Tomáš Kroupa

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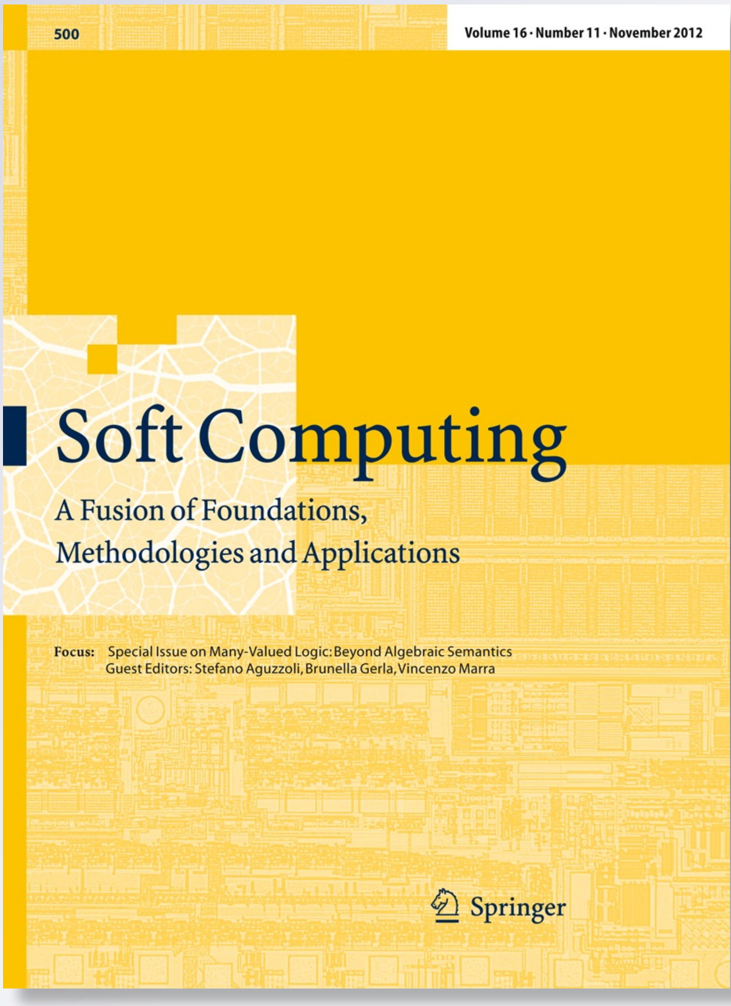

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Extension of belief functions to infinite-valued events

Tomáš Kroupa

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Abstract We generalise belief functions to many-valued events which are represented by elements of Lindenbaum algebra of infinite-valued Łukasiewicz propositional logic. Our approach is based on mass assignments used in the Dempster–Shafer theory of evidence. A generalised belief function is totally monotone and it has Choquet integral representation with respect to a unique belief measure on Boolean events.

Keywords Belief function · MV-algebra · Choquet integral · Łukasiewicz logic · Möbius transform

1 Introduction

The main goal of this paper is to study belief functions on formulas in Łukasiewicz logic. This is in line with an increasing interest in the generalisation of classical probability towards "infinite-valued" events, such as those resulting from Lindenbaum algebra of formulas in Łukasiewicz infinite-valued logic. An algebra of such many-valued events is called an MV-algebra (Cignoli et al. 2000). The counterpart of a probability on a Boolean algebra is the so-called state on an MV-algebra—see (Mundici 1995; Riečan and Mundici 2002; Mundici 2006) for a detailed discussion of probability on MV-algebras including its interpretation in terms of bookmaking over many-valued events. The recent articles (Flaminio et al. 2011b; Fedel et al. 2011; Kroupa 2009) focus on more general functionals on MV-algebras, such as upper (lower) probabilities and possibility (necessity) measures. The connection of belief functions to many-valued logics was explored already in the paper (Godo et al. 2003), where a fuzzy modal logic for belief functions on Boolean formulas was investigated.

The presented paper is a continuation of the previous works (Kroupa 2009, 2010, 2011) of the author. In the first paper (Kroupa 2009), we "guessed" the right form of a generalised belief function to be the Choquet integral with respect to a classical belief measure. Although no reference to basic assignments was made therein, it was the basic assignment approach which enabled construction of belief functions in the case of finitely-valued Łukasiewicz logic (Kroupa 2010). All those results paved the way for the synthesis (Kroupa 2011b), in which we established the existence of Möbius transform for the class of functions in the Choquet integral form. This article attempts to complete the circle of ideas leading from Choquet integral to basic state assignments: all the above-mentioned definitions of belief functions are equivalent. Moreover, we will make an effort to interpret the presented concept of belief functions within Łukasiewicz logic: an elementary belief function or an elementary necessity function is just Pavelka-style truth degree with respect to a deductive theory [see (5.3)].

The paper is structured as follows. The basic definitions and results regarding Łukasiewicz logic and MV-algebras are repeated in Sect. 2. In Sect. 3 we will discuss states on MV-algebras and, in particular, the notion of support of a state: the state is supported by the quotient MV-algebra modulo the filter corresponding to the smallest closed set of measure one (Definition 3.2 and Proposition 3.1). A summary of well-known properties of belief measures on the finite algebras of sets (Shafer 1976) is contained in Sect. 4. This makes it possible to compare those properties

T. Kroupa (✉)
Institute of Information Theory and Automation of the ASCR,
Pod Vodárenskou věží 4, 182 08 Prague, Czech Republic
e-mail: kroupa@utia.cas.cz

with the relevant properties of belief functions introduced in Sect. 5. The main result is a characterisation of belief functions in Theorem 5.1, which enables further description of important subclasses of belief functions such as states and necessity functions. Moreover, every belief function is a lower probability according to Theorem 5.2 [cf. Fedel et al. 2011].

Theory of belief measures and functions is a product of a rich mixture of ideas ranging from MV-algebras and logic to hyperspace topologies on spaces of compact sets and convex analysis. It is barely possible to give even a brief account of all of them in this article. Since we prefer clarity and the overall picture to technical details of the proofs, in many parts of our argumentation we refer the reader to our previous results presented in Kroupa (2011a, b).

2 Preliminaries

In this section we give necessary background on MV-algebras and Łukasiewicz propositional infinite-valued logic. The reader is referred to the book Cignoli et al. (2000) or Chapter 3 in Hájek (1998) for further details.

