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Shape Optimization in 2D Contact Problems with Given Friction and a Solution-Dependent Coefficient of Friction

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Abstract The paper deals with shape optimization of elastic bodies in unilateral contact. The aim is to extend the existing results to the case of contact problems, where the coefficient of friction depends on the solution. We consider the two-dimensional Signorini problem, coupled with the physically less accurate model of given friction, but assume a solution-dependent coefficient of friction. First, we investigate the shape optimization problem in the continuous, infinite-dimensional setting, followed by a suitable finite-dimensional approximation based on the finite-element method. Convergence analysis is presented as well. Next, an algebraic form of the state problem is studied, which is obtained from the discretized problem by further approximating the frictional term by a quadrature rule. It is shown that if the coefficient of friction is Lipschitz continuous with a sufficiently small modulus, then the algebraic state problem is uniquely solvable and its solution is a Lipschitz continuous function of the control variable, describing the shape of the elastic body. For the purpose of numerical solution of the shape optimization problem via the so-called implicit programming approach we perform sensitivity analysis by using the tools from the generalized differential calculus of Mordukhovich. The paper is concluded first order optimality conditions.

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1 Introduction

Shape optimization is a special branch of optimal control theory in which control variables are related to the geometry of optimized systems. The present paper deals with a particular problem of the so-called contact shape optimization, i.e. the optimization of loaded structures composed from several deformable bodies being in a mutual contact. Due to non-penetration and friction conditions prescribed on contact parts, the mathematical models characterizing the behavior of such structures lead to variational inequalities whose solutions represent equilibrium states of the structure. A common feature of such problems is the fact that the control-to-state mappings are not of the class C^1 but only Lipschitz continuous functions of their variables. In simple situations such as frictionless contact problems or problems with the so-called given friction (both having a unique solution), the respective solution maps are directionally differentiable (see [20] for continuous setting and [5] for discrete problems). If, however, more realistic model of friction such as the one obeying the Coulomb law is considered, the situation becomes more involved since the state relations are now represented by implicit variational inequalities. The present paper deals with shape optimization in contact problems with given friction but with the coefficient of friction which depends on the solution, i.e. the case which also leads to an implicit variational inequality. We present a complex study of this problem: from the existence analysis of its continuous setting over its approximation and convergence analysis up to sensitivity analysis of the discrete model.

For stability analysis it is convenient to model the respective algebraic problem as a variational inequality. When this equilibrium arises, however, as a constraint in a finite-dimensional optimization problem, we prefer to model it via a *generalized equation (GE)*. The reason is that this model enables us an efficient treatment via the generalized differential calculus of B. Mordukhovich which will be our main tool in this part of the paper. A similar situation arises in discretized contact problems with Coulomb friction, cf. [1] and [2]. The GE considered here differs, however, considerably from the corresponding GE in the case of 2D contact problems with Coulomb friction. Indeed, its multivalued part is not polyhedral and depends also on the design variable. This makes the analysis substantially more complicated. Nevertheless, under standard assumptions we have succeeded to verify the respective qualification conditions and arrived at a sharp estimate of a subdifferential of the composite objective, resulting from the payoff in the mentioned optimization problem and the considered control-to-state mapping. This enables us

(i) to solve this optimization problem via a suitable method of nondifferentiable optimization, and
(ii) to derive 1st-order necessary optimality conditions.

The numerical tests associated with (i) have been, however, postponed to a next paper devoted to this subject.

The paper is organized as follows. At the beginning of Section 2 we introduce the geometrical and mechanical setting of our state problem, the 2D Signorini problem with given friction and a solution-dependent coefficient of friction. After recalling results from [8] concerning the solvability of the state problem, we define the shape optimization problem and prove its solvability. Section 3 is devoted to the approximation of the optimal shape design problem by developing an appropriate finite element discretization of the state problem. Existence of discrete optimal shapes is established in this section. A natural question to ask is how the solutions of the discrete problems relate to the continuous problem as the discretization parameter h tends to 0. This issue is treated in Section 4. Thereafter in Section 5 the algebraic formulation of the state problem is presented. Unfortunately, it is not equivalent to the discretized problem from Section 3, due to approximating the friction term by a quadrature formula. Therefore, properties of its solution are also investigated in detail. The respective shape optimization problem (using the reduced form of the algebraic state problem) amounts to a *Mathematical Program with Equilibrium Constraints* (MPEC), which is treated by the so-called implicit programming approach (cf. [13]). In particular, Section 6 deals with the computation of Clarke's subgradients of the composite objective functional by means of the generalized differential calculus of Mordukhovich. We conclude the paper with establishing first order optimality conditions for our MPEC.

Throughout the paper we use the following notation: the symbol $H^k(\Omega)$ ($k \geq 0$ integer) stands for the Sobolev space of functions which are together with their derivatives up to order k square integrable in Ω, i.e. elements of $L^2(\Omega)$ (we set $H^0(\Omega) \equiv L^2(\Omega)$). The norm in $H^k(\Omega)$ will be denoted by $\| \cdot \|_{k,\Omega}$. Vector functions and the respective spaces of vector functions will be denoted by bold characters. Bold characters will be used also for vectors in $\mathbb{R}^n$, where we will assume the euclidean scalar product $\langle \cdot, \cdot \rangle_n$ and norm $\| \cdot \|_n$, respectively. For a set $A \subset X$, $\overline{A}$ stands for the *closure* of A with respect to the topology of X. For $X = \mathbb{R}^n$ and $\bar{\mathbf{x}} \in A$ we denote by $\widehat{N}_A(\bar{\mathbf{x}})$ the *Fréchet (regular) normal cone* to A at $\bar{\mathbf{x}}$:

$$\widehat{N}_A\left(\bar{\mathbf{x}}\right) := \left\{ \mathbf{x}^* \in \mathbb{R}^n \;\middle|\; \limsup_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \frac{\langle \mathbf{x}^*, \mathbf{x} - \bar{\mathbf{x}} \rangle_n}{\| \mathbf{x} - \bar{\mathbf{x}} \|_n} \leq 0 \right\},$$

whereas the *limiting (Mordukhovich) normal cone* to A at $\bar{\mathbf{x}}$ will be denoted by $N_A(\bar{\mathbf{x}})$:

$$N_A\left(\bar{\mathbf{x}}\right) := \operatorname*{Lim\,sup}_{\mathbf{x} \xrightarrow{A} \bar{\mathbf{x}}} \widehat{N}_A(\mathbf{x}).$$

Here the symbol "Lim sup" stands for the Kuratowski–Painlevé outer limit of sets (cf. [19]). Given a multifunction $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we denote its graph by $\operatorname{Gr} Q := \{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \mathbf{y} \in Q(\mathbf{x}) \}$. At a reference point $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \operatorname{Gr} Q$ the *regular coderivative* of Q is given by the multifunction $\widehat{D}^* Q(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, which is defined as follows:

$$\widehat{D}^* Q\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}\right)\left(\mathbf{y}^*\right) := \left\{ \mathbf{x}^* \in \mathbb{R}^n \mid \left(\mathbf{x}^*, -\mathbf{y}^*\right) \in \widehat{N}_{\operatorname{Gr} Q}\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}\right) \right\}.$$

Analogously, the multifunction $D^*Q(\bar{\mathbf{x}}, \bar{\mathbf{y}}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$, defined by

$$D^*Q\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}\right)\left(\mathbf{y}^*\right) := \left\{ \mathbf{x}^* \in \mathbb{R}^n \mid \left(\mathbf{x}^*, -\mathbf{y}^*\right) \in N_{\mathrm{Gr}\, Q}\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}\right) \right\}$$

is called the *limiting (Mordukhovich) coderivative* of Q at $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$. Further, we will employ another important notion from the theory of generalized differentiation, namely that of calmness: a multifunction Q is said to be *calm* at $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathrm{Gr}\, Q$ provided $\exists L > 0\ \exists$ neighbourhoods U, V of $\bar{\mathbf{x}}, \bar{\mathbf{y}}$, respectively, such that:

$$Q(\mathbf{x}) \cap V \subset Q(\bar{\mathbf{x}}) + L\|\mathbf{x} - \bar{\mathbf{x}}\|_n \mathbb{B}_m(0, 1) \quad \forall \mathbf{x} \in U,$$

where $\mathbb{B}_m(0, 1)$ stands for the closed unit ball in $\mathbb{R}^m$, centered at the origin.

2 Shape Optimization Problem: Continuous Setting

This section starts with the formulation of the state problem. Let an elastic body be represented by a domain

$$\Omega = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a, b),\ \alpha(x_1) < x_2 < \gamma \right\} \tag{1}$$

and denote

$$\Gamma_c = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (a, b),\ x_2 = \alpha(x_1) \right\}, \tag{2}$$

where $\alpha : [a, b] \to \mathbb{R}^1_+$ is a non-negative, Lipschitz continuous function; $-\infty < a < b < \infty$, $\gamma > 0$ given. The boundary $\partial\Omega$ will be split into three non-empty, disjoint parts Γ_P, Γ_u and Γ_c (given by (2)) with different boundary conditions: on Γ_u the body is fixed, while surface tractions of density $\mathbf{P} = (P_1, P_2)$ act along Γ_P. On Γ_c, representing the contact part of $\partial\Omega$, the body will be unilaterally supported by a rigid foundation $S = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \leq 0 \}$. In this case the non-penetration conditions on Γ_c read as follows:

$$\left.\begin{aligned} &u_2\left(x_1, \alpha\left(x_1\right)\right) \geq -\alpha(x_1), \quad T_2(\mathbf{u})(x_1, \alpha(x_1)) \geq 0, \\ &\left(u_2\left(x_1, \alpha\left(x_1\right)\right) + \alpha\left(x_1\right)\right) T_2\left(\mathbf{u}\right)\left(x_1, \alpha\left(x_1\right)\right) = 0 \end{aligned}\right\} \text{ for } x_1 \in (a, b). \tag{3}$$

Here $\mathbf{u} = (u_1, u_2) : \Omega \to \mathbb{R}^2$ is a displacement vector, $\mathbf{T}(\mathbf{u}) = (T_1(\mathbf{u}), T_2(\mathbf{u})) : \partial\Omega \to \mathbb{R}^2$ is the stress vector associated with $\mathbf{u}$. In addition to (3) we shall consider effects of friction between Ω and S. We use the friction law of Tresca type, i.e. with an a-priori given slip bound $g : \Gamma_c \to \mathbb{R}_+$, but with a coefficient of friction $\mathscr{F}$ which depends on the solution. Thus the friction conditions on Γ_c read as follows:

$$\left.\begin{aligned} &u_1 = 0 \Longrightarrow |T_1(\mathbf{u})| \leq \mathscr{F}(0)g \\ &u_1 \neq 0 \Longrightarrow T_1(\mathbf{u}) = -\mathrm{sgn}(u_1)\mathscr{F}(|u_1|)g \end{aligned}\right\} \text{ on } \Gamma_c. \tag{4}$$

Finally, Ω will be subject to body forces of density $\mathbf{F} = (F_1, F_2)$. The equilibrium state of Ω is characterized by a displacement vector $\mathbf{u}$ which satisfies the system of the linear equilibrium equations in Ω, the classical boundary conditions on Γ_P, Γ_u and the unilateral and friction conditions (3) and (4), respectively on Γ_c.

