möbius transform to semisimple MV-algebras. The center of our interest is to find an appropriate algebra on which such generalized Möbius transform "lives".

3.1. Möbius Transform. In his fundamental paper [**Rot64**], Rota introduced the Möbius inversion formula for any locally finite partially ordered set. His approach unified the classical inclusion-exclusion principle, the number theoretic Möbius inversion, and some graph problems. For the purposes of this paper, we will confine our discussion to the algebra 2^X of all subsets of a finite set X. The chapter [**PS07**, Section 8] or the paper [**DK00**] show the important role which Möbius transform plays in cooperative game theory.

Let v be a game on the clan 2^X, where $X=\{1,\dots,n\}$. The *Möbius transform* of v is the only solution

$$\mathbf{m} : 2^X \to \mathbb{R}$$

of the equation

$$v(A)=\sum_{B\subseteq A}\mathbf{m}(B), \quad \text{for each } A \in 2^X. \tag{3.1}$$

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We will denote the Möbius transform of v by $\mathbf{m}_v$. The function $\mathbf{m}_v$ can be directly recovered from v as

$$\mathbf{m}_v(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} v(B), \quad \text{for each } A \in 2^X.$$

Vice versa, let $\mathbf{m} : 2^X \to \mathbb{R}$ be such that $\mathbf{m}(\emptyset) = 0$. Put

$$v_{\mathbf{m}}(A) = \sum_{B \subseteq A} \mathbf{m}(B), \quad \text{for each } A \in 2^X.$$

Then it follows that

$$\mathbf{m}_{v_{\mathbf{m}}} = \mathbf{m}.$$

Observe that the Möbius transform $\mathbf{m}_v$ of any game v determines a unique (finitely-additive) measure $\boldsymbol{\mu}_v$ on 2^{2^X} by setting

$$\boldsymbol{\mu}_v(\mathcal{A}) = \sum_{A \in \mathcal{A}} \mathbf{m}_v(A), \quad \text{for each } \mathcal{A} \in 2^{2^X}.$$

On the other hand, every measure $\boldsymbol{\mu} : 2^{2^X} \to \mathbb{R}$ such that $\boldsymbol{\mu}(\{\emptyset\}) = 0$ gives rise to a unique mapping $\mathbf{m} : 2^X \to \mathbb{R}$ with $\mathbf{m}(\emptyset) = 0$ by putting $\mathbf{m}(A) = \boldsymbol{\mu}(\{A\})$, for each $A \in 2^X$. Hence the set of all possible Möbius transforms can be identified with the set of all measures on 2^{2^X} supported by a subset of $2^X \setminus \{\emptyset\}$. Note that the set $2^X \setminus \{\emptyset\}$ is in a one-to-one correspondence with the proper filters in the algebra 2^X via the mapping

$$B \in 2^X \setminus \{\emptyset\} \mapsto \left\{ A \in 2^X \;\middle|\; B \subseteq A \right\}.$$

The set

$$\left\{ B \in 2^X \setminus \{\emptyset\} \;\middle|\; B \subseteq A \right\} \tag{3.2}$$

used in the summation (3.1) can be interpreted as the set of those proper filters in 2^X to which $A \in 2^X$ belongs. In the next section, we will first establish a genuine MV-algebraic generalization of (3.2).

3.2. Generalized Möbius Transform. Let M_X be a separating clan of continuous functions over a compact Hausdorff space X. By $\mathcal{K}$ we denote the set of all compact subsets of X. For every $a \in M_X$ and every $A \in \mathcal{K}$, define

$$\rho_a(A) = \begin{cases} \inf \{ a(x) \mid x \in A \}, & A \neq \emptyset, \\ 1, & A = \emptyset. \end{cases} \tag{3.3}$$

In particular, if $M_X = 2^X$ with X finite, then

$$\rho_A(B) = \begin{cases} 1, & B \subseteq A, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for each } A, B \in 2^X.$$

So (3.2) is just a special case of (3.3) since

$$\left\{ B \in 2^X \;\middle|\; B \subseteq A \right\} = \left\{ B \in 2^X \;\middle|\; \rho_A(B) = 1 \right\}.$$

It is useful to think of ρ_a as a continuation of function $a \in M_X$ from X to set $\mathcal{K}$. Such an interpretation is imaginable since $\rho_a(\{x\}) = a(x)$ for every $x \in X$. Moreover, the function ρ_a will become a continuous extension of a once we introduce a suitable topology on $\mathcal{K}$. In general, a rich variety of topologies on $\mathcal{K}$ can be induced from the topology of the underlying space X. However, the additional assumption of second-countability of the space X is later required for the existence of a generalized

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Möbius transform (Theorems 3.5 and 3.6). Under this assumption, the Urysohn metrization theorem [**Kel55**, p. 125, Theorem 16] implies the following.