2.1 MV-algebras

Definition 2.1 An *MV-algebra* is an algebra

$$\langle M, \oplus, \neg, 0 \rangle$$

with a binary operation $\oplus$, a unary operation $\neg$ and a constant 0 such that $\langle M, \oplus, 0 \rangle$ is an abelian monoid and the following equations hold true for every pair of elements $a, b \in M$:

$$\neg\neg a = a,$$
$$a \oplus \neg 0 = \neg 0,$$
$$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a.$$

On every MV-algebra M we define $1 = \neg 0, a \odot b = \neg(\neg a \oplus \neg b)$. For any two elements $a, b \in M$ we write $a \leq b$ if $\neg a \oplus b = 1$. The relation $\leq$ is in fact a partial order. Further, the operations $\vee, \wedge$ defined by $a \vee b = \neg(\neg a \oplus b) \oplus b$ and $a \wedge b = \neg(\neg a \vee \neg b)$, respectively, make the algebraic structure $\langle M, \wedge, \vee, 0, 1 \rangle$ into a distributive lattice with bottom element 0 and top element 1. *Isomorphism* of MV-algebras M_1, M_2 is a bijective mapping $h : M_1 \to M_2$ preserving the MV-algebraic operations $\oplus, \neg$ and the constant 1.

Example 2.1 (*Standard MV-algebra*) The basic example of an MV-algebra is the *standard MV-algebra*, which is the real unit interval [0,1] equipped with operations

$$a \oplus b = \min(1, a + b),$$
$$\neg a = 1 - a.$$

This implies

$$a \odot b = \max(0, a + b - 1)$$

by the definition of operation $\odot$. The partial order $\leq$ of the standard MV-algebra coincides with the usual order of real numbers in the unit interval [0,1].

The set $[0,1]^X$ of all functions $X \to [0,1]$ becomes an MV-algebra if the operations $\oplus, \neg$ and the element 0 are defined pointwise. The corresponding lattice operations $\vee, \wedge$ are then the pointwise maximum and the pointwise minimum of two functions $X \to [0,1]$, respectively. Particular MV-algebras of functions (the so-called clans) are the most frequently encountered instances of MV-algebras.

Definition 2.2 Let X be a nonempty set. A *clan* over X is a collection M_X of functions $X \to [0,1]$ such that the zero function 0 is in M_X and the following conditions are satisfied:

(i) if $a \in M_X$, then $\neg a \in M_X$,
(ii) if $a, b \in M_X$, then $a \oplus b \in M_X$.

In particular, a clan M_X contains the constant function 1 and it is closed with respect to the operation $\odot$. Thus every clan is an MV-algebra. A clan M_X of functions $X \to [0,1]$ is *separating* whenever, for every pair $x, y \in X$ with $x \neq y$, there exists a function $a \in M_X$ such that $a(x) \neq a(y)$.

Let M be an MV-algebra. A *filter* in M is a subset F of M such that

(i) $1 \in F$,
(ii) if $a, b \in F$, then $a \odot b \in F$,
(iii) if $a \in F$ and $a \leq b \in M$, then $b \in F$.

A filter F in M is *proper* if $F \neq M$. We say that a proper filter is *maximal* whenever it is not strictly included in any proper filter. Let X_M be the set of all maximal filters in M. It can be shown that $X_M \neq \emptyset$ for any MV-algebra M. The set X_M can be endowed with a topology whose family of closed sets is given by all sets

$$C_F = \{F' \in X_M \mid F' \not\supseteq F\},$$

where F is a filter in M. Then the topological space X_M becomes compact and Hausdorff.

An MV-algebra is *semisimple* MV-algebra (Cignoli et al 2000, Chapter 3.6) if it is isomorphic to a separating clan of continuous functions over the compact Hausdorff space X_M. In our investigation of belief functions, we confine our discussion to a particular case of a semisimple MV-algebra. Namely, we study belief functions on the Lindenbaum algebra of Łukasiewicz logic. We do so for the following reasons. First, this specialisation exhibits a clear connection between belief functions and logic. Second, the topological space of maximal filters of this Lindenbaum algebra is second-countable, which enables us to find a faithful generalisation of belief measures by employing some results from (Kroupa 2011b).

We will make ample use of the duality between closed sets and filters. Let M_X be a separating clan of continuous functions over a compact Hausdorff space X. There exists a one-to-one correspondence between certain filters in M_X and closed subsets of X. For every set $A \subseteq X$, the subset of M_X defined by

$$F_A = \{a \in M_X | a(x) = 1, \text{ for every } x \in A\} \tag{2.1}$$

is a filter in M_X. In particular, $F_\emptyset = M_X, F_X = \{1\}$. Moreover, the filter $F_{\{x\}}$ is maximal for each $x \in X$. Conversely, a closed subset V_F of X can be assigned to every filter F in M_X by putting

$$V_F = \bigcap \{a^{-1}(1) | a \in F\}, \tag{2.2}$$

since every function $a \in F$ is continuous. The following theorem summarises the relevant results in (Cignoli et al. 2000, Chapter 3.4).

Theorem 2.1 *Let M_X be a separating clan of continuous functions over a compact Hausdorff space X.*

(i) *The mapping $x \in X \mapsto F_{\{x\}}$ is a one-to-one correspondence between X and the set of all maximal filters in the clan M_X.*
(ii) *If $A \subseteq X$ is closed, then $A = V_{F_A}$.*
(iii) *If F is a proper filter that is an intersection of all maximal filters containing F, then $F = F_{V_F}$.*

Thus there is an order-reversing bijection between the set of all nonempty closed subsets of X and the set of all proper filters in M_X that are intersections of maximal filters.

2.2 Łukasiewicz logic

In this section we provide a survey of Łukasiewicz infinite-valued propositional logic (Cignoli et al. 2000, Chapter 4) and its associated Lindenbaum algebra. Formulas $\varphi, \psi, \ldots$ are constructed from propositional variables $A_1, \ldots, A_k$ by applying the standard rules known in Boolean logic. Note that we confine our focus to the language of Łukasiewicz logic with finitely-many variables only. The connectives are negation, disjunction and conjunction, which are denoted by $\neg, \oplus$ and $\odot$, respectively. This is already a complete set of connectives: for example, the implication $\varphi \rightarrow \psi$ can be defined as $\neg\varphi \oplus \psi$. The set of all formulas containing propositional variables $A_1, \ldots, A_k$ is denoted by $\mathrm{Form}\,(A_1, \ldots, A_k)$.