To give the weak form of this problem we first introduce the following function spaces:

$$\left.\begin{aligned} V &= \left\{ v \in H^1(\Omega) \mid v = 0 \ \text{ on } \ \Gamma_u \right\} \\ \mathbf{V} &= V \times V \end{aligned}\right\} \tag{5}$$

and the closed, convex set $\mathbf{K} \subset \mathbf{V}$ of kinematically admissible displacements:

$$\mathbf{K} = \{ \mathbf{v} = (v_1, v_2) \in \mathbf{V} \mid v_2\,(x_1, \alpha\,(x_1)) \geq -\alpha(x_1) \ \text{ a.e. in } \ (a, b) \}\,. \tag{6}$$

Further let $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $L : \mathbf{V} \to \mathbb{R}$ be defined by

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \tau_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})\, dx \\ L(\mathbf{v}) &= \int_\Omega F_i v_i\, dx + \int_{\Gamma_P} P_i v_i\, ds, \qquad \mathbf{F} \in \mathbf{L}^2(\Omega), \mathbf{P} \in \mathbf{L}^2(\partial\Omega), \end{aligned} \tag{7}$$

where $\tau(\mathbf{u}) = \left(\tau_{ij}(\mathbf{u})\right)_{i,j=1}^2$, $\varepsilon(\mathbf{u}) = \left(\varepsilon_{ij}(\mathbf{u})\right)_{i,j=1}^2$ is the stress tensor and the linearized strain tensor, respectively corresponding to $\mathbf{u}$. The constitutive law between $\tau(\mathbf{u})$ and $\varepsilon(\mathbf{u})$ is given by linear Hooke's law:

$$\tau_{ij}(\mathbf{u}) = c_{ijkl} \varepsilon_{kl}(\mathbf{u}). \tag{8}$$

The elasticity coefficients $c_{ijkl} \in L^\infty(\hat{\Omega})$ satisfy the usual symmetry and ellipticity conditions in $\hat{\Omega}$:

$$\left.\begin{aligned} c_{ijkl} = c_{jikl} = c_{klij} \quad &\text{a.e. in } \ \hat{\Omega} \quad \forall i, j, k, l = 1, 2 \\ \exists C_{ell} > 0 : \ c_{ijkl}(x) \xi_{ij} \xi_{kl} \geq C_{ell} \xi_{ij} \xi_{ij} \quad &\text{a.a. } \ x \in \hat{\Omega} \quad \forall \xi_{ij} = \xi_{ji} \in \mathbb{R}, \end{aligned}\right\} \tag{9}$$

where $\hat{\Omega} \supset \Omega$ is a given domain whose choice will be specified later. Recall that the components of the stress vector $\mathbf{T}(\mathbf{u})$ are given by $T_i(\mathbf{u}) = \tau_{ij}(\mathbf{u}) n_j$, $i = 1, 2$, where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal vector to $\partial\Omega$.

The function $\mathscr{F} : \mathbb{R}_+ \to \mathbb{R}_+$ in (4) is assumed to be *continuous* and *bounded* in $\mathbb{R}_+$.

Definition 1 By a *weak solution* to the Signorini problem with Tresca model of friction and a solution dependent coefficient of friction $\mathscr{F}$ we mean any $\mathbf{u} \in \mathbf{K}$ solving the following implicit variational inequality:

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_{\Gamma_c} \mathscr{F}(|u_1|) g\,(|v_1| - |u_1|)\; ds \geq L(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}. \tag{$\mathscr{P}$}$$

From Green's formula it easily follows that ($\mathscr{P}$) is formally equivalent to the classical formulation, in particular we recover (3) and (4).

The following existence and uniqueness results have been established in [8].

Theorem 1 *Let the slip bound $g \in L^2(\Gamma_c)$, $g \geq 0$ on Γ_c. Then ($\mathscr{P}$) has at least one solution.*

Theorem 2 *Let $g \in L^\infty(\Gamma_c)$, $g \geq 0$ on Γ_c and let the coefficient of friction be Lipschitz continuous in $\mathbb{R}_+$:*

$$\exists C_L > 0 : \quad |\mathscr{F}(x) - \mathscr{F}(\bar{x})| \leq C_L |x - \bar{x}| \quad \forall x, \bar{x} \in \mathbb{R}_+.$$

If

$$0 < C_L \|g\|_{0,\infty,\Gamma_c} < \frac{C_{ell} C_K}{C_{tr}^2}, \tag{10}$$

where C_{ell} is the constant from (9), C_K is the constant in Korn's inequality and C_{tr} is the norm of the trace mapping $\mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Gamma_c)$, then $(\mathscr{P})$ has a unique solution.

Up to now the function α defining the contact part Γ_c of the boundary was fixed. From now on we shall consider α to be a *design variable* by means of which one can change the shape of Ω. To emphasize the fact that our state problem $(\mathscr{P})$ is now parametrized by α, we shall write α as the argument. Thus we shall use the following notation: $\Omega(\alpha)$, $\Gamma_c(\alpha)$, $\mathbf{V}(\alpha)$, $\mathbf{K}(\alpha)$, etc. instead of Ω, Γ_c, $\mathbf{V}$, $\mathbf{K}$, etc. Similarly, the bilinear form a and the linear term L on $\Omega(\alpha)$ will be denoted by a_α and L_α, respectively.

In what follows we shall restrict ourselves to α belonging to the *admissible set U_{ad}* defined by

$$\begin{aligned} U_{ad} = \big\{ \alpha \in C^{1,1}([a,b]) \,|\, 0 \leq \alpha \leq C_0 \text{ in } [a,b], \\ |\alpha(x) - \alpha(\bar{x})| \leq C_1 |x - \bar{x}| \ \forall x, \bar{x} \in [a,b], \\ |\alpha'(x) - \alpha'(\bar{x})| \leq C_2 |x - \bar{x}| \ \forall x, \bar{x} \in [a,b], \\ \text{meas}\, \Omega(\alpha) = C_3 \big\}, \end{aligned} \tag{11}$$

i.e. U_{ad} contains all functions which are together with their first derivatives Lipschitz equi-continuous in $[a,b]$ and preserve the constant area of $\Omega(\alpha)$. We shall suppose that the constants C_0, C_1, C_2 and C_3 are chosen in such a way that $U_{ad} \neq \emptyset$.

Remark 1 Since most of the results in the subsequent parts is valid for a larger class of admissible functions we introduce the set

$$\begin{aligned} Q_{ad} = \big\{ \alpha \in C^{0,1}([a,b]) \,|\, 0 \leq \alpha \leq C_0 \text{ in } [a,b], \\ |\alpha(x) - \alpha(\bar{x})| \leq C_1 |x - \bar{x}| \ \forall x, \bar{x} \in [a,b], \\ \text{meas}\, \Omega(\alpha) = C_3 \big\} \supset U_{ad}, \end{aligned} \tag{12}$$

with the same C_0, C_1 and C_3 as in (11).

The set $\mathscr{O}$ of *admissible shapes* is defined by

$$\mathscr{O} = \{ \Omega(\alpha) \,|\, \alpha \in U_{ad} \} \tag{13}$$

with $\Omega(\alpha)$ as in (1). Here and in what follows we take $\hat{\Omega} = (a,b) \times (0,\gamma) \supset \Omega(\alpha)$ $\forall \alpha \in U_{ad}$, where $\gamma > 0$ is from (1). In the sequel we shall suppose that $\mathbf{F} \in \mathbf{L}^2(\hat{\Omega})$, $\mathbf{P} \in \mathbf{L}^2(\partial\hat{\Omega})$ and (9) holds for this choice of $\hat{\Omega}$. Finally, the slip bound g on $\Gamma_c(\alpha)$ will be assumed to be the trace on $\Gamma_c(\alpha)$ of a function $g \in H^1(\hat{\Omega})$, $g \geq 0$ a.e. in $\hat{\Omega}$.

Let $I : \Delta \to \mathbb{R}$, where $\Delta = \{ (\alpha, \mathbf{y}) \,|\, \alpha \in Q_{ad},\ \mathbf{y} \in \mathbf{V}(\alpha) \}$ be a *cost functional* and denote

$$\mathscr{G} = \left\{ (\alpha, \mathbf{u}) \,|\, \alpha \in U_{ad},\ \mathbf{u} \text{ solves } (\mathscr{P}(\alpha)) \right\} \subset \Delta \tag{14}$$

the graph of the respective control-to-state mapping S, which is *multivalued*, in general.

Definition 2 A domain $\Omega(\alpha^*) \in \mathscr{O}$ is said to be *optimal* iff a pair $(\alpha^*, \mathbf{u}^*)$ solves the following problem:

$$\left.\begin{aligned} &\text{Find } \left(\alpha^*, \mathbf{u}^*\right) \in \mathscr{G} \text{ such that:} \\ &I\left(\alpha^*, \mathbf{u}^*\right) \le I(\alpha, \mathbf{u}) \quad \forall (\alpha, \mathbf{u}) \in \mathscr{G}. \end{aligned}\right\} \tag{$\mathbb{P}$}$$

To prove the existence of optimal shapes we first introduce convergence of domains belonging to $\mathscr{O}$. Let $\Omega_n := \Omega(\alpha_n)$, $\Omega := \Omega(\alpha)$, $\alpha_n, \alpha \in U_{ad}$, $n = 1, 2, \ldots$ We say that $\{\Omega_n\}$ tends to Ω and write $\Omega_n \xrightarrow{\mathscr{O}} \Omega$ iff $\alpha_n \to \alpha$ in $C^1([a, b])$.

Since different functions have different domains of definition, we shall need their extension to the common domain $\hat{\Omega}$: let $\mathbf{v} \in \mathbf{H}^1(\Omega(\alpha))$ for some $\alpha \in Q_{ad}$. Its extension from $\Omega(\alpha)$ to $\hat{\Omega}$ will be denoted by $\tilde{\mathbf{v}}$, i.e. $\tilde{\mathbf{v}} \in \mathbf{H}^1(\hat{\Omega})$, $\tilde{\mathbf{v}}|_{\Omega(\alpha)} = \mathbf{v}$ on $\Omega(\alpha)$ and

$$\exists c > 0 : \quad \|\tilde{\mathbf{v}}\|_{1,\hat{\Omega}} \le c\|\mathbf{v}\|_{1,\Omega(\alpha)} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega(\alpha)). \tag{15}$$

Denote

$$\mathscr{M} = \{ \Omega(\alpha) \,|\, \alpha \in Q_{ad} \} \supset \mathscr{O}. \tag{16}$$

Domains belonging to $\mathscr{M}$ possess the uniform extension property hence the constant c in (15) can be chosen to be independent of $\alpha \in Q_{ad}$ (see [3]).

Convention From now on the symbol "$\tilde{}$" above a function $\mathbf{v} \in \mathbf{H}^1(\Omega(\alpha))$ denotes its extension on $\hat{\Omega}$ satisfying (15) with $c > 0$ independent of $\alpha \in Q_{ad}$.

The main result of this section is the following existence theorem.

Theorem 3 *Let the cost functional I be lower semicontinuous in the following sense:*

$$\left.\begin{aligned} &\Omega_n \xrightarrow{\mathscr{O}} \Omega, \quad \Omega_n, \Omega \in \mathscr{O} \\ &\mathbf{y}_n \rightharpoonup \mathbf{y} \ \text{in} \ \mathbf{H}^1(\hat{\Omega}),\ \mathbf{y}_n, \mathbf{y} \in \mathbf{H}^1(\hat{\Omega}) \end{aligned}\right\} \Longrightarrow \liminf_{n\to\infty} I\left(\alpha_n, \mathbf{y}_n|_{\Omega_n}\right) \ge I\left(\alpha, \mathbf{y}|_{\Omega}\right). \tag{17}$$

Then ($\mathbb{P}$) *has a solution.*

The crucial role in the proof of Theorem 3 plays the following compactness result.

Lemma 1 *The set* $\mathscr{G}$ *is compact in the following sense:*

$$\forall \{(\alpha_n, \mathbf{u}_n)\} \subset \mathscr{G}\ \exists\{(\alpha_{n_j}, \mathbf{u}_{n_j})\} \subset \{(\alpha_n, \mathbf{u}_n)\}\ \exists(\alpha, \mathbf{u}) \in \mathscr{G} :$$

$$\alpha_{n_j} \to \alpha \ \text{in} \ C^1([a, b]), \quad \tilde{\mathbf{u}}_{n_j} \rightharpoonup \tilde{\mathbf{u}} \ \text{in} \ \mathbf{H}^1(\hat{\Omega}), \quad j \to \infty.$$

Proof Let $\{(\alpha_n, \mathbf{u}_n)\} \subset \mathscr{G}$ be given. Without loss of generality we may assume that $\alpha_n \to \alpha \in U_{ad}$ in $C^1([a, b])$ as follows from the Arzelà–Ascoli theorem and the definition of U_{ad}. Arguing as in [5, Lemma 7.2] one can show that $\{\tilde{\mathbf{u}}_n\}$ is bounded in $\mathbf{H}^1(\hat{\Omega})$ so that for an appropriate subsequence $\{\tilde{\mathbf{u}}_{n_j}\} \subset \{\tilde{\mathbf{u}}_n\}$ we have $\tilde{\mathbf{u}}_{n_j} \rightharpoonup \tilde{\mathbf{u}}$ in $\mathbf{H}^1(\hat{\Omega})$. Moreover $\tilde{\mathbf{u}}|_{\Omega(\alpha)} \in \mathbf{K}(\alpha)$.