THEOREM 3.2. *Let X be a compact Hausdorff space. If X is second-countable, then X is metrizable.*

Consequently, the most natural choice is to endow the set $\mathcal{K}$ of all compact subsets of a compact Hausdorff second-countable space X with the topology given by Hausdorff distance [**CV77**, Chapter II.1]. Specifically, let d be any metric compatible with the topology of X. Define

$$d(x, A) = \inf \{ d(x, y) \mid y \in A \}, \quad x \in X, A \in \mathcal{K},$$

and

$$e(A, B) = \sup \{ d(x, B) \mid x \in A \}, \quad A, B \in \mathcal{K}.$$

The *Hausdorff distance* H_d on $\mathcal{K}$ is given by

$$H_d(A, B) = \max \{ e(A, B), e(B, A) \}, \quad A, B \in \mathcal{K}.$$

The function H_d makes $\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$ into a metric space but it is only an extended metric on $\mathcal{K}$ as $H_d(A, \emptyset) = \infty$ for every $A \in \mathcal{K}'$. The topology τ_d generated by H_d on $\mathcal{K}$ will be called the *Hausdorff metric topology*. It follows from [**CV77**, Corollary II-7] that $\tau_d = \tau_{d'}$ whenever d and d' are equivalent metrics inducing the topology on X. Thus we may simply refer to "the Hausdorff metric topology" without explicitly mentioning the underlying metric on X. We will make ample use of the following properties of the Hausdorff metric topology — see [**Mol05**, Appendix B-C].

PROPOSITION 3.1. *If X is a compact Hausdorff second-countable space, then the Hausdorff metric topology on $\mathcal{K}$ has the following properties:*

(i) *Both spaces $\mathcal{K}$ and $\mathcal{K}'$ are compact Hausdorff, and the point $\emptyset$ is isolated in $\mathcal{K}$.*
(ii) *The subspace $\{ \{x\} \mid x \in X \}$ of $\mathcal{K}$ is homeomorphic to X.*
(iii) *For each open $G \subseteq X$, the sets $\{ A \in \mathcal{K} \mid A \subseteq G \}$ and $\{A \in \mathcal{K} | A \cap G \neq \emptyset\}$ are open.*
(iv) *The Borel σ-algebra $\mathfrak{B}(\mathcal{K})$ on $\mathcal{K}$ is generated by the sets $\{A \in \mathcal{K} | A \cap G \neq \emptyset\}$ for all open $G \subseteq X$.*

In the sections that follow, we denote by $C_{\mathcal{K}}$ the clan of all continuous (in the Hausdorff metric topology) functions $\mathcal{K} \to [0, 1]$. The clan $C_{\mathcal{K}'}$ is defined analogously. It turns out that the mapping $\rho : a \in M_X \mapsto \rho_a \in [0, 1]^{\mathcal{K}}$ is into $C_{\mathcal{K}}$.

PROPOSITION 3.2. *Let M_X be a separating clan of continuous functions over a compact Hausdorff second-countable space. For every $a, b \in M_X$:*

(i) $\rho_{a \wedge b} = \rho_a \wedge \rho_b$,
(ii) $\rho_a \in C_{\mathcal{K}}$.

PROOF. The first equality is obvious. The restriction of ρ_a to $\mathcal{K}'$ is continuous due to [**Bee93**, Exercise 13, p. 145]. Since the point $\emptyset$ is isolated in $\mathcal{K}$, function ρ_a is continuous on the whole set $\mathcal{K}$. $\square$

In particular, ρ indeed extends a continuous function $a \in M_X$ to the continuous function $\rho_a \in C_{\mathcal{K}}$. For every $A \in \mathcal{K}$, the number $\rho_a(A) \in [0, 1]$ can be viewed as

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a "degree" to which $a \in M_X$ belongs to the filter F_A generated by the compact set A (see (2.1) together with Theorem 2.6). Specifically, this means that

$$\rho_a(A) = 1 \quad \text{if and only if} \quad a \in F_A.$$

The restriction of ρ_a to $C_{\mathcal{K}'}$ is denoted also by ρ_a.

We obtain a natural generalization of Möbius transform once we replace a measure $\boldsymbol{\mu}$ on $2^{2^X}\setminus\{\emptyset\}$ from Section 2.2 by a bounded measure $\mathbf{m}$ on $C_{\mathcal{K}'}$. Hence we focus on the class of games v on M_X for which

$$v(a) = \mathbf{m}(\rho_a), \qquad \text{for every } a \in M_X. \tag{3.4}$$

Only the games v satisfying (3.4) will have the generalized Möbius transform. If X is finite and $M_X = 2^X$, then $C_{\mathcal{K}'} = 2^{2^X}\setminus\{\emptyset\}$, so the equality (3.1) is recovered as a special case. In order to characterize the class of games v given by (3.4), we need the basic concepts of the theory of capacities and Choquet integral theory — see [**Mol05**, Chapter 1] for details.

A *capacity* on $\mathcal{K}$ is a function $\beta : \mathcal{K} \to \mathbb{R}$ such that

(i) $\beta(\emptyset) = 0$,

(ii) if $(A_n) \in \mathcal{K}^{\mathbb{N}}$ is non-increasing, then $\beta\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \beta(A_n)$,

(iii) $\sup\{\, |\beta(A)| \mid A \in \mathcal{K} \,\} < \infty$.

A capacity β on $\mathcal{K}$ is called *totally monotone* when β is monotone and the following inequality holds true for each $n \geq 2$ and every $A_1, \ldots, A_n \in \mathcal{K}$:

$$\beta\Big(\bigcup_{i=1}^{n} A_i\Big) \geq \sum_{\substack{I\subseteq\{1,\ldots n\}\\ I\neq\emptyset}} (-1)^{|I|+1} \beta\Big(\bigcap_{i\in I} A_i\Big).$$

Example 3.3. *If $A \in \mathcal{K}'$, then the function $\delta_A : \mathcal{K} \to \{0,1\}$ defined as*

$$\delta_A(B) = \begin{cases} 1, & A \subseteq B, \\ 0, & \textit{otherwise}, \end{cases} \qquad B \in \mathcal{K}, \tag{3.5}$$

is a totally monotone capacity [**Den97**, Proposition 1.2].