The standard semantics for connectives of Łukasiewicz logic is defined by the corresponding operations of the standard MV-algebra [0,1]. A *valuation* is a mapping

$$V : \mathrm{Form}\,(A_1, \ldots, A_k) \rightarrow [0, 1]$$

such that for each $\varphi, \psi \in \mathrm{Form}\,(A_1, \ldots, A_k)$:

$$V(\neg\varphi) = 1 - V(\varphi),$$
$$V(\varphi \oplus \psi) = V(\varphi) \oplus V(\psi),$$
$$V(\varphi \odot \psi) = V(\varphi) \odot V(\psi).$$

Formulas $\varphi, \psi \in \mathrm{Form}\,(A_1, \ldots, A_k)$ are called *equivalent* when $V(\varphi) = V(\psi)$, for every valuation V. The *equivalence class* of φ is denoted $[\varphi]$. The set of all such equivalence classes endowed with the operations

$$\neg[\varphi] = [\neg\varphi],$$
$$[\varphi] \oplus [\psi] = [\varphi \oplus \psi],$$
$$[\varphi] \odot [\psi] = [\varphi \odot \psi],$$

forms an MV-algebra denoted by L_k. This algebra is the *Lindenbaum algebra* of Łukasiewicz logic over k propositional variables.

Since every valuation V is uniquely determined by its restriction to the propositional variables

$$V \mapsto V(A_1, \ldots, A_k) \in [0, 1]^k,$$

every "possible world" V is matched with a unique point x_V from the k-dimensional unit cube $[0, 1]^k$ and vice versa. Let V_x be the valuation corresponding to $x \in [0, 1]^k$. Put $[\varphi](x) = V_x(\varphi)$, for every $x \in [0, 1]^k$. Hence the equivalence class $[\varphi]$ of every $\varphi \in \mathrm{Form}\,(A_1, \ldots, A_k)$ can be viewed as a function $[0, 1]^k \rightarrow [0, 1]$. Since the Lindenbaum algebra L_k coincides with the free MV-algebra over k generators (Cignoli et al. 2000, Proposition 4.5.5), McNaughton theorem (McNaughton 1951) yields that L_k is precisely the MV-algebra of all functions $[0, 1]^k \rightarrow [0, 1]$ that are continuous and piecewise linear, where each linear piece has integer coefficients.

A *deductive theory* of Łukasiewicz logic is a set Θ of formulas such that

(i) all axioms of Łukasiewicz logic belong to Θ,
(ii) if $\varphi, \varphi \rightarrow \psi \in \Theta$, then $\psi \in \Theta$.

If Θ is a deductive theory, then put

$$F_\Theta = \{[\varphi] \in L_k | \varphi \in \Theta\}.$$

By $\Theta^\models$ we denote the set of all formulas

$$\varphi \in \mathrm{Form}\,(A_1, \ldots, A_k)$$

such that $V(\varphi) = 1$, whenever V is a truth valuation with $V(\psi) = 1$, for every $\psi \in \Theta$. The following theorem is a consequence of (Cignoli et al. 2000, Theorem 4.6.3) together with (Cignoli et al. 2000, Theorem 4.6.6).

Theorem 2.2 *The mapping $\Theta \mapsto F_\Theta$ is a bijection between the deductive theories Θ satisfying $\Theta = \Theta^{\models}$ on the one hand and the proper filters in L_k that are intersections of maximal filters on the other hand.*

Putting together (2.1)–(2.2) with Theorems 2.2–2.1, we get the following corollary.

Corollary 2.1 *Let $K \subseteq [0,1]^k$ be nonempty and closed. The mapping*

$$K \mapsto \Theta_K = \{\varphi \in \mathrm{Form}\,(A_1,\ldots,A_k) | [\varphi](x) = 1, x \in K\} \tag{2.3}$$

is a one-to-one correspondence between the nonempty closed subsets of $[0,1]^k$ and the deductive theories Θ such that $\Theta = \Theta^{\models}$.

3 State and its support

Definition 3.1 Let M be an MV-algebra. A *state s* on M is a function $M \to [0,1]$ with $s(1) = 1$ and satisfying

$$s(f \oplus g) = s(f) + s(g),$$

for every $f, g \in M$ with $f \odot g = 0$.

In particular, every state s is a *modular function* with respect to $\oplus$ and $\odot$, that is, the identity

$$s(f \oplus g) + s(f \odot g) = s(f) + s(g) \tag{3.1}$$

is true for every $f, g \in M$. The following theorem, which was proved independently in (Kroupa 2006) and (Panti 2008), is the integral representation of states. Let $\mathfrak{B}(X)$ be the σ-algebra of Borel subsets of X.

Theorem 3.1 *Let M_X be a separating clan of continuous functions over a compact Hausdorff space X. There is a one-to-one correspondence between the states on M_X and regular Borel probability measures on $\mathfrak{B}(X)$: if s is a state on M_X, then there exists a unique regular Borel probability measure μ on $\mathfrak{B}(X)$ such that*

$$s(f) = \int f \,\mathrm{d}\mu, \quad f \in M_X. \tag{3.2}$$

The well-known identity (based on the inclusion-exclusion principle) can be applied to any finite subset $\{f_1,\ldots,f_n\}$ of M_X:

$$\bigvee_{i=1}^{n} f_i = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \bigwedge_{i \in I} f_i. \tag{3.3}$$

Due to (3.2) and (3.3), we can, for every state s on M_X, write

$$s\left(\bigvee_{i=1}^{n} f_i\right) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} s\left(\bigwedge_{i \in I} f_i\right). \tag{3.4}$$

Let μ be a Borel probability measure on a compact Hausdorff space X. By $\mathcal{K}$ we denote the set of all compact subsets of X. The *support* of μ is a set

$$\bigcap \{K | K \in \mathcal{K},\ \mu(K) = 1\}.$$

This corresponds to the usual meaning of "support" as the smallest closed set on which the measure is concentrated. The measure-theoretic notion of support can be expressed algebraically and extended to states, which will prove useful later for describing various types of belief functions.