It remains to show that $(\alpha, \tilde{\mathbf{u}}|_{\Omega(\alpha)}) \in \mathscr{G}$. Let $\xi \in \mathbf{K}(\alpha)$. Then one can find a sequence $\{\xi_l\} \subset \mathbf{H}^1(\hat{\Omega})$ such that $\xi_l \to \tilde{\xi}$ in $\mathbf{H}^1(\hat{\Omega})$ and for every $l \in \mathbb{N}$ there exists $n_0(l) \in \mathbb{N}$ such that $\xi_l|_{\Omega_n} \in \mathbf{K}(\alpha_n) \quad \forall n \geq n_0(l)$ (see [5]). Let $l \in \mathbb{N}$ be fixed. Since $\xi_l|_{\Omega_{n_j}} \in \mathbf{K}(\alpha_{n_j})$ for j large enough, the definition of $(\mathscr{P}(\alpha_{n_j}))$ yields:

$$a_{\alpha_{n_j}}\left(\mathbf{u}_{n_j}, \xi_l - \mathbf{u}_{n_j}\right) + \int_{\Gamma_c\left(\alpha_{n_j}\right)} \mathscr{F}\left(|u_{n_j 1}|\right) g\left(|\xi_{l1}| - |u_{n_j 1}|\right)\, ds \geq L_{\alpha_{n_j}}\left(\xi_l - \mathbf{u}_{n_j}\right). \tag{18}$$

We now pass to the limit first with $j \to \infty$ and then with $l \to \infty$ in (18). The limit passage in the first and the third term has been already done in [5]:

$$\lim_{l\to\infty}\left(\limsup_{j\to\infty} a_{\alpha_{n_j}}\left(\mathbf{u}_{n_j}, \xi_l - \mathbf{u}_{n_j}\right)\right) \leq a_\alpha\left(\tilde{\mathbf{u}}, \xi - \tilde{\mathbf{u}}\right)$$

$$\lim_{l\to\infty}\left(\liminf_{j\to\infty} L_{\alpha_{n_j}}\left(\xi_l - \mathbf{u}_{n_j}\right)\right) \geq L_\alpha\left(\xi - \tilde{\mathbf{u}}\right). \tag{19}$$

To complete the proof we show that

$$\lim_{l\to\infty}\left(\lim_{j\to\infty}\int_{\Gamma_c\left(\alpha_{n_j}\right)} \mathscr{F}\left(|u_{n_j 1}|\right) g\left(|\xi_{l1}| - |u_{n_j 1}|\right)\, ds\right) = \int_{\Gamma_c(\alpha)} \mathscr{F}\left(|\tilde{u}_1|\right) g\left(|\xi_1| - |\tilde{u}_1|\right)\, ds. \tag{20}$$

Denote the integral on the left of (20) as $I^{(j)}$. Then

$$\begin{aligned} I^{(j)} &:= \int_a^b \mathscr{F}\left(|u_{n_j 1}\circ\alpha_{n_j}|\right) g\circ\alpha_{n_j}\left(|\xi_{l1}\circ\alpha_{n_j}| - |u_{n_j 1}\circ\alpha_{n_j}|\right)\sqrt{1+\left(\alpha'_{n_j}\right)^2}\, dx_1 \\ &= \int_a^b \mathscr{F}\left(|u_{n_j 1}\circ\alpha_{n_j}|\right) g\circ\alpha\left(|\xi_{l1}\circ\alpha| - |u_1\circ\alpha|\right)\sqrt{1+\left(\alpha'_{n_j}\right)^2}\, dx_1 \\ &\quad + \int_a^b \mathscr{F}\left(|u_{n_j 1}\circ\alpha_{n_j}|\right)\left(g\circ\alpha_{n_j}|\xi_{l1}\circ\alpha_{n_j}| - g\circ\alpha|\xi_{l1}\circ\alpha|\right)\sqrt{1+\left(\alpha'_{n_j}\right)^2}\, dx_1 \\ &\quad - \int_a^b \mathscr{F}\left(|u_{n_j 1}\circ\alpha_{n_j}|\right)\left(g\circ\alpha_{n_j}|u_{n_j 1}\circ\alpha_{n_j}| - g\circ\alpha|u_1\circ\alpha|\right)\sqrt{1+\left(\alpha'_{n_j}\right)^2}\, dx_1 \\ &=: I_1^{(j)} + I_2^{(j)} - I_3^{(j)}. \end{aligned}$$

Next we show that $I_3^{(j)} \to 0,\ j \to \infty$. From the definition of U_{ad} and the fact that $\max_{x\in\mathbb{R}_+} \mathscr{F}(x) \leq \overline{\mathscr{F}}$ for some $\overline{\mathscr{F}} > 0$ we obtain:

$$\left|I_3^{(j)}\right| \leq \overline{\mathscr{F}}\sqrt{1+C_1^2}\int_a^b g\circ\alpha_{n_j}\left(|u_{n_j}\circ\alpha_{n_j}| - g\circ\alpha|u_1\circ\alpha|\right) dx_1.$$

Adding and subtracting the term $g \circ \alpha_{n_j} |u_1 \circ \alpha|$ we have:

$$\begin{aligned}\left|I_3^{(j)}\right| &\leq \overline{\mathscr{F}}\sqrt{1+C_1^2}\big(\|g\circ\alpha_{n_j}\|_{L^2(a,b)}\|u_{n_j1}\circ\alpha_{n_j} - u_1\circ\alpha\|_{L^2(a,b)} \\ &\quad + \|u_1\circ\alpha\|_{L^2(a,b)}\|g\circ\alpha_{n_j} - g\circ\alpha\|_{L^2(a,b)}\big) \\ &\leq \overline{\mathscr{F}}\sqrt{1+C_1^2}\big(\hat{C}_{tr}\|g\|_{1,\hat{\Omega}}\|u_{n_j1}\circ\alpha_{n_j} - u_1\circ\alpha\|_{L^2(a,b)} \\ &\quad + \|u_1\circ\alpha\|_{L^2(a,b)}\|g\circ\alpha_{n_j} - g\circ\alpha\|_{L^2(a,b)}\big)\end{aligned}$$

where $\hat{C}_{tr}$ is the norm of the trace mapping $H^1(\hat{\Omega}) \to L^2(a,b)$ and it can be estimated independently of $\alpha \in U_{ad}$ (see [18, Theorem A.4]). Since $\|u_{n_j1}\circ\alpha_{n_j} - u_1 \circ \alpha\|_{L^2(a,b)} \to 0$, $\|g\circ\alpha_{n_j} - g\circ\alpha\|_{L^2(a,b)} \to 0$, $j\to\infty$ (see [18, Lemma 2.1]) we obtain that $I_3^{(j)} \to 0$, $j\to\infty$.

Similarly one can show that $I_2^{(j)} \to 0$, $j\to\infty$. Finally, from the Lebesgue dominated convergence theorem and the fact that $\alpha_{n_j} \to \alpha$ in $C^1([a,b])$ we arrive at

$$I_1^{(j)} \underset{j\to\infty}{\longrightarrow} \int_{\Gamma_c(\alpha)} \mathscr{F}(|u_1|)g(|\xi_l| - |u_1|)\,ds \underset{l\to\infty}{\longrightarrow} \int_{\Gamma_c(\alpha)} \mathscr{F}(|u_1|)g(|\xi| - |u_1|)\,ds.$$

From this and (19) we arrive at the assertion of Lemma 1. □

Proof (Theorem 3) It follows from (17) and Lemma 1. □

From Theorem 2 we know that the solution of $(\mathscr{P}(\alpha))$ is unique provided that (10) is satisfied. All the constants appearing on the right of (10) can be chosen independently of $\alpha \in U_{ad}$. In particular, for the constant C_K in Korn's inequality this property has been proven in [16]. Thus one can establish sufficient conditions independent of $\alpha \in U_{ad}$ under which $(\mathscr{P}(\alpha))$ has a unique solution.

3 Discretization of $(\mathbb{P})$

In this section we shortly describe a discretization of problem $(\mathbb{P})$ by using a piecewise linear approximation of U_{ad} and a finite element approximation of the state problem.

Let $d \geq 1$ be a given integer and set $h := (b-a)/d$. By δ_h we denote the equidistant partition of $[a,b]$:

$$\delta_h: \quad a \equiv a_0 < a_1 < \cdots < a_{d(h)} \equiv b, \qquad a_j = a + jh,\ j = 0,\ldots,d. \tag{21}$$

With any δ_h we associate the set U_{ad}^h defined by

$$\begin{aligned}U_{ad}^h := \Big\{\, &\alpha_h \in C([a,b]) \mid \alpha_h|_{[a_{i-1},a_i]} \in P_1([a_{i-1},a_i]) \quad \forall i = 1,\ldots,d, \\ &0 \leq \alpha_h(a_i) \leq C_0 \quad \forall i = 0,\ldots,d, \\ &|\alpha_h(a_i) - \alpha_h(a_{i-1})| \leq C_1 h \quad \forall i = 1,\ldots,d, \\ &|\alpha_h(a_{i+1}) - 2\alpha_h(a_i) + \alpha_h(a_{i-1})| \leq C_2 h^2, \quad \forall i = 1,\ldots,d-1, \\ &\text{meas}\,\Omega(\alpha_h) = C_3 \,\Big\},\end{aligned} \tag{22}$$

where $C_0, \dots, C_3$ are the same as in (11). Notice that $U_{ad}^h \subset Q_{ad}$ but $U_{ad}^h \not\subset U_{ad}$, i.e. U_{ad}^h is the *external approximation* of U_{ad}. We denote the set of all *discrete admissible* shapes by

$$\mathscr{O}_h := \left\{ \Omega(\alpha_h) \,|\, \alpha_h \in U_{ad}^h \right\} \tag{23}$$

with $\Omega(\alpha_h)$ defined by (1) replacing α by α_h.

Since $\Omega(\alpha_h)$ is a polygonal domain, one can construct its triangulation $\mathscr{T}(h, \alpha_h)$ whose nodes lie on the lines $\{a_i\} \times \mathbb{R}_+$, $i = 0, \dots, d$ for $\forall \alpha_h \in U_{ad}^h$.

Let $h > 0$ be fixed. Next we shall use the system $\{\mathscr{T}(h, \alpha_h)\}$, $\alpha_h \in U_{ad}^h$ which consists of *topologically equivalent* triangulations, i.e.

($\mathscr{T}$1) the number of the nodes in $\mathscr{T}(h, \alpha_h)$ as well as the neighbours of each triangle from $\mathscr{T}(h, \alpha_h)$ are the *same* for all $\alpha_h \in U_{ad}^h$;

($\mathscr{T}$2) the position of the nodes of $\mathscr{T}(h, \alpha_h)$ depends *continuously* on changes of $\alpha_h \in U_{ad}^h$;

($\mathscr{T}$3) the triangulations $\mathscr{T}(h, \alpha_h)$ are *compatible* with the decomposition of $\partial\Omega(\alpha_h)$ into $\Gamma_c(\alpha_h)$, $\Gamma_P(\alpha_h)$ and $\Gamma_u(\alpha_h)$ for any $h > 0$ and any $\alpha_h \in U_{ad}^h$.

In order to establish convergence results we shall also need:

($\mathscr{T}$4) the system $\{\mathscr{T}(h, \alpha_h)\}$ is *uniformly regular* with respect to $h > 0$ as well as $\alpha_h \in U_{ad}^h$, i.e. there exists a constant $\theta_0 > 0$ such that

$$\theta(h, \alpha_h) \geq \theta_0 \quad \forall h > 0 \; \forall \alpha_h \in U_{ad}^h,$$

where $\theta(h, \alpha_h)$ denotes the minimal interior angle of all triangles from $\mathscr{T}(h, \alpha_h)$.

The domain $\Omega(\alpha_h)$ with the triangulation $\mathscr{T}(h, \alpha_h)$ will be denoted by $\Omega_h(\alpha_h)$, or just shortly Ω_h in what follows.

On $\Omega_h(\alpha_h)$, $\alpha_h \in U_{ad}^h$ we construct the following piecewise linear approximations of $V(\alpha_h)$, $\mathbf{V}(\alpha_h)$ and $\mathbf{K}(\alpha_h)$:

$$V_h(\alpha_h) := \left\{ v_h \in C\left(\overline{\Omega}_h\right) \,|\, v_h|_T \in P_1(T) \;\; \forall T \in \mathscr{T}(h, \alpha_h), \;\; v_h = 0 \;\text{ on }\; \overline{\Gamma}_u(\alpha_h) \right\},$$
$$\mathbf{V}_h(\alpha_h) := V_h(\alpha_h) \times V_h(\alpha_h),$$

and

$$\mathbf{K}_h(\alpha_h) := \{\mathbf{v}_h = (v_{h1}, v_{h2}) \in \mathbf{V}_h(\alpha_h) \,|\, v_{h2}(a_i, \alpha_h(a_i)) \geq -\alpha_h(a_i) \; \forall a_i \in \mathscr{N}_h\},$$

respectively, where $\mathscr{N}_h$ is the set of all contact nodes, i.e. $a_i \in \mathscr{N}_h$ iff $(a_i, \alpha_h(a_i)) \in \overline{\Gamma}_c(\alpha_h) \setminus \overline{\Gamma}_u(\alpha_h)$. Observe, that $\mathbf{K}_h(\alpha_h) \subset \mathbf{K}(\alpha_h) \;\; \forall h > 0 \; \forall \alpha_h \in U_{ad}^h$.