Let M_X be a clan of continuous functions over a compact Hausdorff space X. If β is a monotone capacity on $\mathcal{K}$, then the *Choquet integral* of a function $a \in M_X$ over X with respect to β is given by

$$\oint a \,\mathrm{d}\beta = \int_0^1 \beta(a^{-1}([t,1]))\,\mathrm{d}t. \tag{3.6}$$

The Riemann integral on the right-hand side of (3.6) exists since the function

$$t \in [0,1] \mapsto \beta(a^{-1}([t,1])) \in [0,\infty)$$

is well-defined (that is, $a^{-1}([t,1]) \in \mathcal{K}$) and non-increasing.

Example 3.4. *In light of Example 3.3, observe that for every $A \in \mathcal{K}'$ and every $a \in M_X$,*

$$\oint a \,\mathrm{d}\delta_A = \rho_a(A). \tag{3.7}$$

We introduce the following notations:

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$\mathbf{M}^+(C_{\mathcal{K}'})$	the set of all nonnegative measures on the MV-algebra $C_{\mathcal{K}'}$,
CAP_∞	the set of all totally monotone capacities on $\mathcal{K}$,
$\mathfrak{B}(\mathcal{K})$	the Borel σ-algebra generated by the Hausdorff metric topology on $\mathcal{K}$,
$\mathfrak{B}(\mathcal{K}')$	the Borel σ-algebra generated by the Hausdorff metric topology on $\mathcal{K}'$.

THEOREM 3.5. *Let M_X be a separating clan of continuous functions over a compact Hausdorff second-countable space X.*

(i) *There is an injective maping $\mathbf{m} \in \mathbf{M}^+(C_{\mathcal{K}'}) \mapsto \beta_{\mathbf{m}} \in \mathrm{CAP}_\infty$ satisfying*

$$\mathbf{m}(\rho_a) = \oint a \,\mathrm{d}\beta_{\mathbf{m}}, \quad \textit{for every } a \in M_X. \tag{3.8}$$

(ii) *There is a mapping $\beta \in \mathrm{CAP}_\infty \mapsto \mathbf{m}_\beta \in \mathbf{M}^+(C_{\mathcal{K}'})$ such that*

$$\oint a \,\mathrm{d}\beta = \mathbf{m}_\beta(\rho_a), \quad \textit{for every } a \in M_X. \tag{3.9}$$

(iii) *If $\mathbf{m} \in \mathbf{M}^+(C_{\mathcal{K}'})$, then $\mathbf{m}_{\beta_{\mathbf{m}}} = \mathbf{m}$.*

PROOF. (i) Let $\mathbf{m} \in \mathbf{M}^+(C_{\mathcal{K}'})$. Then Theorem 2.8 yields a unique nonnegative regular Borel measure $\boldsymbol{\mu}_{\mathbf{m}}$ on $\mathfrak{B}(\mathcal{K}')$ such that

$$\mathbf{m}(f) = \int_{\mathcal{K}'} f(A) \,\mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A), \quad f \in C_{\mathcal{K}'}.$$

We may think of $\boldsymbol{\mu}_{\mathbf{m}}$ as a Borel measure on $\mathfrak{B}(\mathcal{K})$ supported by a subset of $\mathcal{K}'$. Put

$$\beta_{\mathbf{m}}(A) = \boldsymbol{\mu}_{\mathbf{m}}\left(\{\, B \in \mathcal{K} \mid B \subseteq A \,\}\right), \quad A \in \mathcal{K}. \tag{3.10}$$

The function $\beta_{\mathbf{m}}$ is well-defined since the set

$$\{\, B \in \mathcal{K} \mid B \subseteq A \,\}$$

is closed for every $A \in \mathcal{K}$ as a consequence of Proposition 3.1(iii). It can be routinely checked that $\beta_{\mathbf{m}} \in \mathrm{CAP}_\infty$. The mapping $\mathbf{m} \mapsto \beta_{\mathbf{m}}$ is injective: Theorem 2.8 says that $\mathbf{m} \to \boldsymbol{\mu}_{\mathbf{m}}$ is a one-to-one correspondence and the mapping $\boldsymbol{\mu}_{\mathbf{m}} \mapsto \beta_{\mathbf{m}}$ given by (3.10) is injective due to Proposition 3.1(iv). We will show that the equality (3.8) holds true. For every $a \in M_X$:

$$\begin{aligned} \mathbf{m}(\rho_a) &= \int_{\mathcal{K}'} \rho_a(A) \,\mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A) = \int_{\mathcal{K}'} \oint a \,\mathrm{d}\delta_A \,\mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A) \\ &= \int_{\mathcal{K}'} \int_0^1 \delta_A(a^{-1}([t,1])) \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A), \end{aligned} \tag{3.11}$$

where the second equality follows from (3.7) and the third from the definition of the Choquet integral. We will show that the function

$$b : \mathcal{K}' \times [0,1] \to \{0,1\}$$

defined by

$$b(A,t) = \delta_A(a^{-1}([t,1])), \quad A \in \mathcal{K}',\ t \in [0,1],$$

is measurable with respect to the product σ-algebra $\mathfrak{B}(\mathcal{K}') \times \mathfrak{B}([0,1])$. It is enough to verify that the set

$$\{\, (A,t) \in \mathcal{K}' \times [0,1] \mid b(A,t) = 1 \,\} = \left\{\, (A,t) \in \mathcal{K}' \times [0,1] \;\middle|\; A \subseteq a^{-1}([t,1]) \,\right\}$$