Throughout this section, let M_X be a separating clan of continuous functions over a given compact Hausdorff space X. If s is a state on M_X, then it follows from (3.1) that the set

$$\{f \in M_X | s(f) = 1\}$$

is a proper filter in M_X. Hence the next definition makes sense, as it is based on the standard notion of a quotient MV-algebra modulo filter (see Cignoli et al. 2000, p. 89).

Definition 3.2 Let s be a state on M_X and

$$F_s = \{f \in M_X | s(f) = 1\}.$$

The *support* of s is the quotient MV-algebra M_X/F_s.

The support of a state s is related to the support of the representing Borel measure μ (Theorem 3.1) as follows. We claim that the closed set V_{F_s} is the support of μ. Indeed, given any $f \in F_s$, take $K := f^{-1}(1)$ and observe that $K \in \mathcal{K}$ and $\mu(K) = 1$. Conversely, if $K \in \mathcal{K}$ is such that $\mu(K) = 1$, then any $f \in F_K$ necessarily satisfies $s(f) = 1$. Hence

$$\begin{aligned} V_{F_s} &= \bigcap \{f^{-1}(1) | s(f) = 1\} \\ &= \bigcap \{f^{-1}(1) | \mu(f^{-1}(1)) = 1\} \\ &= \bigcap \{K | K \in \mathcal{K},\ \mu(K) = 1\}. \end{aligned}$$

This implies that the integral (3.2) can be restricted to V_{F_s} so that

$$s(f) = \int_{V_{F_s}} f \,\mathrm{d}\mu, \quad f \in M_X. \tag{3.5}$$

For every nonempty set $Y \subseteq X$, let $M_{X/Y}$ be the MV-algebra of the restrictions to Y of all the functions in the clan M_X.

Proposition 3.1 *The support of every state s on M_X is isomorphic to the MV-algebra $M_{X/V_{F_s}}$.*

Proof Due to (Cignoli et al. 2000, Proposition 3.4.5) we only need to show that F_s is an intersection of maximal filters. We will in fact show that

$$F_s = \bigcap_{x \in V_{F_s}} F_{\{x\}}.$$

Let $f \in F_{\{x\}}$, for each $x \in V_{F_s}$. Since V_{F_s} is the support of the probability measure μ associated with s, we get

$$s(f) = \int f \, d\mu \geq \int_{V_{F_s}} 1 \, d\mu = \mu(V_{F_s}) = 1.$$

On the other hand, let

$$s(f) = \int_{V_{F_s}} f \, d\mu = 1, \quad f \in M_X.$$

This implies that measure μ of the set

$$Y = \{x \in V_{F_s} | f(x) < 1\}$$

is 0. Since Y is open, the compact set $V_{F_s} \setminus Y$ has a measure of 1. But Y must be empty: otherwise there exists a proper closed subset of the support V_{F_s} whose measure is 1. Thus $f(x) = 1$, for each $x \in V_{F_s}$. □

The equality (3.5) says that $s(f) = s(g)$, for each pair of functions $f, g \in M_X$ with $f(x) = g(x)$, for every $x \in V_{F_s}$. The state s' on the support M_X/F_s of s is thus well-defined by letting

$$s'(f/F_s) = s(f), \quad f \in M_X. \tag{3.6}$$

Example 3.1 Let $x \in X$. Then the function

$$s_x(f) = f(x), \quad f \in M_X \tag{3.7}$$

is a state on the clan M_X. Since the set

$$F_{s_x} = \{f \in M_X | f(x) = 1\}$$

is a maximal filter in M_X, the quotient M_X/F_{s_x} is isomorphic to a subalgebra of the standard MV-algebra [0,1].

Example 3.2 Consider the *Lebesgue state* s_λ on the MV-algebra L_k of k-variable McNaughton functions:

$$s_\lambda(f) = \int f \, dx, \quad f \in M_X$$

where the integral on the right-hand side is Riemann. Then $F_{s_\lambda} = \{1\}$, so that L_k/F_{s_λ} is isomorphic to L_k.

In conclusion: the support of every state s is the smallest quotient of M_X whose elements fully determine the values of s. Due to (3.6) every state s on M_X can be viewed as the state s' on the quotient algebra M_X/F_s.

4 Belief measures on finite Boolean algebras

The main goal of this section is to repeat the basic definitions and results concerning belief measures on finite algebras of sets (Shafer 1976). This will enable us to compare their properties directly with those of belief functions on formulas introduced in the next section. Let X be a finite nonempty set.

Definition 4.1 Let m be a function $2^X \to [0,1]$ such that $m(\emptyset) = 0$ and

$$\sum_{A \in 2^X} m(A) = 1.$$

A *belief measure* β is a function $2^X \to$ [0,1] defined by

$$\beta(A) = \sum_{B \subseteq A} m(B), \quad A \in 2^X.$$

The function m is called a *basic assignment*. The finitely additive probability measure P on 2^{2^X} given by

$$P(\mathcal{A}) = \sum_{A \in \mathcal{A}} m(A), \quad \mathcal{A} \in 2^{2^X}$$

is said to be a *probability assignment*.

The value $m(A)$ is the belief that one commits exactly to A, while $\beta(A)$ is the total belief committed to A. A *plausibility measure* $\gamma : 2^X \to [0,1]$ is defined by $\gamma(A) = 1 - \beta(\bar{A})$, $A \in 2^X$. The properties of a plausibility measure are completely determined by those of the corresponding belief measure. So we consider only belief measures in the sequel. The terminological remark is in order at this point. Since the definition of "belief measure" or "belief function" appears in various contexts in the literature (cf. Halpern 2003), we make the following stipulation: the former is used for the belief degrees of events in a Boolean algebra, while the latter is preferred in the general many-valued setting introduced in Sect. 5. The same remark applies to plausibility measures (functions).

Every belief measure is *normalised*, that is,

$$\beta(\emptyset) = 0, \quad \beta(X) = 1,$$

and *monotone*:

$$\beta(A) \leq \beta(B), \quad \text{whenever } A \subseteq B \text{ for } A, B \in 2^X.$$

Observe that the basic assignment m is exactly the *Möbius transform* of β (Rota 1964). This implies that we can recover m from β as

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \beta(B), \quad A \in 2^X.$$

The following two examples appear in many different variations throughout the literature about belief measures.