The *discrete state problem* reads as follows:

$$\left.\begin{array}{l} \text{Find } \; \mathbf{u}_h := \mathbf{u}_h(\alpha_h) \in \mathbf{K}_h(\alpha_h) \; \text{ such that:} \\ a_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\ \quad + \displaystyle\int_a^b \mathscr{F}(r_h|u_{h1} \circ \alpha_h|) g \circ \alpha_h \left(|v_{h1} \circ \alpha_h| - |u_{h1} \circ \alpha_h|\right) \sqrt{1 + \left(\alpha_h'\right)^2}\, dx_1 \\ \qquad \geq L_h(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in \mathbf{K}_h(\alpha_h), \end{array}\right\} \qquad (\mathscr{P}_h(\alpha_h))$$

where $r_h : C([a, b]) \to C([a, b])$ stands for the piecewise linear Lagrange interpolation operator on δ_h and $a_h := a_{\alpha_h}$, $L_h := L_{\alpha_h}$.

As far the existence and uniqueness of solutions to ($\mathscr{P}_h(\alpha_h)$) is concerned, the following results are available.

Theorem 4 *Let $\mathscr{F}$ be a non-negative, bounded and continuous function in $\mathbb{R}_+$. Then ($\mathscr{P}_h(\alpha_h)$) has a solution for any $h > 0$ and $\alpha_h \in U^h_{ad}$.*

Proof See [8]. □

Theorem 5 *Let $g \in C(\overline{\hat{\Omega}})$, $g \geq 0$, and $\mathscr{F}$ be non-negative, bounded and Lipschitz continuous in $\mathbb{R}_+$:*

$$|\mathscr{F}(x) - \mathscr{F}(\bar{x})| \leq C_L|x - \bar{x}| \quad \forall x, \bar{x} \in \mathbb{R}_+. \tag{24}$$

There exists a constant $\bar{C} > 0$ such that if

$$C_L \|g\|_{C(\bar{\hat{\Omega}})} < \bar{C}$$

then the discrete state problems ($\mathscr{P}_h(\alpha_h)$) admit a unique solution for any $h > 0$ and any $\alpha_h \in U^h_{ad}$.

The proof and the explicit form of $\bar{C}$ can be found in [18].

Let

$$\mathscr{G}_h = \left\{ (\alpha_h, \mathbf{u}_h) \,|\, \alpha_h \in U^h_{ad},\ \mathbf{u}_h \text{ solves } (\mathscr{P}_h(\alpha_h)) \right\}$$

be the graph of the discrete, generally multivalued control-to-state mapping. The *discrete shape optimization problem* is defined by:

$$\left.\begin{array}{l} \text{Find } \left(\alpha_h^*, \mathbf{u}_h^*\right) \in \mathscr{G}_h \text{ such that:} \\ I\left(\alpha_h^*, \mathbf{u}_h^*\right) \leq I(\alpha_h, \mathbf{u}_h) \quad \forall (\alpha_h, \mathbf{u}_h) \in \mathscr{G}_h. \end{array}\right\} \tag{$\mathbb{P}_h$}$$

The standard way of proving the existence of a solution to ($\mathbb{P}_h$) is to use its algebraic formulation. In our case, however, this formulation *is not* fully equivalent to ($\mathbb{P}_h$) since it contains the frictional term evaluated only approximately by using the numerical integration. To avoid this discrepancy we use the original setting of ($\mathbb{P}_h$).

Modifying the approach from the previous section to the discrete case one can show that the graph $\mathscr{G}_h$ is *compact* for any $h > 0$ (for the detailed proof see [18]) so that the following result is straightforward.

Theorem 6 *Let the cost functional I be lower semicontinuous in the following sense:*

$$\left.\begin{array}{l} \alpha_h^{(n)} \to \alpha_h \text{ in } C([a,b]),\ \alpha_h^{(n)}, \alpha_h \in U^h_{ad} \\ \mathbf{y}^{(n)} \rightharpoonup \mathbf{y} \quad \text{in } \mathbf{H}^1(\hat{\Omega}),\ \mathbf{y}^{(n)}, \mathbf{y} \in \mathbf{H}^1(\hat{\Omega}),\ n \to \infty \end{array}\right\} \Longrightarrow \liminf_{n\to\infty} I\left(\alpha_h^{(n)}, \mathbf{y}^{(n)}|_{\Omega_h^{(n)}}\right) \geq I(\alpha, \mathbf{y}|_{\Omega_h}),$$

where $\Omega_h^{(n)} := \Omega_h(\alpha_h^{(n)})$, $\Omega_h := \Omega_h(\alpha_h)$. Then ($\mathbb{P}_h$) has a solution.

4 Convergence Analysis

In this section we shall analyze the mutual relation between solutions to ($\mathbb{P}_h$) and ($\mathbb{P}$) as $h \to 0_+$ aiming to show that the discrete optimal shapes converge in some sense to an optimal shape in the continuous setting.

We start by recalling two auxiliary lemmas concerning the relationship of the discrete admissible sets U_{ad}^h, $h \to 0_+$ and U_{ad} whose proof can be found in [10] (see also [18]).

Lemma 2 *For any $\alpha \in U_{ad}$ there exists a sequence $\{\alpha_h\}$, $\alpha_h \in U_{ad}^h$ such that $\alpha_h \to \alpha$ in $C([a,b])$, $h \to 0_+$.*

Lemma 3 *Let $\{\alpha_h\}$, $\alpha_h \in U_{ad}^h$ be such that $\alpha_h \to \alpha$ in $C([a,b])$, $h \to 0_+$. Then $\alpha \in U_{ad}$ and there exists a subsequence $\{\alpha_{h_m}\} \subset \{\alpha_h\}$ satisfying:*

$$\alpha'_{h_m} \to \alpha' \ \ in \ \ L^{\infty}(a,b), \quad h_m \to 0_+.$$

We shall also need the following result on the approximation of a function $\xi \in \mathbf{K}(\alpha)$, $\alpha \in U_{ad}$.

Lemma 4 *Let $\alpha_h \to \alpha$ in $C([a,b])$, $h \to 0_+$, where $\alpha_h \in U_{ad}^h$, $\alpha \in U_{ad}$ and let $\xi \in \mathbf{K}(\alpha)$ be given. Then there exists a sequence $\{\xi_l\} \subset \mathbf{H}^2(\hat{\Omega})$ such that*

$$\xi_l \to \tilde{\xi} \ \ in \ \ \mathbf{H}^1(\hat{\Omega}),\ l \to \infty \tag{25}$$

and $\forall l \in \mathbb{N}\ \exists h_0 := h_0(l) > 0$ such that

$$\xi_l|_{\Omega(\alpha_h)} \in \mathbf{K}(\alpha_h) \quad \forall h < h_0. \tag{26}$$

Proof See Lemma 3.1 in [7]. □

In order to pass to the limit in the frictional term we now prove the following result.

Lemma 5 *Let $\{\alpha_h\}$, $\alpha_h \in U_{ad}^h$ and $\{v_h\}$, $v_h \in V_h(\alpha_h)$, $h \to 0_+$ be such that*

$$\alpha_h \to \alpha \ \ in \ \ C([a,b]) \ \ and \ \ \tilde{v}_h \rightharpoonup v \ \ in \ \ H^1(\hat{\Omega}),$$

where $\alpha \in U_{ad}$ and $v \in H^1(\hat{\Omega})$. Then

$$r_h|v_h \circ \alpha_h| \to |v \circ \alpha| \ \ in \ \ L^2(a,b),\ h \to 0_+. \tag{27}$$

Proof From the triangle inequality

$$\|r_h|v_h \circ \alpha_h| - |v \circ \alpha|\|_{L^2(a,b)} \le \|r_h|v_h \circ \alpha_h| - |v_h \circ \alpha_h|\|_{L^2(a,b)} + \||v_h \circ \alpha_h| - |v \circ \alpha|\|_{L^2(a,b)}$$

we see that it is sufficient to prove that the first term tends to zero (the second one tends to zero as follows from Lemma 2.1 in [18]). Throughout the rest of the proof let $c > 0$ denote a generic constant independent of $h > 0$.

Using the approximation property of the linear Lagrange interpolation operator r_h and the inverse inequality between $H^1(a,b)$ and $H^{1/2}(a,b)$ we may write:

$$\|r_h|v_h \circ \alpha_h| - |v_h \circ \alpha_h|\|_{L^2(a,b)} \le ch|v_h \circ \alpha_h|_{H^1(a,b)} \le ch^{1/2}\|v_h \circ \alpha_h\|_{H^{1/2}(a,b)}. \tag{28}$$

If we prove that there exists a constant $c > 0$ such that

$$\|v_h \circ \alpha_h\|_{H^{1/2}(a,b)} \le c\|v_h\|_{H^1(\Omega(\alpha_h))} \quad \forall h > 0\ \forall \alpha_h \in U_{ad}^h \tag{29}$$

then by the uniform extension property and the boundedness of $\{\tilde{v}_h\} \subset H^1(\hat{\Omega})$ we obtain from (28):

$$\|r_h|v_h \circ \alpha_h| - |v_h \circ \alpha_h|\|_{L^2(a,b)} \le ch^{1/2} \to 0, \quad h \to 0_+.$$

To prove (29), first define the mapping

$$\varphi_{\alpha_h}(x_1, x_2) := \left(x_1, \frac{2\gamma^2 - \gamma(x_2 + \alpha_h(x_2))}{\gamma - \alpha_h(x_2)}\right)^T$$

which maps $\Omega(\alpha_h)$ *onto* $\Theta = (a, b) \times (\gamma, 2\gamma)$. Its inverse is given by

$$\varphi_{\alpha_h}^{-1}(x_1, x_2) := \left(x_1, 2\gamma - x_2\left(1 - \frac{\alpha_h(x_1)}{\gamma}\right) - \alpha_h(x_1)\right)^T, \quad (x_1, x_2) \in \Theta.$$

It is straightforward to verify that the Jacobian $\nabla\varphi_{\alpha_h}^{-1}$ satisfies:

(i) $\exists c > 0 : \|\nabla\varphi_{\alpha_h}^{-1}\| \le c$ a.e. in $\Theta, \forall h > 0\ \forall \alpha_h \in U_{ad}^h$,
(ii) $|\det \nabla\varphi_{\alpha_h}^{-1}| = 1 - \alpha_h(x_1)/\gamma \ge 1 - C_0/\gamma$ a.e. in Θ,

where $\|\cdot\|$ in (i) stands for the Frobenius norm of the matrix. For each $h > 0$ define the function

$$w_h := v_h \circ \varphi_{\alpha_h}^{-1} \in H^1(\Theta).$$

From (i) and (ii) it easily follows:

(k) $w_h(x_1, 2\gamma) = v_h(x_1, \alpha_h(x_1)) \quad \forall x_1 \in (a, b)\ \forall h > 0$,
(kk) $\exists c > 0 : \quad \|w_h\|_{H^1(\Theta)} \le c\|v_h\|_{H^1(\Omega(\alpha_h))} \quad \forall h > 0$.

Finally, denoting $\Gamma := (a, b) \times \{2\gamma\}$ and exploiting the embedding $H^1(\Theta) \hookrightarrow H^{1/2}(\Gamma)$ and (k), (kk) we obtain

$$\|v_h \circ \alpha_h\|_{H^{1/2}(a,b)} = \|w_h\|_{H^{1/2}(\Gamma)} \le c\|w_h\|_{H^1(\Theta)} \le c\|v_h\|_{H^1(\Omega(\alpha_h))}.$$

Thus (29) is satisfied and the proof is complete. □

To establish a convergence result we shall need the following analogy of Lemma 1.