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belongs to $\mathfrak{B}(\mathcal{K}') \times \mathfrak{B}([0,1])$. Let $\mathcal{G}_0$ be a countable base for the topology of X. Then it follows that

$$\begin{aligned}
&\left\{ (A,t) \in \mathcal{K}' \times [0,1] \;\middle|\; A \subseteq a^{-1}([t,1]) \right\} \\
&= \bigcup_{G \in \mathcal{G}_0} \left\{ (A,t) \in \mathcal{K}' \times [0,1] \;\middle|\; G \cap A = \emptyset,\, G \supseteq a^{-1}([0,t)) \right\} \\
&= \bigcup_{G \in \mathcal{G}_0} \Big(\{ A \in \mathcal{K}' \mid A \cap G = \emptyset \} \times \big[0, \sup\{ a(x) \mid x \in G \}\big] \Big) \in \mathfrak{B}(\mathcal{K}') \times \mathfrak{B}([0,1]).
\end{aligned}$$

Hence b is measurable, so the Fubini theorem can be applied to the last integral in (3.11). This gives

$$\int_{\mathcal{K}'} \int_0^1 \delta_A(a^{-1}([t,1]))\, \mathrm{d}t\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A) = \int_0^1 \int_{\mathcal{K}'} \delta_A(a^{-1}([t,1]))\, \mathrm{d}\boldsymbol{\mu}_{\mathbf{m}}(A)\, \mathrm{d}t. \tag{3.12}$$

Observe that, for every $t \in [0,1]$,

$$\left\{ A \in \mathcal{K}' \;\middle|\; \delta_A(a^{-1}([t,1])) = 1 \right\} = \left\{ A \in \mathcal{K}' \;\middle|\; A \subseteq a^{-1}([t,1]) \right\}.$$

Therefore the Riemann integral on the right-hand side of (3.12) becomes

(3.13)
$$\int_0^1 \boldsymbol{\mu}_{\mathbf{m}} \left(\left\{ A \in \mathcal{K}' \;\middle|\; A \subseteq a^{-1}([t,1]) \right\} \right) \mathrm{d}t = \int_0^1 \beta_{\mathbf{m}}(a^{-1}([t,1]))\, \mathrm{d}t = \oint a \, \mathrm{d}\beta_{\mathbf{m}},$$

where the first equality is a consequence of (3.10) and the second one follows from the definition of the Choquet integral. This finishes the proof of (i).

(ii) Let $\beta \in \mathrm{CAP}_\infty$. If $\beta(X) = 0$, then put $\mathbf{m}_\beta = 0$ and observe that (3.9) is satisfied. Otherwise define $\gamma : \mathcal{K} \to [0, \infty)$ as

$$\gamma = \frac{\beta}{\beta(X)}.$$

This implies $\gamma \in \mathrm{CAP}_\infty$. In particular, γ takes on values in $[0,1]$ and $\gamma(X) = 1$. A dual version of the Choquet theorem [**Mol05**, Theorem 1.13] yields a unique regular Borel probability measure $\boldsymbol{\mu}_\gamma$ on $\mathfrak{B}(\mathcal{K})$ satisfying

$$\boldsymbol{\mu}_\gamma (\{ B \in \mathcal{K} \mid B \subseteq A \}) = \gamma(A), \quad \text{for every } A \in \mathcal{K}.$$

Note that $\boldsymbol{\mu}_\gamma$ is necessarily supported by a subset of $\mathcal{K}'$. Setting $\boldsymbol{\mu}_\beta = \beta(X)\boldsymbol{\mu}_\gamma$, we get

$$\boldsymbol{\mu}_\beta (\{ B \in \mathcal{K} \mid B \subseteq A \}) = \beta(A), \quad \text{for every } A \in \mathcal{K}. \tag{3.14}$$

Let

$$\mathbf{m}_\beta(f) = \int_{\mathcal{K}'} f(A)\, \mathrm{d}\boldsymbol{\mu}_\beta(A), \quad \text{for every } f \in C_{\mathcal{K}'}.$$

Proceeding in the same way as in (3.11)-(3.13), we come to

$$\mathbf{m}_\beta(\rho_a) = \oint a \, \mathrm{d}\beta, \quad \text{for every } a \in M_X.$$

This concludes the proof of (ii).

(iii) We must show that the Borel measures corresponding to $\mathbf{m}$ and $\mathbf{m}_{\beta_{\mathbf{m}}}$ via Theorem 2.8, respectively, are identical. Due to Proposition 3.2(iv), we need only to check that the two representing Borel measures agree on each set

$$\{ B \in \mathcal{K} \mid B \subseteq A \}, \quad A \in \mathcal{K},$$

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since this is a set of generators for $\mathfrak{B}(\mathcal{K})$. But this follows directly by combining (3.10) with (3.14). $\square$

The canonical example of a second-countable player set X in game theory is the continuum of players $[0,1]$, which was previously used in [**AS74, BK93**]. On the one hand, the second-countability assumption is not needed to apply the Choquet theorem in the proof above: its more general version proved recently in [**Ter10**] relaxes this assumption. On the other hand, this assumption made it possible to check the measurability of the function b in the proof of Theorem 3.5(i). The transformation of the integral from (3.10) to (3.13) can be formulated in the theory of random sets as a so-called Robbins' theorem [**Mol05**].