Example 4.1 (Negligent coin tossing) A coin is tossed so that the result can be unknown: the coin is out of sight. There is either heads or tails on the coin yet we do not know which one is the actual outcome. The frame of discernment is $X = \{h, t\}$, since there are precisely two admissible physical states of the coin. However, our personal belief must admit the situation in which the result is not perceptible: let the real numbers $p_h, p_t \geq 0$ satisfy $p_h + p_t < 1$. Put

$$m(A) = \begin{cases} p_h, & A = \{h\}, \\ p_t, & A = \{t\}, \\ 1 - p_h - p_t, & A = X. \end{cases}$$

The corresponding belief measure β is thus

$$\beta(A) = \begin{cases} p_h, & A = \{h\}, \\ p_t, & A = \{t\}, \\ 1, & A = X. \end{cases}$$

Example 4.2 (*Laplace principle*) You are in a position to assess which of three presented paintings is genuine or counterfeit. All you know is that exactly one of the three is genuine and you are no expert on paintings. Let $X = \{a, b, c\}$. The Laplace principle of insufficient reason gives the basic assignment and the belief function

$$m(A) = \begin{cases} 1, & A = X, \\ 0, & \text{otherwise}, \end{cases} \qquad \beta(A) = \begin{cases} 1, & A = X, \\ 0, & \text{otherwise}, \end{cases}$$

respectively.

In general, if A is a nonempty subset of any finite set X, then the function

$$\beta_A(B) = \begin{cases} 1, & A \subseteq B \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

is a belief function. The corresponding basic assignment $m_A(B)$ vanishes iff $B \neq A$.

Various equivalent properties of belief measures are summarised below—see Shafer (1979) for proofs and details.

Theorem 4.1 *Let X be a finite nonempty set and let β: $2^X \to [0,1]$. Then the following assertions are equivalent:*

(i) *β is a belief measure,*
(ii) *there is a finitely additive probability measure P on 2^{2^X} with $P(\emptyset) = 0$ such that*

$$\beta(A) = P(\{B | B \subseteq A\}), \quad \textit{for every } A \in 2^X,$$

(iii) *β is a totally monotone function on 2^X, i.e., β is monotone and the following inequality holds true for each $n \geq 2$ and every $A_1, \ldots, A_n \in 2^X$:*

$$\beta\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq 0}} (-1)^{|I|+1} \beta\left(\bigcap_{i \in I} A_i\right),$$

β lies in the convex hull of $\{\beta_A | A \in 2^X \setminus \{\emptyset\}\}$.

In particular, total monotonicity yields that β is *supermodular*,

$$\beta(A \cup B) + \beta(A \cap B) \geq \beta(A) + \beta(B), \quad A, B \in 2^X,$$

and thus also *superadditive:*

$$\beta(A \cup B) \geq \beta(A) + \beta(B), \quad \text{whenever } A \cap B = \emptyset.$$

This means that the belief of A and the belief of its complement $\bar{A}$ do not sum to one in general:

$$\beta(A) + \beta(\bar{A}) \leq 1.$$

Belief measures can also be viewed as "lower probabilities", that is, lower envelopes of certain sets of probability measures.

Theorem 4.2 *If β is a belief measure on 2^X, then the set*

$$\mathcal{C}(\beta) = \{\pi | \pi(A) \geq \beta(A), \ A \in 2^X\}$$

of probability measures π on 2^X is nonempty, compact and convex. For every $A \in 2^X$,

$$\beta(A) = \min\{\pi(A) | \pi \in \mathcal{C}(\beta)\}.$$

Probabilities can easily be characterised within the class of belief measures.

Proposition 4.1 *If β is a belief measure, then the following are equiveridical*:

(i) *β is a finitely additive probability,*
(ii) *the basic assignment m satisfies $\sum_{x \in X} m(\{x\}) = 1$,*
(iii) *the support of the probability assignment P is included in $\{\{x\} | x \in X\}$,*
(iv) *β coincides with the plausibility measure γ,*
(v) *$\beta(A) + \beta(\bar{A}) = 1$.*

A *necessity measure* is a function ν: $2^X \to [0,1]$ such that $\nu(\emptyset) = 0$, $\nu(X) = 1$ and, for every $A, B \in 2^X$,

$$\nu(A \cap B) = \min\{\nu(A), \nu(B)\}. \tag{4.2}$$

The belief measure β_A defined in (4.1) is a necessity measure. It can be shown that every necessity measure is a belief measure (Shafer 1976). The class of necessity measures can be characterised within all belief measures as follows.

Proposition 4.2 *If β is a belief measure, then the following are equiveridical:*

(i) *β is a necessity measure,*
(ii) *the set $\{A \in 2^X | m(A) > 0\}$ is a chain in 2^X,*
(iii) *β is a convex combination of the belief measures $\beta_{A_1}, \ldots, \beta_{A_n}$, where $\emptyset \neq A_1 \subseteq \cdots \subseteq A_n$,*
(iv) *the support of the probability assignment P forms a chain in 2^X.*

Proposition 4.2 says that the evidence underlying necessity measures is homogeneous: an event A implies another event B whenever both $m(A)$ and $m(B)$ are positive.

This explains why necessity measures are also termed *consonant belief functions*. Observe that (4.2) restricts the values $v(A)$ since each consonant belief measure v reveals the least possible conflict in evidence: either $v(A) = 0$ or $v(\bar{A}) = 0$, for every $A \in 2^X$.

5 Belief functions over infinite-valued events

A starting point for our introduction of belief functions in the many-valued setting is the characterisation of belief measures in Theorem 4.1 (ii). In the discussion that follows, we will substitute the finitely-additive probability P by a state $\mathbf{s}$. The crucial question is, however, which MV-algebra is the right domain of such a state $\mathbf{s}$. We propose one possible solution to this issue by exploring an algebraic feature of the mapping

$$A \in 2^X \mapsto \{B \in 2^X | B \subseteq A\} \tag{5.1}$$

that is employed in Theorem 4.1 (ii). Namely, the set on the right-hand side of (5.1) can be given the following interpretation: since each $B \in 2^X$ determines a filter in the Boolean algebra 2^X and vice versa, the set

$$\{B \in 2^X | B \subseteq A\} \in 2^{2^X}$$

is viewed as the set of all filters in 2^X to which the element $A \in 2^X$ belongs. This means that (5.1) becomes

$$A \in 2^X \mapsto \{F \text{ filter } | A \in F\}.$$

How do we modify (5.1) in a many-valued case?