Lemma 6 *Let a sequence* $\{(\alpha_h, \mathbf{u}_h)\}$, $h \to 0_+$, *where* $(\alpha_h, \mathbf{u}_h) \in \mathscr{G}_h$, *be given. Then there exists a subsequence* $\{(\alpha_{h_j}, \mathbf{u}_{h_j})\}$ *and functions* $\alpha \in U_{ad}$, $\mathbf{u} \in \mathbf{H}^1(\hat{\Omega})$ *such that*

$$\alpha_{h_j} \to \alpha \text{ in } C([a, b]) \quad \tilde{\mathbf{u}}_{h_j} \rightharpoonup \mathbf{u} \text{ in } \mathbf{H}^1(\hat{\Omega}),\ h_j \to 0_+. \tag{30}$$

Moreover

$$(\alpha, \mathbf{u}|_{\Omega(\alpha)}) \in \mathscr{G}. \tag{31}$$

Proof The existence of a subsequence $\{\alpha_{h_j}\}$ and a function $\alpha \in U_{ad}$ such that $\alpha_{h_j} \to \alpha$ in $C([a, b])$ follows from the Arzelà–Ascoli theorem and Lemma 3. Again, it is easy to show that $\{\|\tilde{\mathbf{u}}_h\|_{H^1(\hat{\Omega})}\}$ is bounded. Thus (30) holds for an appropriate subsequence $\{(\alpha_{h_j}, \tilde{\mathbf{u}}_{h_j})\}$. To prove (31) we have to show that $\mathbf{u}|_{\Omega(\alpha)}$ solves $(\mathscr{P}(\alpha))$. The fact that $\mathbf{u}|_{\Omega(\alpha)} \in \mathbf{K}(\alpha)$ is obvious (see [5, Lemma 7.2]). Let $\xi \in \mathbf{K}(\alpha)$ be arbitrary and find a sequence $\{\xi_l\} \subset \mathbf{H}^2(\hat{\Omega})$ satisfying (25) and (26). Fix $l \in \mathbb{N}$ and denote by $\pi_{h_j}\xi_l$ the piecewise linear Lagrange interpolant of $\xi_l|_{\Omega_{h_j}}$ on $\mathscr{T}(h_j, \alpha_{h_j})$. From assumption $(\mathscr{T}4)$ on the system $\{\mathscr{T}(h, \alpha_h)\}$, $h \to 0_+$, $\alpha_h \in U_{ad}^h$ the following estimate holds:

$$\|\pi_{h_j}\xi_l - \xi_l\|_{H^1\left(\Omega_{h_j}\right)} \le ch_j \|\xi_l\|_{H^1\left(\hat{\Omega}\right)} \quad \forall h_j > 0, \tag{32}$$

where $c > 0$ does not depend on h_j. From (26) we see that $\pi_{h_j}\xi_l \in \mathbf{K}_{h_j}(\alpha_{h_j})$ for h_j small enough so that it can be used as a test function in $(\mathscr{P}(\alpha_{h_j}))$:

$$\begin{aligned} &a_{h_j}\left(\mathbf{u}_{h_j}, \pi_{h_j}\xi_l - \mathbf{u}_{h_j}\right) \\ &\quad + \int_a^b \mathscr{F}\left(|r_{h_j}|u_{h_j1} \circ \alpha_{h_j}|\right) g \circ \alpha_{h_j}\left(|\pi_{h_j}\xi_{l1} \circ \alpha_{h_j}| - |u_{h_j1} \circ \alpha_{h_j}|\right)\sqrt{1 + \left(\alpha'_{h_j}\right)^2}\, dx_1 \\ &\quad \ge L_{h_j}\left(\pi_{h_j}\xi_l - \mathbf{u}_{h_j}\right). \end{aligned} \tag{33}$$

The limit passages in the first and the third term are obvious (see [5]):

$$\lim_{l\to\infty}\left(\limsup_{h_j\to 0_+} a_{h_j}\left(\mathbf{u}_{h_j}, \pi_{h_j}\xi_l - \mathbf{u}_{h_j}\right)\right) \le a_\alpha(\mathbf{u}, \xi - \mathbf{u}), \tag{34}$$

$$\lim_{l\to\infty}\left(\liminf_{h_j\to 0_+} L_{h_j}\left(\pi_{h_j}\xi_l - \mathbf{u}_{h_j}\right)\right) \ge L_\alpha(\xi - \mathbf{u}). \tag{35}$$

The frictional term in (33) will be denoted by $I^{(h_j)}$. As in the proof of Lemma 1 this term can be written in the form: $I^{(h_j)} = I_1^{(h_j)} + I_2^{(h_j)} - I_3^{(h_j)}$, where $I_k^{(h_j)}$, $k = 1, 2, 3$ are the straightforward modifications of $I_k^{(j)}$, $k = 1, 2, 3$ from there. Arguing as in the proof of Lemma 1 one can show that

$$I_2^{(h_j)} \to 0 \ \text{ and } \ I_3^{(h_j)} \to 0 \ \text{ as } \ h_j \to 0_+$$

making use of (32). Without loss of generality we may assume that

$$\alpha'_{h_j} \to \alpha' \ \text{ a.e. in } [a, b], \quad h_j \to 0_+$$

and

$$r_{h_j}|u_{h_j1} \circ \alpha_{h_j}| \to |u_1 \circ \alpha| \ \text{ a.e. in } [a, b], \quad h_j \to 0_+$$

as a consequence of (27). From this, continuity of $\mathscr{F}$, uniform boundedness of $\{\alpha'_{h_j}\}$ and Lebesgue's dominated convergence theorem we get:

$$\begin{aligned} I_1^{(h_j)} &= \int_a^b \mathscr{F}\left(r_{h_j}|u_{h_j1} \circ \alpha_{h_j}|\right) g \circ \alpha\left(|\xi_{l1} \circ \alpha| - |u_1 \circ \alpha|\right)\sqrt{1 + \left(\alpha'_{h_j}\right)^2}\, dx_1 \\ &\xrightarrow[h_j\to 0_+]{} \int_{\Gamma_c(\alpha)} \mathscr{F}(|u_1|) g(|\xi_{l1}| - |u_1|)\, ds \xrightarrow[l\to\infty]{} \int_{\Gamma_c(\alpha)} \mathscr{F}(|u_1|) g(|\xi_1| - |u_1|)\, ds. \end{aligned}$$

This, together with (34) and (35) show that $\left(\alpha, \mathbf{u}|_{\Omega(\alpha)}\right) \in \mathscr{G}$. □

In what follows, we shall suppose that the cost functional I is *continuous* in the following sense:

$$\left.\begin{array}{l}\alpha_h \to \alpha, \text{ in } C([a,b]),\ \alpha_h \in U^h_{ad}, \alpha \in U_{ad},\\ \tilde{\mathbf{u}}_h \rightharpoonup \tilde{\mathbf{u}}, \text{ in } \mathbf{H}^1(\hat{\Omega}), \text{ where } \mathbf{u}_h, \mathbf{u} \text{ solves } (\mathscr{P}_h(\alpha_h)) \text{ and } (\mathscr{P}(\alpha)), \text{ resp.}\end{array}\right\} \Longrightarrow$$
$$\Longrightarrow \lim_{h\to 0_+} I(\alpha_h, \mathbf{u}_h) = I(\alpha, \mathbf{u}). \tag{36}$$

Further, denote

$$\overline{\mathscr{G}} := \left\{(\alpha, \mathbf{u}) \in \mathscr{G} \mid \forall\{h\},\ h \to 0_+\ \exists\{h_j\} \subset \{h\}\ \exists\left\{(\alpha_{h_j}, \mathbf{u}_{h_j})\right\},\ \left(\alpha_{h_j}, \mathbf{u}_{h_j}\right) \in \mathscr{G}_{h_j} :\right.$$
$$\left.\alpha_{h_j} \to \alpha \text{ in } C([a,b]) \text{ and } \tilde{\mathbf{u}}_{h_j} \rightharpoonup \tilde{\mathbf{u}} \text{ in } \mathbf{H}^1(\hat{\Omega}),\ h_j \to 0_+\right\}.$$

Then the following convergence result holds.

Theorem 7 *Let the cost functional I satisfy (36) and let $\{(\alpha^*_h, \mathbf{u}^*_h)\}$, $h \to 0_+$ be a sequence of optimal pairs, i.e. $(\alpha^*_h, \mathbf{u}^*_h) \in \mathscr{G}_h$ is a solution to $(\mathbb{P}_h)$. Then there exists a subsequence $\{(\alpha^*_{h_j}, \mathbf{u}^*_{h_j})\}$ and functions $\alpha^* \in U_{ad}$, $\mathbf{u} \in \mathbf{H}^1(\hat{\Omega})$ such that:*

$$\alpha^*_{h_j} \to \alpha^* \text{ in } C([a,b]) \text{ and } \tilde{\mathbf{u}}^*_{h_j} \rightharpoonup \mathbf{u} \text{ in } \mathbf{H}^1\left(\hat{\Omega}\right),\ h_j \to 0_+. \tag{37}$$

Moreover $(\alpha^*, \mathbf{u}^*) \in \overline{\mathscr{G}}$, where $\mathbf{u}^* := \mathbf{u}|_{\Omega(\alpha^*)}$ satisfies:

$$I(\alpha^*, \mathbf{u}^*) \le I(\overline{\alpha}, \overline{\mathbf{u}}) \quad \forall(\overline{\alpha}, \overline{\mathbf{u}}) \in \overline{\mathscr{G}}. \tag{38}$$

Proof The fact that $(\alpha^*, \mathbf{u}^*) \in \overline{\mathscr{G}}$ follows easily from Lemma 6 and the definition of $\overline{\mathscr{G}}$.

Now choose arbitrary $(\overline{\alpha}, \overline{\mathbf{u}}) \in \overline{\mathscr{G}}$ and an approximating sequence $\{(\overline{\alpha}_{h_j}, \overline{\mathbf{u}}_{h_j})\}$, $h_j \to 0_+$ from the definition of $\overline{\mathscr{G}}$. One has:

$$I\left(\alpha^*_{h_j}, \mathbf{u}^*_{h_j}\right) \le I\left(\overline{\alpha}_{h_j}, \overline{\mathbf{u}}_{h_j}\right) \quad \forall j = 1, 2, \ldots$$

Using (36) and passing to the limit with $h_j \to 0_+$ we immediately obtain (38). □

The set $\overline{\mathscr{G}}$ represents those optimal pairs $(\alpha, \mathbf{u}) \in \mathscr{G}$ that can be approximated by a subsequence $\{(\alpha_{h_j}, \mathbf{u}_{h_j})\}$ of discrete optimal pairs. Theorem 7 then states that from a sequence of discrete optimal pairs one can always extract a subsequence converging to a generally *sub-optimal* pair $(\alpha^*, \mathbf{u}^*) \in \overline{\mathscr{G}}$, i.e. the optimal one with respect to $\overline{\mathscr{G}}$.

It is readily seen that $(\alpha^*, \mathbf{u}^*)$ from Theorem 7 will be *optimal* in the sense of Definition 2 if and only if $\overline{\mathscr{G}} = \mathscr{G}$.

A sufficient condition is formulated in the next lemma.

Lemma 7 *Let $(\mathscr{P}(\alpha))$ be uniquely solvable $\forall \alpha \in U_{ad}$. Then $\overline{\mathscr{G}} = \mathscr{G}$.*

Proof Since $\overline{\mathscr{G}} \subset \mathscr{G}$, we need to prove the opposite inclusion.

Choose any $\alpha \in U_{ad}$ and denote by $\mathbf{u}(\alpha)$ the unique solution corresponding to $(\mathscr{P}(\alpha))$. By Lemma 2 we can find a sequence $\{\alpha_h\}$, $\alpha_h \in U^h_{ad}$ such that

$$\alpha_h \to \alpha \quad \text{in } C([a,b]),\ h \to 0_+.$$

For each h denote by $\mathbf{u}_h$ an *arbitrary* solution to the discrete state problem ($\mathscr{P}_h(\alpha_h)$), i.e. $(\alpha_h, \mathbf{u}_h) \in \mathscr{G}_h$. Due to Lemma 6 we are able to extract a subsequence so that

$$\alpha_{h_j} \to \alpha \text{ in } C([a,b]) \text{ and } \tilde{\mathbf{u}}_{h_j} \rightharpoonup \chi \text{ in } \mathbf{H}^1(\widehat{\Omega}),\ j \to \infty,$$

where $\chi|_{\Omega(\alpha)} = \mathbf{u}(\alpha)$ as follows from (31) and the uniqueness of $\mathbf{u}(\alpha)$. Thus $(\alpha, \mathbf{u}(\alpha)) \in \overline{\mathscr{G}}$ and the proof is complete. □

5 Shape Optimization Problem: Algebraic Setting

Next we shall introduce the algebraic formulation of the discretized contact problem ($\mathscr{P}_h(\alpha_h)$) and establish some basic properties of its solution. The shape optimization problem is then defined using the so-called reduced form of the state problem. Note that from now on the discretization parameter $h > 0$ is supposed to be *fixed*.