Now we are going to extend our investigations to the whole set of bounded measures on $C_{\mathcal{K}'}$. Further notations:

$\mathbf{M}(C_{\mathcal{K}'})$	the set of all bounded measures on the MV-algebra $C_{\mathcal{K}'}$,
CAP	$= \{\, \beta^1 - \beta^2 \mid \beta^1, \beta^2 \in \mathrm{CAP}_\infty \,\}$.

If $\alpha, \beta \in \mathrm{CAP}_\infty$, then it follows from the definition of the Choquet integral (3.6) that

$$\oint a \,\mathrm{d}(\alpha+\beta) = \oint a \,\mathrm{d}\alpha + \oint a \,\mathrm{d}\beta, \quad a \in M_X. \tag{3.15}$$

Let $\beta \in \mathrm{CAP}$. Due to the (3.15), Choquet integral of $a \in M_X$ with respect to β can be unambiguously defined as

$$\oint a \,\mathrm{d}\beta = \oint a \,\mathrm{d}\beta^1 - \oint a \,\mathrm{d}\beta^2, \tag{3.16}$$

where $\beta = \beta^1 - \beta^2$ for $\beta^1, \beta^2 \in \mathrm{CAP}_\infty$. It is elementary to check that the bijection established in Theorem 3.5 can be extended to $\mathbf{M}(C_{\mathcal{K}'})$ and CAP by using (3.16) together with Theorem 2.10. Hence we obtain the final result.

THEOREM 3.6. *Let M_X be a separating clan of continuous functions over a compact Hausdorff second-countable space X. Then there exists a one-to-one correspondence $\mathbf{m} \mapsto \beta_{\mathbf{m}}$ between $\mathbf{M}(C_{\mathcal{K}'})$ and* CAP *such that*

$$\mathbf{m}(\rho_a) = \oint a \,\mathrm{d}\beta_{\mathbf{m}}, \quad \textit{for every } a \in M_X. \tag{3.17}$$

Theorem 3.6 answers the question which games are of the form (3.4). So the next definition makes sense.

DEFINITION 3.7. Let M_X be a separating clan of continuous functions over a compact Hausdorff second-countable space X. Let $\beta \in \mathrm{CAP}$, and v be a game on M_X such that

$$v(a) = \oint a \,\mathrm{d}\beta, \quad \text{for every } a \in M_X.$$

The *generalized Möbius transform* of v is the unique $\mathbf{m}_\beta \in \mathbf{M}(C_{\mathcal{K}'})$ satisfying

$$v(a) = \mathbf{m}_\beta(\rho_a), \quad \text{for every } a \in M_X. \tag{3.18}$$

Thus a game v has a generalized Möbius transform if and only if it arises as the Choquet integral with respect to a capacity from CAP. Some examples of such games follow.

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Example 3.8. *Let $A \in \mathcal{K}'$ and*

$$v_A(a) = \rho_a(A), \quad \textit{for every } a \in M_X.$$

The function δ_A defined by (3.5) *belongs to* CAP_∞ *and* (3.7) *implies*

$$v_A(a) = \oint a \, \mathrm{d}\delta_A.$$

Let ε_A be the Dirac measure on $\mathfrak{B}(\mathcal{K}')$ concentrated at the point A:

$$\varepsilon_A(\mathcal{A}) = \begin{cases} 1, & A \in \mathcal{A}, \\ 0, & A \notin \mathcal{A}, \end{cases} \quad \textit{for every } \mathcal{A} \in \mathfrak{B}(\mathcal{K}').$$

The generalized Möbius transform $\mathbf{m}_{\delta_A}$ of v_A is the integral with respect to ε_A since

$$\mathbf{m}_{\delta_A}(\rho_a) = \int_{\mathcal{K}'} \rho_a(B) \, \mathrm{d}\varepsilon_A(B) = \rho_a(A) = v_A(a), \quad \textit{for every } a \in M_X.$$

The previous example suggests a possible interpretation of the generalized Möbius transform that will be further pursued in Section 4. Namely, the values $\mathbf{m}_{\delta_A}(\rho_a)$ are "degrees of power" of coalitions $a \in M_X$ in the game v_A. We can think of $A \in \mathcal{K}'$ as a group of "veto" players in the game v_A since $\mathbf{m}_{\delta_A}(\rho_a) = 1$ if and only if a lies in the filter F_A.

Example 3.9 (Totally monotone game). *Let $\beta \in \mathrm{CAP}_\infty$ and*

$$v(a) = \oint a \, \mathrm{d}\beta, \quad \textit{for every } a \in M_X.$$

Then the generalized Möbius transform $\mathbf{m}_\beta$ of v is nonnegative. Because the operator ρ preserves infima (Proposition 3.2*(i)) and $v = \mathbf{m}_\beta \circ \rho$, Lemma 6 in* [**dCTM08**] *yields that the game v is* totally monotone *on (the lattice reduct of) the MV-algebra M_X. This means that v is monotone and for each $n \geq 2$ and every $a_1, \dots, a_n \in M_X$:*

$$v\left(\bigvee_{i=1}^{n} a_i\right) \geq \sum_{\substack{I \subseteq \{1,\dots n\} \\ I \neq \emptyset}} (-1)^{|I|+1} v\left(\bigwedge_{i \in I} a_i\right).$$

Example 3.9 implies a particular necessary condition for the existence of the generalized Möbius transform of a game v: the game v must be a difference of two totally monotone functions on M_X.