First, we replace the set $A \in 2^X$ by a McNaughton function $f \in L_k$. Second, instead of the set

$$\{B \in 2^X | B \subseteq A\} \in 2^{2^X},$$

we take the set of all filters in L_k and consider a degree to which the McNaughton function f belongs to each of them. Specifically, we consider only each filter F that is an intersection of maximal filters; the degree of membership of f to F is then

$$\min\{f(x) | x \in V_F\}, \tag{5.2}$$

where V_F is the same as in (2.2). Conversely, the duality expressed by Theorem 2.1 says that (5.2) determines a degree of membership of f to a unique filter given by a nonempty closed set $K \subseteq [0,1]^k$ in place of V_F. Since f is fixed and the closed set V_F varies in (5.2), we will consider the mapping

$$\rho : L_k \times \mathcal{K}' \to [0,1], \quad \rho_f(A) := \min\{f(x) | x \in A\},$$

for every $f \in L_k, A \in \mathcal{K}'$, where $\mathcal{K}'$ denotes the set of all nonempty closed subsets of $[0,1]^k$. Hence it is the function $\rho_f : \mathcal{K}' \to [0,1]$ that plays the role of the set

$$\{B \in 2^X | B \subseteq A\} \in 2^{2^X}$$

in the many-valued setting: indeed, it follows from the definition that

$$\rho_A = \{B \in 2^X | B \subseteq A\}, \quad A \in 2^X,$$

provided the set A is identified with its characteristic function. Consequently, the Boolean algebra 2^{2^X} is replaced with any MV-algebra including the image of L_k via ρ, the set $\{\rho_f | f \in L_k\}$. A state $\mathbf{s}$ on such an MV-algebra is then a many-valued analogue of the probability assignment

$$P : 2^{2^X} \to [0,1].$$

In order to fully exploit the integral representation of states (Theorem 3.1) in the developed theory, we need to endow the space of nonempty closed sets $\mathcal{K}'$ with a compact Hausdorff topology, the considered MV-algebra of functions $\mathcal{K}' \to [0,1]$ becoming a separating clan of continuous functions. Both tasks were solved in Kroupa (2011). Namely, the hyperspace $\mathcal{K}'$ is endowed with the so-called *Hausdorff metric topology*. It is the topology induced by the *Hausdorff metric* H on $\mathcal{K}'$:

$$H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\},$$

where $A, B \in \mathcal{K}'$. It can be shown that the Hausdorff metric topology on $\mathcal{K}'$ is indeed compact Hausdorff and, moreover, that function $\rho_f : \mathcal{K}' \to [0,1]$ is continuous for each $f \in L_k$ [see Kroupa (2011b) and the references therein]. In conclusion, in a many-valued case we can consider the MV-algebra $C_{\mathcal{K}'}$ of all continuous functions $\mathcal{K}' \to [0,1]$ as a substitute for 2^{2^X}. The preceding explanation paved the way for the following definition.

Definition 5.1 A *state assignment* $\mathbf{s}$ is a state on $C_{\mathcal{K}'}$. Let $\mathbf{s}$ be a state assignment. A *belief function* is a function b: $L_k \to [0,1]$ such that

$$b(f) = \mathbf{s}(\rho_f), \quad \text{for every } f \in L_k.$$

On the one hand, a state on the algebra of McNaughton functions L_k can be seen as an averaging process for truth degrees in Łukasiewicz logic [cf. Mundici (1995) and Theorem 3.1]. On the other hand, state assignment $\mathbf{s}$ defines averaging over "relative" truth degrees: this point of view is made possible by adopting the terminology used in Rational Pavelka logic (Hájek 1998, Chapter 3.3). Indeed, due to Corollary 2.1, we can write

$$\rho_f(A) = \inf\{V(\varphi) | V \text{ is a model of } \Theta_A\}, \tag{5.3}$$

for every $f \in L_k, A \in \mathcal{K}'$, where φ is a formula corresponding to McNaughton function f and Θ_A is as in (2.3).

The expression on the right-hand side of (5.3) is called the *truth degree* of φ over Θ_A. In conclusion: belief $b(f)$ is a result of averaging the relative truth degrees of φ over the theories corresponding to all nonempty closed sets of valuations. Whereas the basic example of a state on L_k is a state s_x determined by point $x \in [0,1]^k$ (Example 3.1), the basic example of a belief function is the belief function determined by a nonempty closed set of points in $[0,1]^k$.

Example 5.1 Let $A \in \mathcal{K}'$. Put

$$b_A(f) = \rho_f(A), \quad f \in L_k. \tag{5.4}$$

Then the function b_A is a belief function with state assignment $\mathbf{s}_A$ such that

$$\mathbf{s}_A(\xi) = \xi(A), \quad \xi \in C_{\mathcal{K}'}. \tag{5.5}$$

Belief function b_A is a many-valued analogue of belief measure β_A from (4.1). The support of state assignment $\mathbf{s}_A$ is a subalgebra of [0,1] due to Example 3.1.

In the classical case of a finite Boolean algebra, every finitely additive probability is a belief measure. Similarly, every state on L_k is a belief function.

Proposition 5.1 *Every state s on L_k is a belief function whose state assignment is supported by the MV-algebra L_k/F_s.*

Proof Observe that s can be extended to the MV-algebra of all continuous functions $[0,1]^k \to [0,1]$ by Theorem 3.1. We will denote this extension by s as well. The Hausdorff metric topology of $\mathcal{K}'$ is compatible with the Euclidean topology of the k-cube $[0,1]^k$ in this sense: the set $\hat{X} = \{\{x\} | x \in [0,1]^k\} \subset \mathcal{K}'$ is homeomorphic to $[0,1]^k$. Therefore, if $\xi \in C_{\mathcal{K}'}$, then $\xi/F_{\hat{X}}$ can be identified with the restriction of ξ on $[0,1]^k$. Put

$$\mathbf{s}(\xi) = s(\xi/F_{\hat{X}}), \quad \xi \in C_{\mathcal{K}'}.$$

This gives

$$\mathbf{s}(\rho_f) = s(\rho_f/F_{\hat{X}}) = s(f), \quad f \in L_k,$$

since $\rho_f(\{x\}) = f(x)$. The second assertion follows from the definition of $\mathbf{s}$. □

In the next section we will investigate the properties of belief functions and compare them with the corresponding properties of belief measures in Sect. 4.