Let us set $n := \dim \mathbf{V}_h(\alpha_h)$ and $p := \operatorname{card} \mathscr{N}_h$, i.e. p is the number of the contact nodes. For the sake of simplicity let us further assume that $p = d(h) + 1$ (cf. (21)). Considering the Courant basis of the space of all piecewise linear functions over the partition δ_h of $[a,b]$, the set U^h_{ad} is isomorphic to a convex compact set $\mathscr{U}_{ad} \subset \mathbb{R}^p_+$, i.e. $\alpha_h \in U^h_{ad}$ iff $\alpha = (\alpha_1, \ldots, \alpha_p) \in \mathscr{U}_{ad}$, where $\alpha_i = \alpha_h(a_{i-1})$, $i = 1, \ldots, p$. Also, by means of the Courant basis of $\mathbf{V}_h(\alpha_h)$, the set $\mathbf{K}_h(\alpha_h)$ may be identified with the closed convex set:

$$\mathscr{K}(\alpha) := \left\{ \mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}_\nu \geq -\alpha \right\}, \quad \alpha \in \mathscr{U}_{ad},$$

where $\mathbf{v}_\nu \in \mathbb{R}^p$ stands for the subvector of $\mathbf{v} \in \mathbb{R}^n$ consisting of the second components of the displacement vector $\mathbf{v}$ at all contact nodes, i.e. $(\mathbf{v}_\nu)_i = v_{h2}(a_{i-1}, \alpha_h(a_{i-1}))\ \forall i = 1, \ldots, p$. Analogously, $\mathbf{v}_\tau \in \mathbb{R}^p$ consists of the first components of $\mathbf{v}$ at the contact nodes.

The frictional term in ($\mathscr{P}_h(\alpha_h)$) will be *approximated* by a quadrature formula whose integration nodes coincide with the contact nodes:

$$\int_a^b \mathscr{F}\left(r_h|u_{h1} \circ \alpha_h|\right) g \circ \alpha_h \left(|v_{h1} \circ \alpha_h| - |u_{h1} \circ \alpha_h|\right) \sqrt{1 + \left(\alpha_h'\right)^2}\, dx_1$$
$$\approx \sum_{i=1}^{p} \omega_i(\alpha) \mathscr{F}\left(|(\mathbf{u}_\tau)_i|\right) \left(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\right).$$

The algebraic formulation of the contact problem with a solution-dependent coefficient of friction reads as:

$$\left.\begin{aligned} &\text{Find } \mathbf{u} \in \mathscr{K}(\alpha) \text{ such that:} \\ &\langle \mathbb{A}(\alpha)\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_n + \sum_{i=1}^{p} \omega_i(\alpha) \mathscr{F}\left(|(\mathbf{u}_\tau)_i|\right)\left(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\right) \\ &\quad \geq \langle \mathbf{L}(\alpha), \mathbf{v} - \mathbf{u}\rangle_n \quad \forall \mathbf{v} \in \mathscr{K}(\alpha). \end{aligned}\right\} \qquad (\mathscr{P}'(\alpha))$$

Here $\mathbb{A} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^{n\times n})$ and $\mathbf{L} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^n)$ denote the matrix and vector-valued function, respectively associating with any $\alpha \in \mathscr{U}_{ad}$ the stiffness matrix $\mathbb{A}(\alpha)$ and the load vector $\mathbf{L}(\alpha)$, respectively. Note that the functions ω_i, $i = 1, \ldots, p$, depend on the weights of the quadrature rule and the values of α_h and g at the contact nodes, as well. We shall assume that $\omega_i \in C^1(\mathscr{U}_{ad}; (0, \infty))\ \forall i = 1, \ldots, p$.

Instead of dealing with ($\mathscr{P}'(\alpha)$), we shall be working with its equivalent formulation ($\mathscr{M}(\alpha)$), which involves Lagrange multipliers releasing the constraint $\mathbf{v} \in \mathscr{K}(\alpha)$. Let us begin with the following auxiliary problem.

For a given $\varphi \in \mathbb{R}^p_+$ consider the following problem:

$$\left.\begin{aligned} &\text{Find } (\mathbf{u}, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p_+ \text{ such that:} \\ &\langle \mathbb{A}(\alpha)\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_n + \sum_{i=1}^{p} \omega_i(\alpha)\mathscr{F}(\varphi_i)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \\ &\qquad \geq \langle \mathbf{L}(\alpha), \mathbf{v} - \mathbf{u}\rangle_n + \langle \lambda, \mathbf{v}_\nu - \mathbf{u}_\nu\rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n, \\ &\langle \mu - \lambda, \mathbf{u}_\nu + \alpha\rangle_p \geq 0 \quad \forall \mu \in \mathbb{R}^p_+, \end{aligned}\right\} \qquad (\mathscr{M}(\alpha, \varphi))$$

which represents a contact problem with given friction and a coefficient of friction, which does *not* depend on the solution. The following result is very easy to prove (see e.g. [6]).

Theorem 8 *For any* $(\alpha, \varphi) \in \mathscr{U}_{ad} \times \mathbb{R}^p_+$ *there exists a unique solution* $(\mathbf{u}, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p_+$ *to* ($\mathscr{M}(\alpha, \varphi)$).

Next, let us define the mapping:

$$\Gamma : \mathscr{U}_{ad} \times \mathbb{R}^p_+ \to \mathbb{R}^p_+, \quad (\alpha, \varphi) \mapsto |\mathbf{u}_\tau|,$$

where $\mathbf{u}$ is the first component of the solution to ($\mathscr{M}(\alpha, \varphi)$).

Corollary 1 *For* $\alpha \in \mathscr{U}_{ad}$ *given,* $\mathbf{u}$ *solves* ($\mathscr{P}'(\alpha)$) *iff there exist* $\lambda \in \mathbb{R}^p_+$ *and* $\varphi \in \mathbb{R}^p_+$ *such that* $(\mathbf{u}, \lambda)$ *is a solution to* ($\mathscr{M}(\alpha, \varphi)$) *and* φ *is a* fixed point *of* $\Gamma(\alpha, \cdot)$. *The pair* $(\mathbf{u}, \lambda)$ *is then a solution to:*

$$\left.\begin{aligned} &\textit{Find } (\mathbf{u}, \lambda) \in \mathbb{R}^n \times \mathbb{R}^p_+ \textit{ such that:} \\ &\langle \mathbb{A}(\alpha)\mathbf{u}, \mathbf{v} - \mathbf{u}\rangle_n + \sum_{i=1}^{p} \omega_i(\alpha)\mathscr{F}(|(\mathbf{u}_\tau)_i|)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \\ &\qquad \geq \langle \mathbf{L}(\alpha), \mathbf{v} - \mathbf{u}\rangle_n + \langle \lambda, \mathbf{v}_\nu - \mathbf{u}_\nu\rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n, \\ &\langle \mu - \lambda, \mathbf{u}_\nu + \alpha\rangle_p \geq 0 \quad \forall \mu \in \mathbb{R}^p_+. \end{aligned}\right\} \qquad (\mathscr{M}(\alpha))$$

Theorem 9 *For each* $\alpha \in \mathscr{U}_{ad}$*, there exists at least one solution to* ($\mathscr{M}(\alpha)$).

Proof There exists $R > 0$, such that for all $(\alpha, \varphi) \in \mathscr{U}_{ad} \times \mathbb{R}^p_+$:

$$\|\mathbf{u}\|_n \leq R, \qquad \|\lambda\|_p \leq R, \tag{39}$$

where $(\mathbf{u}, \lambda)$ is the (unique) solution of ($\mathscr{M}(\alpha, \varphi)$)—see [2, Proposition 3.3]. It is straightforward to prove that $\Gamma(\alpha, \cdot) : \mathbb{R}^p_+ \to \mathbb{R}^p_+$ is continuous, hence the existence of a solution to ($\mathscr{M}(\alpha)$) follows from Brouwer's fixed point theorem. □

Proposition 1 *Let* $\alpha \in \mathscr{U}_{ad}$ *be fixed, and* $\mathscr{F}$ *be Lipschitz continuous in* $\mathbb{R}_+$ (*cf.* (24)). *Then there exists a real* $q > 0$*, independent of* α*, such that:*

$$\|(\mathbf{u}, \lambda) - (\bar{\mathbf{u}}, \bar{\lambda})\|_{n+p} \leq q\|\varphi - \bar{\varphi}\|_p \quad \forall \varphi, \bar{\varphi} \in \mathbb{R}^p_+, \tag{40}$$

where $(\mathbf{u}, \lambda)$ *and* $(\bar{\mathbf{u}}, \bar{\lambda})$ *are the (unique) solutions to* ($\mathscr{M}(\alpha, \varphi)$) *and* ($\mathscr{M}(\alpha, \bar{\varphi})$).

Proof One may follow the steps (3.13)–(3.15) on p. 424 of [2] to obtain:

$$\left.\begin{aligned} \|\mathbf{u}-\bar{\mathbf{u}}\|_n &\le C_1 C_L \|\varphi-\bar{\varphi}\|_p, \\ \|\lambda-\bar{\lambda}\|_p &\le C_2 C_L \|\varphi-\bar{\varphi}\|_p, \end{aligned}\right\} \tag{41}$$

where the constants $C_1, C_2 > 0$ do not depend on $\alpha \in \mathscr{U}_{ad}$, $\varphi \in \mathbb{R}^p_+$ and C_L comes from (24). □

An immediate consequence of Proposition 1 is the following result on the unique solvability of ($\mathscr{M}(\alpha)$).

Theorem 10 *Suppose that $\mathscr{F}$ is Lipschitz continuous (cf. (24)) with $C_L > 0$ sufficiently small. Then ($\mathscr{M}(\alpha)$) has exactly one solution for all $\alpha \in \mathscr{U}_{ad}$.*

Proof If $C_L < C_1^{-1}$, then $(41)_1$ implies contractivity of $\Gamma(\alpha, \cdot)$ for all $\alpha \in \mathscr{U}_{ad}$. □

Next we show that under the assumptions of Theorem 10 the solution of ($\mathscr{M}(\alpha)$) is Lipschitz continuous on $\mathscr{U}_{ad}$. To do so, we shall need another auxiliary result.

Proposition 2 *Let $\varphi \in \mathbb{R}^p_+$ be fixed. Then there exists a constant $C > 0$ which does not depend on φ and such that:*

$$\left\|(\mathbf{u}, \lambda) - (\bar{\mathbf{u}}, \bar{\lambda})\right\|_{n+p} \le C\|\alpha-\bar{\alpha}\|_p \quad \forall \alpha, \bar{\alpha} \in \mathscr{U}_{ad}, \tag{42}$$

where $(\mathbf{u}, \lambda)$ and $(\bar{\mathbf{u}}, \bar{\lambda})$ are the solutions to ($\mathscr{M}(\alpha, \varphi)$) and ($\mathscr{M}(\bar{\alpha}, \varphi)$), respectively.

Proof Note that $\mathbf{u} \in \mathscr{K}(\alpha)$ and $\bar{\mathbf{u}} \in \mathscr{K}(\bar{\alpha})$ solve the following variational inequalities:

$$\langle \mathbb{A}(\alpha)\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle_n + \sum_{i=1}^{p} \omega_i(\alpha)\mathscr{F}(\varphi_i)\big(|(\mathbf{v}_\tau)_i| - |(\mathbf{u}_\tau)_i|\big) \ge \langle \mathbf{L}(\alpha), \mathbf{v}-\mathbf{u}\rangle_n \quad \forall \mathbf{v} \in \mathscr{K}(\alpha),$$

$$\langle \mathbb{A}(\bar{\alpha})\bar{\mathbf{u}}, \mathbf{v}-\bar{\mathbf{u}}\rangle_n + \sum_{i=1}^{p} \omega_i(\bar{\alpha})\mathscr{F}(\varphi_i)\big(|(\mathbf{v}_\tau)_i| - |(\bar{\mathbf{u}}_\tau)_i|\big) \ge \langle \mathbf{L}(\bar{\alpha}), \mathbf{v}-\bar{\mathbf{u}}\rangle_n \quad \forall \mathbf{v} \in \mathscr{K}(\bar{\alpha}),$$

respectively. Also note that the sets $\mathscr{K}(\alpha)$ and $\mathscr{K}(\bar{\alpha})$ may be written in the following way: $\mathscr{K}(\alpha) = \mathbf{a} + \mathscr{K}(\mathbf{0})$ and $\mathscr{K}(\bar{\alpha}) = \bar{\mathbf{a}} + \mathscr{K}(\mathbf{0})$, where the vector $\mathbf{a} \in \mathbb{R}^n$ is such that $\mathbf{a}_\nu = -\alpha$ and all its other elements are 0 (analogously for $\bar{\mathbf{a}}$). Thus:

$$\exists \mathbf{w}, \bar{\mathbf{w}} \in \mathscr{K}(\mathbf{0}): \quad \mathbf{u} = \mathbf{a} + \mathbf{w} \quad \text{and} \quad \bar{\mathbf{u}} = \bar{\mathbf{a}} + \bar{\mathbf{w}}. \tag{43}$$

Let us notice that the matrices $\mathbb{A}(\alpha)$ are positively definite uniformly with respect to $\alpha \in \mathscr{U}_{ad}$, i.e. there exists $\bar{\gamma} > 0$ such that $\langle \mathbb{A}(\alpha)\mathbf{v}, \mathbf{v}\rangle_n \ge \bar{\gamma}\|\mathbf{v}\|_n^2$ $\forall \mathbf{v} \in \mathbb{R}^n$ and $\forall \alpha \in \mathscr{U}_{ad}$. This follows from the fact that the constant C_K of Korn's inequality can be chosen independently of $\alpha \in Q_{ad}$ and the definition of topological equivalency of $\{\mathscr{T}(h, \alpha_h)\}$, $\alpha_h \in U_{ad}^h$ (see ($\mathscr{T}1$)–($\mathscr{T}4$)).