Example 3.10 (Measure). *Every bounded measure m on M_X is a game. Theorem* 2.10 *says that, for every $a \in M_X$,*

$$m(a) = \int_X a \, \mathrm{d}\mu,$$

where μ is a unique regular Borel measure on $\mathfrak{B}(X)$. The restriction of μ to $\mathcal{K}$ is clearly a capacity on $\mathcal{K}$. This restriction is also denoted by μ. Moreover, the Jordan decomposition of μ yields that the capacity μ belongs to CAP. *Then the Choquet integral of $a \in M_X$ with respect to μ coincides with the Lebesgue integral:*

$$m(a) = \oint a \, \mathrm{d}\mu, \quad \textit{for every } a \in M_X.$$

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Hence the measure m has the generalized Möbius transform $\mathbf{m}_\mu$. Since the capacity μ is additive, the unique Borel probability measure $\boldsymbol{\mu}$ on $\mathcal{K}$ representing $\mathbf{m}_\mu$ is supported only by a subset of the set $\mathcal{S} = \{\, \{x\} \mid x \in X \,\}$. Thus,

$$\mathbf{m}_\mu(f) = \int_{\mathcal{S}} f(A)\,\mathrm{d}\boldsymbol{\mu}(A), \quad \textit{for every } f \in C_{\mathcal{K}'}.$$

Examples of games without the generalized Möbius transform are found easily even on the clan $M_X = [0,1]^X$, where the player set X is finite.

Example 3.11. *If $X = \{1,2\}$, then the clan $M_X = [0,1]^X$ can be identified with the unit square $[0,1]^2$. Since $\mathcal{K}' = \big\{\{1\},\{2\},X\big\}$, the clan $C_{\mathcal{K}'} = [0,1]^{\mathcal{K}'}$ can be viewed as the unit cube $[0,1]^3$. Let*

$$v(a) = a_1^2 + a_2^2, \quad \textit{for each } a = (a_1,a_2) \in M_X.$$

The game v does not have the generalized Möbius transform. Indeed, every $\mathbf{m} \in C_{\mathcal{K}'}$ is just a linear mapping on $[0,1]^3$. Hence

$$\mathbf{m}(\rho_a) = \mu_1 a_1 + \mu_2 a_2 + \mu_3 \min\{a_1,a_2\}, \quad a \in M_X,$$

for some $\mu_1,\mu_2,\mu_3 \in \mathbb{R}$. In conclusion, the equality (3.18) *cannot be satisfied.*

4. Core, Enlarged Core and Cimmino-type Algorithm

Solving a game amounts to predicting a final profit distribution among the players. It is usually assumed that all the coalitions in a game comply with the basic criteria of economical rationality. The concept of a core solution is based on two such premises:

(i) the coalition comprising all the players is formed and the players are able to redistribute its profit,
(ii) no coalition will accept a smaller profit distribution than the one generated by its own members.

Quoting Shapley [**Sha53**, p. 11], “the core is the set of feasible outcomes that cannot be improved upon by any coalition of players”. These principles lead to the following definition, which unifies the concept of core for games on various coalition structures (cf. [**Sha72, Aub74, AS74, BK93**]).

Definition 4.1. Let v be a game on a semisimple MV-algebra M and $\mathbf{M}(M)$ be the set of all bounded measures on M. The *core* of the game v is the set

$$\mathcal{C}(v) = \{\, m \in \mathbf{M}(M) \mid m(1) = v(1),\ m(a) \geq v(a),\ \text{for every } a \in M \setminus \{1\} \,\}.$$

An empty core indicates that no coalitions are able to arrive at any agreement about the joint distribution of profits. The question of non-emptiness of the core is omnipresent in coalition game theory. This question is non-trivial even for games on the clan 2^X with the finite player set X: the core of any game on 2^X is the intersection of an affine hyperplane with $2^{|X|}-2$ half-spaces in $\mathbb{R}^{|X|}$. If $M_X = [0,1]^X$ with X finite, then a fortiori, checking non-emptiness is hard as the core is the intersection of infinitely-many half-spaces and affine hyperplanes. Properties of the core solution for games on $[0,1]^X$ were, among others, studied by Aubin [**Aub74**] and Azrieli and Lehrer [**AL07**]. In Section 4.2, we will apply the generalized Möbius transform to checking nonemptiness of the core. In particular, the proposed procedure will always find at least one profit distribution provided the core is not

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empty. The procedure utilizes a bargaining power of coalitions introduced in the next section.