5.1 Properties of belief functions

It follows from Definition 5.1 that $b(0) = 0$, $b(1) = 1$, and b is monotone: $b(f) \le b(g)$ for each $f, g \in L_k$ with $f \le g$. Moreover, since $f \odot g = 0$ implies

$$\rho_{f \oplus g} = \rho_{f+g} \ge \rho_f + \rho_g,$$

every belief function is superadditive with respect to the operations $\oplus, \odot$:

$$b(f \oplus g) \ge b(f) + b(g), \quad \text{whenever } f \odot g = 0.$$

This also shows that $b(f) + b(\neg f) \le 1$.

Proposition 5.2 *Every belief function b is a totally monotone function on the lattice reduct of L_k, that is, b is monotone and satisfies*

$$b\left(\bigvee_{i=1}^n f_i\right) \ge \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \ne \emptyset}} (-1)^{|I|+1} b\left(\bigwedge_{i \in I} f_i\right),$$

for each $n \ge 2$ and every $f_1, \ldots, f_n \in L_k$.

Proof The definition of belief function b makes it possible to write $b = \mathbf{s} \circ \rho$, where $\rho : f \in L_k \mapsto \rho_f \in C_{\mathcal{K}'}$. According to Lemma 6 from de Cooman et al. (2008), it suffices to show that $\mathbf{s}$ is totally monotone on the lattice reduct of $C_{\mathcal{K}'}$ and that

$$\rho_{f \wedge g} = \rho_f \wedge \rho_g, \quad \text{for every } f, g \in L_k.$$

The former assertion is a consequence of (3.4) and the latter follows directly from the definition of ρ. □

In particular, setting $n = 2$ in Proposition 5.2 shows that b is supermodular with respect to the lattice operations of the clan L_k:

$$b(f \vee g) + b(f \wedge g) \ge b(f) + b(g), \quad f, g \in L_k.$$

Belief functions on L_k can be described in a number of equivalent ways: this will be conveyed by Theorem 5.1 below. In order to state the theorem, we will need to repeat a few definitions regarding belief measures on compact subsets of $[0,1]^k$ and their associated Choquet integrals [see Shafer (1979) and Denneberg (1994), respectively]. A *belief measure* on $\mathcal{K} = \mathcal{K}' \cup \{\emptyset\}$ is function $\beta : \mathcal{K} \to [0,1]$ such that:

(i) $\beta(\emptyset) = 0$, $\beta([0,1]^k) = 1$,
(ii) β is totally monotone on $\mathcal{K}$,
(iii) if $(A_n)_{n \in \mathbb{N}} \in \mathcal{K}^{\mathbb{N}}$ is non-increasing, then

$$\beta\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \beta(A_n).$$

When β is a belief measure on $\mathcal{K}$, the *Choquet integral* of $f \in L_k$ with respect to β is given by

$$\oint f \, d\beta = \int_0^1 \beta(f^{-1}([t,1])) \, dt.$$

This is well-defined since the Riemann integral on the right-hand side exists due to continuity of f and monotonicity of β. By $\mathfrak{B}(\mathcal{K}')$ we denote Borel σ-algebra generated by Hausdorff metric topology of $\mathcal{K}'$. Although $\mathfrak{B}(\mathcal{K}')$ is a σ-algebra of sets of subsets, it can be described explicitly [see Molchanov (2005, Chapter 1)].

Theorem 5.1 *Let b be a function $L_k \to [0,1]$. Then the following assertions are equivalent:*

(i) *b is a belief function,*
(ii) *there is a belief measure β on $\mathcal{K}$ such that*

$$b(f) = \oint f \, d\beta, \quad f \in L_k,$$

(iii) *there is a Borel probability measure μ on $\mathfrak{B}(\mathcal{K}')$ such that*

$$b(f) = \int_{\mathcal{K}'} \rho_f(A) \, d\mu(A), \quad f \in L_k,$$

(iv) *b lies in the closed convex hull of $\{b_A | A \in \mathcal{K}'\}$, where the closure is considered in the product topology of $[0,1]^{L_k}$.*

Proof The equivalence of the first and the second assertion is formulated as the main result (Theorem 3.5) in Kroupa (2011). Theorem 3.1 yields that (i) holds true if and only if (iii) is satisfied. We will show that (i) implies (iv). If $\mathbf{s}$ is an arbitrary state assignment, put $\Phi(\mathbf{s}) = b$, where b is the belief function corresponding to $\mathbf{s}$. The mapping Φ is affine: for every $\alpha \in [0,1]$ and every pair of state assignments $\mathbf{s}_1, \mathbf{s}_2$, we have

$$\Phi(\alpha\mathbf{s}_1 + (1-\alpha)\mathbf{s}_2) = \alpha\Phi(\mathbf{s}_1) + (1-\alpha)\Phi(\mathbf{s}_2).$$

Moreover, the mapping Φ is continuous from the set of all state assignments (considered with the product topology of $[0,1]^{C_{\mathcal{K}'}}$) to the set of all belief functions on L_k (considered with the product topology of $[0,1]^{L_k}$). It follows from (Mundici 1995) that $\mathbf{s}$ is in the closed convex hull of the set $\{\mathbf{s}_A | A \in \mathcal{K}'\}$, where $\mathbf{s}_A$ is the same as in (5.5). Since $\Phi(\mathbf{s}_A) = b_A$ for each $A \in \mathcal{K}'$, the function b lies in the closed convex hull of $\{b_A | A \in \mathcal{K}'\}$. The implication from (iv) to (i) is proven analogously. □

Theorem 5.2 (Belief function as a lower probability) *If b is a belief function on L_k, then the set*

$$\mathcal{C}(b) = \{s | s(f) \geq b(f), \ f \in L_k\}$$

of states on L_k is nonempty, compact and convex. For every $f \in L_k$,

$$b(f) = \min\{s(f) | s \in \mathcal{C}(b)\}.$$

Proof Let μ be the Borel probability measure on $\mathfrak{B}(\mathcal{K}')$ representing the state assignment $\mathbf{s}$. By $C[0,1]^k$ we denote the Banach space of all real continuous functions on $[0,1]^k$ with the supremum norm. For each nonnegative function $f \in C[0,1]^k$, put

$$\bar{b}(f) = \int_{\mathcal{K}'} \bar{\rho}_f(A) \, d\mu(A),$$

where $\bar{\rho}_f(A) := \min\{f(x) | x \in A\}$. Then $\bar{b}(f) = b(f)$ for each $f \in L_k$. It can be routinely shown that the function $\bar{b}$ is continuous, concave and positively homogeneous on the subspace $\{f \in C[0,1]^k | f \geq 0\}$. The assertion then follows from Proposition 3 (i) in (Kroupa 2011). □