Inserting $\mathbf{v} := \mathbf{a} + \bar{\mathbf{w}} \in \mathscr{K}(\alpha)$ into the first inequality, $\mathbf{v} := \bar{\mathbf{a}} + \mathbf{w} \in \mathscr{K}(\bar{\alpha})$ into the second one and summing the two inequalities yields:

$$\begin{aligned}\bar{\gamma}\|\mathbf{w}-\bar{\mathbf{w}}\|_n^2 &\le \langle \mathbb{A}(\alpha)(\mathbf{w}-\bar{\mathbf{w}}), \mathbf{w}-\bar{\mathbf{w}}\rangle_n \\ &\le \langle \mathbb{A}(\alpha)(\bar{\mathbf{a}}-\mathbf{a}), \mathbf{w}-\bar{\mathbf{w}}\rangle_n + \langle \big(\mathbb{A}(\bar{\alpha})-\mathbb{A}(\alpha)\big)\bar{\mathbf{u}}, \mathbf{w}-\bar{\mathbf{w}}\rangle_n \\ &\quad + \sum_{i=1}^{p} \mathscr{F}(\varphi_i)\big(\omega_i(\alpha)-\omega_i(\bar{\alpha})\big)\big(|(\bar{\mathbf{w}}_\tau)_i| - |(\mathbf{w}_\tau)_i|\big) + \langle \mathbf{L}(\alpha)-\mathbf{L}(\bar{\alpha}), \mathbf{w}-\bar{\mathbf{w}}\rangle_n \\ &\le c\|\alpha-\bar{\alpha}\|_p\|\mathbf{w}-\bar{\mathbf{w}}\|_n,\end{aligned}$$

using that $\mathbb{A}$, $\mathbf{L}$, ω_i are Lipschitz continuous in $\mathscr{U}_{ad}$ and $\mathscr{F}$ is bounded. Hence:

$$\|\mathbf{w}-\bar{\mathbf{w}}\|_n \le c\|\alpha-\bar{\alpha}\|_p,$$

and finally:

$$\|\mathbf{u}-\bar{\mathbf{u}}\|_n \le \|\mathbf{a}-\bar{\mathbf{a}}\|_n + \|\mathbf{w}-\bar{\mathbf{w}}\|_n \le (1+c)\|\alpha-\bar{\alpha}\|_p. \tag{44}$$

To estimate the Lagrange multipliers, we proceed as follows. From $(\mathscr{M}(\alpha,\varphi))_1$ and $(\mathscr{M}(\bar{\alpha},\varphi))_1$ one easily obtains (see (3.9) in [2, p. 423]):

$$\begin{aligned}\langle \mathbb{A}(\alpha)\mathbf{u}, \mathbf{v}\rangle_n &= \langle \mathbf{L}(\alpha), \mathbf{v}\rangle_n + \langle \lambda, \mathbf{v}_\nu\rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n,\ \mathbf{v}_\tau = \mathbf{0}, \\ \langle \mathbb{A}(\bar{\alpha})\bar{\mathbf{u}}, \mathbf{v}\rangle_n &= \langle \mathbf{L}(\bar{\alpha}), \mathbf{v}\rangle_n + \langle \bar{\lambda}, \mathbf{v}_\nu\rangle_p \quad \forall \mathbf{v} \in \mathbb{R}^n,\ \mathbf{v}_\tau = \mathbf{0}.\end{aligned}$$

By subtracting the two equations we get:

$$\begin{aligned}\big\langle \lambda-\bar{\lambda}, \mathbf{v}_\nu\big\rangle_p &= \big\langle \big(\mathbb{A}(\alpha)-\mathbb{A}(\bar{\alpha})\big)\bar{\mathbf{u}}, \mathbf{v}\big\rangle_n + \big\langle \mathbb{A}(\bar{\alpha})\big(\mathbf{u}-\bar{\mathbf{u}}\big), \mathbf{v}\big\rangle_n \\ &\quad + \langle \mathbf{L}(\alpha)-\mathbf{L}(\bar{\alpha}), \mathbf{v}\rangle_n \quad \forall \mathbf{v} \in \mathbb{R}^n,\ \mathbf{v}_\tau = \mathbf{0}.\end{aligned}$$

Now, dividing this equation by $\|\mathbf{v}\|_n$ and taking supremum over the set $\{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}_\nu \neq \mathbf{0}$ and the remaining components of $\mathbf{v}$ are $0\}$, we arrive at:

$$\|\lambda-\bar{\lambda}\|_p \le c\|\alpha-\bar{\alpha}\|_p,$$

making use of (44) and the fact that $\mathbb{A}$ and $\mathbf{L}$ are Lipschitz continuous on $\mathscr{U}_{ad}$. □

Theorem 11 *Assume that $\mathscr{F}$ is Lipschitz continuous with a sufficiently small modulus C_L, so that Proposition* 1 *holds for $q < 1$. Then the (unique) solution $(\mathbf{u}(\alpha), \lambda(\alpha))$ to $(\mathscr{M}(\alpha))$ is a Lipschitz continuous function of $\alpha \in \mathscr{U}_{ad}$.*

Proof Let $\alpha, \bar{\alpha} \in \mathscr{U}_{ad}$ be given and denote the solutions to $(\mathscr{M}(\alpha))$ and $(\mathscr{M}(\bar{\alpha}))$ by $(\mathbf{u}, \lambda)$ and $(\bar{\mathbf{u}}, \bar{\lambda})$, respectively. Since the corresponding mappings $\Gamma(\alpha, \cdot)$ and $\Gamma(\bar{\alpha}, \cdot)$ are contractive, these solutions may be revealed by the method of successive approximations in the following way.

Choose an arbitrary $\varphi^{(0)} \in \mathbb{R}_+^p$ and compute the solutions to $(\mathscr{M}(\alpha, \varphi^{(0)}))$ and $(\mathscr{M}(\bar{\alpha}, \varphi^{(0)}))$ - denote them by $(\mathbf{u}^{(0)}, \lambda^{(0)})$ and $(\bar{\mathbf{u}}^{(0)}, \bar{\lambda}^{(0)})$. Set $\varphi^{(1)} := \Gamma(\alpha, \varphi^{(0)})$ and $\bar{\varphi}^{(1)} := \Gamma(\bar{\alpha}, \varphi^{(0)})$. By Proposition 2 we already know that:

$$\big\|\big(\mathbf{u}^{(0)}, \lambda^{(0)}\big) - \big(\bar{\mathbf{u}}^{(0)}, \bar{\lambda}^{(0)}\big)\big\|_{n+p} \le C\|\alpha-\bar{\alpha}\|_p, \tag{45}$$

and hence also:

$$\left\|\varphi^{(1)} - \bar{\varphi}^{(1)}\right\|_p \le \left\|\mathbf{u}^{(0)} - \bar{\mathbf{u}}^{(0)}\right\|_n \le C\|\alpha - \bar{\alpha}\|_p. \tag{46}$$

Now, solve problems $(\mathscr{M}(\alpha, \varphi^{(1)}))$ and $(\mathscr{M}(\bar{\alpha}, \bar{\varphi}^{(1)}))$ to obtain $(\mathbf{u}^{(1)}, \lambda^{(1)})$ and $(\bar{\mathbf{u}}^{(1)}, \bar{\lambda}^{(1)})$. Further, denote the solution to $(\mathscr{M}(\alpha, \bar{\varphi}^{(1)}))$ by $(\mathbf{U}^{(1)}, \Lambda^{(1)})$. Thus, we may estimate:

$$\begin{aligned}\left\|\left(\mathbf{u}^{(1)}, \lambda^{(1)}\right) - \left(\bar{\mathbf{u}}^{(1)}, \bar{\lambda}^{(1)}\right)\right\|_{n+p} &\le \left\|\left(\mathbf{u}^{(1)}, \lambda^{(1)}\right) - \left(\mathbf{U}^{(1)}, \Lambda^{(1)}\right)\right\|_{n+p} \\ &\quad + \left\|\left(\mathbf{U}^{(1)}, \Lambda^{(1)}\right) - \left(\bar{\mathbf{u}}^{(1)}, \bar{\lambda}^{(1)}\right)\right\|_{n+p} \\ &\le q\left\|\varphi^{(1)} - \bar{\varphi}^{(1)}\right\|_p + C\|\alpha - \bar{\alpha}\|_p \\ &\le C(1+q)\|\alpha - \bar{\alpha}\|_p,\end{aligned}$$

as follows from Propositions 1, 2 and (46). Continuing this iterative process, in the kth step one has $(\mathbf{u}^{(k)}, \lambda^{(k)})$ and $(\bar{\mathbf{u}}^{(k)}, \bar{\lambda}^{(k)})$, the solutions to $(\mathscr{M}(\alpha, \varphi^{(k)}))$ and $(\mathscr{M}(\bar{\alpha}, \bar{\varphi}^{(k)}))$, respectively, along with the estimate:

$$\begin{aligned}\left\|\left(\mathbf{u}^{(k)}, \lambda^{(k)}\right) - \left(\bar{\mathbf{u}}, \bar{\lambda}^{(k)}\right)\right\|_{n+p} &\le C\left(1 + q + q^2 + \cdots + q^k\right)\|\alpha - \bar{\alpha}\|_p \\ &\le \frac{C}{1-q}\|\alpha - \bar{\alpha}\|_p.\end{aligned} \tag{47}$$

Then one sets $\varphi^{(k+1)} := \Gamma(\alpha, \varphi^{(k)})$, $\bar{\varphi}^{(k+1)} := \Gamma(\bar{\alpha}, \bar{\varphi}^{(k)})$, and starts the iteration loop with $k := k + 1$.

The sequences $\{\varphi^{(k)}\}$ and $\{\bar{\varphi}^{(k)}\}$ generated by this process converge to the unique fixed points of the mappings $\Gamma(\alpha, \cdot)$ and $\Gamma(\bar{\alpha}, \cdot)$, respectively, and the sequences $\{(\mathbf{u}^{(k)}, \lambda^{(k)})\}$, $\{(\bar{\mathbf{u}}^{(k)}, \bar{\lambda}^{(k)})\}$ converge to the (unique) solutions of $(\mathscr{M}(\alpha))$ and $(\mathscr{M}(\bar{\alpha}))$, respectively. Thus it is sufficient to pass to the limit $k \to \infty$ in (47) to obtain the assertion of the theorem. □

Modifying Proposition 2 and Theorem 11 appropriately, one may easily prove the following result (cf. also [12, Theorem 3.2]), which will be useful in the next section.