4.1. Enlarged Core. We consider a negotiation scheme for a game v whose aim is to reach a consensus. The consensus in the game is any distribution of profit lying in the core $\mathcal{C}(v)$. We will confine our discussion to the finite player set X. One can thus write $X = \{1, \dots, n\}$ and identify the clan $M_X = [0,1]^X$ with the n-cube $[0,1]^n$. Then every bounded measure on $[0,1]^n$ is just a linear function on $[0,1]^n$, which corresponds to a unique vector $x \in \mathbb{R}^n$. Thus $\mathbf{M}(M_X) = \mathbb{R}^n$. Let $\langle .,. \rangle$ be the standard scalar product of vectors in $\mathbb{R}^n$. Then we can express the core $\mathcal{C}(v)$ of any game v on $[0,1]^n$ as

$$\mathcal{C}(v) = \{\, x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1),\ \langle a, x \rangle \geq v(a),\ \text{for every } a \in M_X \setminus \{1\} \,\}.$$

Put

$$\mathcal{C}_a(v) = \begin{cases} \{\, x \in \mathbb{R}^n \mid \langle 1, x \rangle = v(1) \,\}, & \text{if } a = 1, \\ \{\, x \in \mathbb{R}^n \mid \langle a, x \rangle \geq v(a) \,\}, & \text{if } a \in M \setminus \{1\}. \end{cases}$$

Then we can write

$$\mathcal{C}(v) = \bigcap_{a \in [0,1]^n} \mathcal{C}_a(v).$$

The concept of an enlarged core was proposed in [**BK09**] as a solution for games on $[0,1]^n$. While the core of v is the set of common points of all the sets $\mathcal{C}_a(v)$, the enlarged core of v will be defined as the set of points in $\mathbb{R}^n$ belonging to all but "negligibly many" sets $\mathcal{C}_a(v)$. What "negligible" means depends on the assessment of the bargaining power of coalitions $a \in [0,1]^n$ in the given game.

Definition 4.2. A *coalitional assessment* of a game v on $[0,1]^n$ is a complete probability measure P defined on the σ-algebra $\mathfrak{A}$ of the Lebesgue measurable subsets of $[0,1]^n$.

For each $A \in \mathfrak{A}$, the number $P(A)$ can be thought of as a relative degree of influence of the coalitions in $A \in \mathfrak{A}$ on the final distribution of profit in the game v. If $P(A) = 0$, then the set A of coalitions has a negligible impact on the bargaining about distributions of profit. Hence the conditions imposed by $\mathcal{C}_a(v)$ can be disregarded for each $a \in A$ when $P(A) = 0$. This naturally leads to the following definition.

Definition 4.3. The *enlarged core* in the game v with a coalitional assessment P is the set

$$\mathcal{C}_P(v) = \bigcup_{\substack{A \in \mathfrak{A} \\ P(A)=0}} \bigcap_{a \in [0,1]^n \setminus A} \mathcal{C}_a(v).$$

It is clear that

$$\mathcal{C}(v) \subseteq \mathcal{C}_P(v). \tag{4.1}$$

It can happen that $\mathcal{C}_P(v)$ is much larger than $\mathcal{C}(v)$ for a game v and a coalitional assessment P [**BK09**, Example 3]. Interestingly enough, the existence of the non-negative generalized Möbius transform of a game v yields a large class of coalitional assessments for which the inclusion (4.1) can be strengthened to an identity.

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THEOREM 4.4. *Let β be a totally monotone capacity on 2^X, where $X = \{1, \dots, n\}$. If v is a game on $[0,1]^n$ such that*

$$v(a) = \oint a \,\mathrm{d}\beta, \quad a \in [0,1]^n, \tag{4.2}$$

then the generalized Möbius transform $\mathbf{m}_\beta$ is nonnegative. Moreover, if the measure $\boldsymbol{\mu}_\beta$ corresponding to $\mathbf{m}_\beta$ satisfies $\boldsymbol{\mu}_\beta(\{X\}) > 0$, then there is a coalitional assessment P such that $\mathcal{C}(v) = \mathcal{C}_P(v)$.

PROOF. As β is totally monotone, the generalized Möbius transform $\mathbf{m}_\beta$ of v is indeed nonnegative (Example 3.9). Since $\mathbf{m}_\beta$ is nonnegative and nonzero, we may suppose without loss of generality that

$$\sum_{A \in 2^X \setminus \{\emptyset\}} \boldsymbol{\mu}_\beta(\{A\}) = 1.$$

The function $\boldsymbol{\mu}_\beta$ is nonnegative and sums to one over all the vertices of the n-cube $[0,1]^n$ except the origin. Hence it can be seen as a probability measure on $\mathfrak{A}$ supported by a subset of $\{0,1\}^n$. Assume that λ is any complete probability measure on $\mathfrak{A}$ such that $\lambda(A) > 0$ for every nonempty open set $A \subseteq [0,1]^n$. Case in point: consider the Lebesgue measure on $\mathfrak{A}$. Let $\alpha \in (0,1)$. Then the function

$$P = \alpha\lambda + (1-\alpha)\boldsymbol{\mu}_\beta$$

is a coalitional assessment on $\mathfrak{A}$ satisfying $P(A) > 0$ whenever $A \in \mathfrak{A}$ is open or A contains the vector 1. The conclusion $\mathcal{C}(v) = \mathcal{C}_P(v)$ is then a consequence of Theorem 1 in [**BK93**], which says that the existence of P with those properties is sufficient for the equality in (4.1). $\square$

The class of games v on $[0,1]^n$ satisfying (4.2) is easily described. For each $a = (a_1, \dots, a_n) \in [0,1]^n$, let π be a permutation of $\{1, \dots, n\}$ such that

$$a_{\pi(1)} \leq \cdots \leq a_{\pi(n)}.$$

Put $a_{\pi(0)} = 0$. Then

$$v(a) = \oint a \,\mathrm{d}\beta = \sum_{i=1}^{n} \left(a_{\pi(i)} - a_{\pi(i-1)}\right) \beta\left(\{\pi(i), \dots \pi(n)\}\right).$$

4.2. Bargaining Schemes. The concepts of enlarged core and coalitional assessment are, together with Theorem 4.4, main ingredients in introducing the so-called bargaining schemes for games on $[0,1]^n$, which were proposed by Butnariu and the author in [**BK09**].