Plausibility functions are dual to belief functions. Specifically, for every belief function b, let

$$p(f) = 1 - b(\neg f), \quad \text{for every } f \in L_k.$$

Since the negation $\neg$ is involutive, the properties of plausibility function p are dual to those of the associated belief function b. We will use plausibility functions to achieve a characterisation of states within belief functions, which is analogous to Proposition 4.1.

Proposition 5.3 (Characterisation of states) *If b is a belief function with the state assignment* $\mathbf{s}$, *then the following are equiveridical*:

(i) *b is a state,*
(ii) *if $f, g \in L_k$ are such that $f \odot g = 0$, then*

$$\mathbf{s}(\rho_{f \oplus g}) = \mathbf{s}(\rho_f) + \mathbf{s}(\rho_g),$$

(iii) *the support of the state assignment* $\mathbf{s}$ *is (up to isomorphism) the MV-algebra L_k/F_A, for some $A \in \mathcal{K}$,*
(iv) *b coincides with the associated plausibility functionp,*
(v) *$b(f) + b(\neg f) = 1$, for every $f \in L_k$.*

Proof The equivalence of the first three assertions directly follows from the assertion and the proof of Proposition 5.1. Clearly, (iv) and (v) are equivalent, and (i) implies (v). We will show that the implication (iii) follows from (v). Let $b(f) + b(\neg f) = 1$, for each $f \in L_k$. This provides

$$\int_{\mathcal{K}'} (\rho_f + \rho_{\neg f}) \, d\mu = 1,$$

where μ is the unique Borel probability measure on $\mathfrak{B}(\mathcal{K}')$ from Theorem 5.1 (iii). Let

$$\hat{X} = \{\{x\} | x \in [0,1]^k\}.$$

The integral above is split into the following terms:

$$\begin{aligned}&\int\limits_{\mathcal{K}'} (\rho_f + \rho_{\neg f})\, d\mu \\ &\quad = \int\limits_{\hat{X}} (\rho_f + \rho_{\neg f})\, d\mu + \int\limits_{\mathcal{K}'\setminus\hat{X}} (\rho_f + \rho_{\neg f})\, d\mu \\ &\quad = \mu(\hat{X}) + \int\limits_{\mathcal{K}'\setminus\hat{X}} (\rho_f + \rho_{\neg f})\, d\mu.\end{aligned}$$

The last sum is equal to 1 for each $f \in L_k$. This necessarily implies $\mu(\mathcal{K}' \setminus \hat{X}) = 0$. So $\mu(\hat{X}) = 1$ and the support of μ is thus included in $\hat{X}$, which yields (iii). □

The characterisation of necessity measures in Proposition 4.2 is a plausible starting point for the introduction of necessity functions as special belief functions. The property (iv) in Proposition 4.2 can be directly rephrased in the many-valued setting as follows.

Definition 5.2 A *necessity function* is a belief function Nec whose state assignment is supported by the MV-algebra $C_{\mathcal{K}'}/F_{\mathcal{A}}$, for a compact set $\mathcal{A}$ whose elements form a chain in $\mathcal{K}'$.

It is worth mentioning that our necessity function Nec is not min-preserving in general. This is a point of dissimilarity with the defining property (4.2) of classical necessity measures. On the one hand, there are other concepts of necessity functions in a many-valued framework (Flaminio et al. 2011b), which consider the identity

$$\mathrm{Nec}\,(f \wedge g) = \mathrm{Nec}\,(f) \wedge \mathrm{Nec}\,(g), \quad f, g \in L_k \tag{5.6}$$

as the constituting property of necessity functions. On the other hand, a definition of necessity function violating (5.6) already appeared in the paper (Dubois and Prade 1985) by Dubois and Prade. So it seems that there are more possible generalisations of necessity functions outside Boolean setting depending on the particular property of classical necessity functions chosen. These issues are discussed in detail in (Flaminio and Kroupa 2011).

Example 5.2 Assume that $\mathcal{A} = (A_1, \ldots, A_n)$ and $A_1 \subseteq \cdots \subseteq A_n$, where $A_i \in \mathcal{K}'$. For any $\alpha_1, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, put

$$\mathrm{Nec}\,(f) = \sum_{i=1}^{n} \alpha_i \rho_f(A_i).$$

This is a necessity function with a state assignment

$$\mathbf{s}(\xi) = \sum_{i=1}^{n} \alpha_i \xi(A_i), \quad \xi \in C_{\mathcal{K}'}.$$

Observe that if $\alpha_i = 1$ for some $i = 1, \ldots, n$, then (5.6) is satisfied. This means that at least the basic necessity functions b_A from (5.4) are min-preserving.

6 Open problems

We will state two problems open for further investigation. First, it is not known whether total monotonicity implies the existence of a state assignment. Specifically, can we achieve equivalence in Proposition 5.2 so that a totally monotone normalised function is a belief function? This is true in the Boolean case (cf. Theorem 4.1).

Second, we made no attempts to study Dempster's rule of combination. Let m_1, m_2 be two basic assignments on 2^X. We say that basic assignment m on 2^X results from *Dempster's rule* of combination on m_1 and m_2 whenever

$$m(A) = \begin{cases} k^{-1} \sum\limits_{\substack{B,C \in 2^X \\ A = B\cap C}} m_1(B) m_2(C), & A \neq \emptyset, \\ 0, & A = \emptyset, \end{cases}$$

where

$$k = 1 - \sum_{\substack{B,C \in 2^X \\ \emptyset = B\cap C}} m_1(B) m_2(C).$$

The main difficulty in our setting is how to construct a combined state assignment over the infinite MV-algebra $C_{\mathcal{K}'}$. Initial efforts in this direction were recently made in Flaminio et al. (2011a) in a framework very similar to ours.

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