DEFINITION 4.5. A *bargaining scheme* for the core (or for the enlarged core) is an iterative procedure that

(i) starts from an arbitrarily chosen initial distribution of profit $x^0 \in \mathbb{R}^n$ among the players,
(ii) generates a sequence $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ converging to a point of the core or the enlarged core, provided such a point exists.

In this context, each vector x^{k+1} is seen as a redistribution of wealth emerging as the result of a bargaining process in which the terms of the distribution of wealth x^k are renegotiated at each step k according to specific rules. These rules are determined by the Cimmino-style procedure [**Cim38**] generating the sequence

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$(x^k)_{k\in\mathbb{N}}$. Finding an element of the core is thus interpreted as a convex feasibility problem [**BB96**] in which the number of conditions is infinite. The algorithm, which was originally developed in [**BS08**], is designed as follows.

We consider a game v on $[0,1]^n$ meeting the assumptions of Theorem 4.4 and a coalitional assessment P from the conclusion of the same theorem. This implies that $\mathcal{C}(v) = \mathcal{C}_P(v)$. Let $p_a(x)$ be the metric projection of $x \in \mathbb{R}^n$ onto the set $\mathcal{C}_a(v)$, which is given by

$$p_a(x) = \begin{cases} x, & \text{if } a = 0, \\ x + \frac{v(1)-\langle 1,x\rangle}{n} 1, & \text{if } a = 1, \\ x + \frac{\max\{0, v(a)-\langle a,x\rangle\}}{\sum_{i=1}^n a_i^2}, & \text{otherwise.} \end{cases}$$

The following vector integral $p(x)$ is thus well-defined:

$$p(x) = \int_{[0,1]^n} p_a(x)\,\mathrm{d}P(a), \quad x \in \mathbb{R}^n. \tag{4.3}$$

Definition 4.6. The *Cimmino-type bargaining scheme* in the game v is the following rule of generating sequences $(x^k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$:

$$x^0 \in \mathbb{R}^n \quad \text{and} \quad x^{k+1} = p(x^k), \quad \text{for every } k = 0,1,2,\dots .$$

Starting from an arbitrary initial distribution of profit x^0, every subsequent vector x^{k+1} is computed according to (4.3) as the amalgamated projection with respect to the coalitional assessment P. The question of convergence of this procedure to a point in $\mathcal{C}(v)$ is discussed in [**BK09**] in detail. Define

$$g(x) = \tfrac{1}{2} \int_{[0,1]^n} \|p_a(x) - x\|^2\,\mathrm{d}P(a), \quad x \in \mathbb{R}^n.$$

Then the nonnegative function g is everywhere finite, convex, and continuously differentiable with $\nabla g(x) = p(x) - x$. The behavior of g with respect to a sequence $(x^k)_{k\in\mathbb{N}}$ indicates the speed of convergence of the sequence $(x^k)_{k\in\mathbb{N}}$ to a point in $\mathcal{C}(v)$. The two most important cases are singled out.

Theorem 4.7. *Let x^0 be any initial point and $(x^k)_{k\in\mathbb{N}}$ be the sequence generated by the Cimmino-type bargaining scheme. Then:*

(i) *if the sequence $(x^k)_{k\in\mathbb{N}}$ is bounded, then the limit x^* of $(x^k)_{k\in\mathbb{N}}$ exists and $x^* \in \mathcal{C}(v)$ provided $g(x^*) = 0$;*
(ii) *if the sequence $(x^k)_{k\in\mathbb{N}}$ is unbounded or the limit x^* of $(x^k)_{k\in\mathbb{N}}$ exists with $g(x^*) > 0$, then $\mathcal{C}(v) = \emptyset$.*

These two criteria thus enable us to determine nonemptiness of the core or to find an element from the core. The examples of runs of this procedure together with a discussion of emerging computational issues can be found in [**BK09**].

5. Open Problems

The existence of the generalized Möbius transform is proven only for a class of games defined on a clan over a compact Hausdorff second-countable space (cf. Theorem 3.5 and Theorem 3.6). The second-countability assumption was a key component in the proof of Theorem 3.5, which enabled the representation of the

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Choquet integral by way of the Fubini theorem in (3.12). It is an interesting question whether the second-countability assumption can be relaxed in the statement of Theorem 3.5.

To the best of the author's knowledge, the application of the Möbius transform to Cimmino-style projection techniques has not previously appeared in the literature. In Section 4, we investigated such an application for games with finitely many players. The restriction to clans over a finite set was necessary to utilize the result from [**BK09**] that the core and the enlarged core coincide for specific coalitional assessments in games with finitely many players (see the proof of Theorem 4.4). It is therefore an open problem whether or not the presented approach to the Cimmino-style bargaining scheme can also be carried over to games with infinitely many players.

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