Theorem 12 *Let* $\alpha \in \mathscr{U}_{ad}$ *be fixed and* $\mathbf{L}$, $\bar{\mathbf{L}} \in \mathbb{R}^n$ *be arbitrary. Let the assumption of Theorem* 11 *be satisfied and denote by* $(\mathbf{u}, \lambda)$, $(\bar{\mathbf{u}}, \bar{\lambda})$ *the (unique) solutions to* $(\mathscr{M}(\alpha))$ *with the load vectors* $\mathbf{L}$ *and* $\bar{\mathbf{L}}$*, respectively. Then there exists* $C > 0$*, independent of* α*,* $\mathbf{L}$*,* $\bar{\mathbf{L}}$ *such that:*

$$\left\|(\mathbf{u}, \lambda) - \left(\bar{\mathbf{u}}, \bar{\lambda}\right)\right\|_{n+p} \le C\left\|\mathbf{L} - \bar{\mathbf{L}}\right\|_n.$$

Let us conclude this section with the shape optimization problem which uses the reduced algebraic form of the state problem only. The reduction of $(\mathscr{M}(\alpha))$ consists in eliminating all components of the displacement field $\mathbf{u}$ corresponding to the non-contact nodes of the finite element partition of $\overline{\Omega}(\alpha_h)$. One obtains a variational inequality in terms of the variables $\mathbf{u}_\tau$, $\mathbf{u}_\nu$ and λ, defined on the contact zone, which may be formulated as the following *generalized equation* (GE) (for details see section 3 of [1]):

$$\left.\begin{aligned} \mathbf{0} &\in \mathbb{A}_{\tau\tau}(\alpha)\mathbf{u}_\tau + \mathbb{A}_{\tau\nu}(\alpha)\mathbf{u}_\nu - \mathbf{L}_\tau(\alpha) + Q_1(\alpha, \mathbf{u}_\tau) \\ \mathbf{0} &= \mathbb{A}_{\nu\tau}(\alpha)\mathbf{u}_\tau + \mathbb{A}_{\nu\nu}(\alpha)\mathbf{u}_\nu - \mathbf{L}_\nu(\alpha) - \lambda \\ \mathbf{0} &\in \mathbf{u}_\nu + \alpha + N_{\mathbb{R}^p_+}(\lambda). \end{aligned}\right\} \tag{48}$$

Here the multifunction $Q_1 : \mathscr{U}_{ad} \times \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is defined as:

$$\big(Q_1(\alpha, \mathbf{u}_\tau)\big)_i := \omega_i(\alpha)\mathscr{F}(|(\mathbf{u}_\tau)_i|)\partial|(\mathbf{u}_\tau)_i| \quad \forall i = 1, \dots, p,$$

where "∂" denotes the subdifferential of convex functions, $N_{\mathbb{R}^p_+}(\cdot)$ is the normal cone in the sense of convex analysis and submatrices $\mathbb{A}_{\tau\tau}$, $\mathbb{A}_{\tau\nu}$, $\mathbb{A}_{\nu\nu} \in C^1(\mathscr{U}_{ad}; \mathbb{R}^{p\times p})$ are parts of the Schur complement to the stiffness matrix with $\mathbb{A}_{\nu\tau} = \mathbb{A}^T_{\tau\nu}$. In addition, $\mathbb{A}_{\tau\tau}$ and $\mathbb{A}_{\nu\nu}$ are positive definite uniformly with respect to $\alpha \in \mathscr{U}_{ad}$.

Introducing the state variable $\mathbf{y} := (\mathbf{u}_\tau, \mathbf{u}_\nu, \lambda) \in (\mathbb{R}^p)^3$, the GE (48) may be written in the compact form:

$$\mathbf{0} \in F(\alpha, \mathbf{y}) + Q(\alpha, \mathbf{y}), \tag{49}$$

with $\alpha \in \mathscr{U}_{ad}$ being the control variable and

$$F(\alpha, \mathbf{y}) := \begin{pmatrix} \mathbb{A}_{\tau\tau}(\alpha) & \mathbb{A}_{\tau\nu}(\alpha) & 0 \\ \mathbb{A}_{\nu\tau}(\alpha) & \mathbb{A}_{\nu\nu}(\alpha) & -\mathbb{I} \\ 0 & \mathbb{I} & 0 \end{pmatrix} \mathbf{y} - \begin{pmatrix} \mathbf{L}_\tau(\alpha) \\ \mathbf{L}_\nu(\alpha) \\ -\alpha \end{pmatrix}, \quad Q(\alpha, \mathbf{y}) := \begin{pmatrix} Q_1(\alpha, \mathbf{y}_1) \\ 0 \\ N_{\mathbb{R}^p_+}(\mathbf{y}_3) \end{pmatrix}.$$

Note that F is single-valued, continuously differentiable in its domain of definition and Q is a closed-graph multifunction. Further, denote the corresponding control-to-state map by $S : \mathscr{U}_{ad} \rightrightarrows (\mathbb{R}^p)^3$:

$$S : \alpha \mapsto \left\{\mathbf{y} \in (\mathbb{R}^p)^3 \,|\, \mathbf{0} \in F(\alpha, \mathbf{y}) + Q(\alpha, \mathbf{y})\right\}.$$

Let $J : \mathscr{U}_{ad} \times (\mathbb{R}^p)^3 \to \mathbb{R}$ be a continuously differentiable cost functional. The algebraic form of our *shape optimization problem* leads to the following MPEC:

$$\left.\begin{array}{ll} \text{minimize} & J(\alpha, \mathbf{y}) \\ \text{subj. to} & \mathbf{0} \in F(\alpha, \mathbf{y}) + Q(\alpha, \mathbf{y}) \\ & \alpha \in \mathscr{U}_{ad}. \end{array}\right\} \tag{$\mathbb{P}$}$$

In the sequel we shall assume that the assumptions of Theorem 11 are satisfied, implying that S is single-valued and Lipschitz continuous in $\mathscr{U}_{ad}$. Then ($\mathbb{P}$) may be equivalently reformulated as the following nonlinear program:

$$\left.\begin{array}{ll} \text{minimize} & \mathscr{J}(\alpha) := J(\alpha, S(\alpha)) \\ \text{subj. to} & \alpha \in \mathscr{U}_{ad}. \end{array}\right\} \tag{$\tilde{\mathbb{P}}$}$$

Since the composite cost functional $\mathscr{J}$ is locally Lipschitz continuous, ($\tilde{\mathbb{P}}$) can be solved by standard algorithms of nonsmooth optimization. Such algorithms, however, require knowledge of some subgradient information, usually in the form of one (arbitrary) subgradient from the Clarke subdifferential $\bar{\partial}\mathscr{J}$ (cf. [4, Theorem 2.5.1]) at each iteration step. This can be facilitated by the chain rule in [4, Theorem 2.6.6]:

$$\bar{\partial}\mathscr{J}(\bar{\alpha}) = \nabla_\alpha J\big(\bar{\alpha}, \bar{\mathbf{y}}\big) + \big(\bar{\partial} S(\bar{\alpha})\big)^T \nabla_y J\big(\bar{\alpha}, \bar{\mathbf{y}}\big), \tag{50}$$

valid at any reference point $\bar{\alpha} \in \mathscr{U}_{ad}$, $\bar{\mathbf{y}} := S(\bar{\alpha})$. Thus, for the required subgradient information it is sufficient to determine an element from $\left(\overline{\partial} S(\bar{\alpha})\right)^T \nabla_y J(\bar{\alpha}, \bar{\mathbf{y}})$. In (50), $\overline{\partial} S(\bar{\alpha})$ stands for the *generalized Jacobian* of Clarke, defined in [4, Definition 2.6.1]. The rest of the paper is devoted to this task.

6 Sensitivity Analysis

As already indicated, instead of working with the generalized differential calculus of Clarke, we are going to employ the substantially richer differential calculus of Mordukhovich ([15]). The reason is that the computation of the generalized Jacobian of S for the GE (48) via [4, Definition 2.6.1] requires quite a considerable effort, whereas by using the Mordukhovich theory one can employ a number of efficient rules, e.g. [19, Chapter 10] and [15, Chapter 3]. These rules do not have any counterpart in the Clarke's calculus and suit very well to sensitivity and stability issues [15, Chapter 4].

To this aim, observe first that by Lipschitz continuity of S and formula (2.23) in [14]:

$$\forall \mathbf{y}^* \in (\mathbb{R}^p)^3 : \quad \left(\overline{\partial} S(\bar{\alpha})\right)^T \mathbf{y}^* = \operatorname{conv} D^* S(\bar{\alpha}(\mathbf{y}^*)).$$

Comparing with (50), we immediately see that it is sufficient to determine one element from the set

$$D^* S(\bar{\alpha})(\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}}))$$

and we are done. The computation of the coderivative $D^* S(\bar{\alpha})$ in terms of the data of our problem is facilitated on the basis of Theorem 2 in [11]. Before being able to use the aforementioned result, however, the following property has to be verified:

Lemma 8 *Let $\bar{\alpha} \in \mathscr{U}_{ad}$ be fixed, $\bar{\mathbf{y}} := S(\bar{\alpha})$ and introduce the mapping $\Phi : \mathbb{R}^p \times (\mathbb{R}^p)^3 \to \mathbb{R}^p \times (\mathbb{R}^p)^3 \times (\mathbb{R}^p)^3$:*

$$\Phi : (\alpha, \mathbf{y}) \mapsto \big(\alpha, \mathbf{y}, -F(\alpha, \mathbf{y})\big).$$

Then the multifunction $M : \mathbb{R}^p \times (\mathbb{R}^p)^3 \times (\mathbb{R}^p)^3 \rightrightarrows \mathbb{R}^p \times (\mathbb{R}^p)^3$ defined by

$$M : \mathbf{p} \mapsto \left\{ (\alpha, \mathbf{y}) \,|\, \mathbf{p} + \Phi(\alpha, \mathbf{y}) \in Gr\, Q \right\}$$

is calm at $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \bar{\alpha}, \bar{\mathbf{y}})$.

Proof If M is not calm at $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \bar{\alpha}, \bar{\mathbf{y}})$, one can easily disprove calmness of the following multifunction $\tilde{M} : (\mathbb{R}^p)^3 \rightrightarrows \mathbb{R}^p \times (\mathbb{R}^p)^3$ at $(\mathbf{0}, \bar{\alpha}, \bar{\mathbf{y}})$:

$$\tilde{M} : \tilde{\mathbf{p}} \mapsto \left\{(\alpha, \mathbf{y}) \,|\, (\mathbf{0}, \mathbf{0}, \tilde{\mathbf{p}}) + \Phi(\alpha, \mathbf{y}) \in \mathrm{Gr}\, Q\right\}.$$

Therefore it is sufficient to show that $\tilde{M}$ is calm at $(\mathbf{0}, \bar{\alpha}, \bar{\mathbf{y}})$.

Let $\tilde{\mathbf{p}} \in (\mathbb{R}^p)^3$ be given. Then $(\alpha, \mathbf{y}) \in \tilde{M}(\tilde{\mathbf{p}}) \;\Leftrightarrow\; \tilde{\mathbf{p}} \in F(\alpha, \mathbf{y}) + Q(\alpha, \mathbf{y})$, i.e.

$$\left.\begin{aligned}
&\tilde{\mathbf{p}}_1 \in \mathbb{A}_{\tau\tau}(\alpha)\mathbf{y}_1 + \mathbb{A}_{\tau\nu}(\alpha)\mathbf{y}_2 - \mathbf{L}_\tau(\alpha) + Q_1(\alpha, \mathbf{y}_1)\\
&\tilde{\mathbf{p}}_2 = \mathbb{A}_{\nu\tau}(\alpha)\mathbf{y}_1 + \mathbb{A}_{\nu\nu}(\alpha)\mathbf{y}_2 - \mathbf{y}_3 - \mathbf{L}_\nu(\alpha)\\
&\tilde{\mathbf{p}}_3 \in \mathbf{y}_2 + \alpha + N_{\mathbb{R}^p_+}(\mathbf{y}_3).
\end{aligned}\right\} \tag{51}$$

Introducing the new variable $\tilde{\mathbf{y}} := (\mathbf{y}_1, \mathbf{y}_2 - \tilde{\mathbf{p}}_2, \mathbf{y}_3)$, we see that $(\alpha, \tilde{\mathbf{y}})$ solves (48) with the load vector $\tilde{\mathbf{l}} := (\mathbf{L}_\tau(\alpha) + \tilde{\mathbf{p}}_1 - \mathbb{A}_{\tau\nu}(\alpha)\tilde{\mathbf{p}}_3,\ \mathbf{L}_\nu(\alpha) + \tilde{\mathbf{p}}_2 - \mathbb{A}_{\nu\nu}(\alpha)\tilde{\mathbf{p}}_3,\ -\alpha)^T$. From Theorem 10 it follows that there exists a unique solution $(\alpha, \tilde{\mathbf{y}})$ to the perturbed GE (51). Denoting $(\alpha, \mathbf{y})$ the solution to (48) with the original load vector $\mathbf{l}$, we obtain from Theorem 12:

$$\|(\alpha, \tilde{\mathbf{y}}) - (\alpha, \mathbf{y})\| \le \left\|\tilde{\mathbf{l}} - \mathbf{l}\right\| \le c\|\tilde{\mathbf{p}}\|,$$

where $c > 0$ does not depend on α. From this the required calmness property follows easily. □

Theorem 13 *Consider a reference pair* $(\bar{\alpha}, \bar{\mathbf{y}}) \in Gr\, S$.

(i) *Let* $(\mathbf{p}^*, \mathbf{v}^*) \in \mathbb{R}^p \times (\mathbb{R}^p)^3$ *be a solution to the regular adjoint GE:*

$$\begin{pmatrix} \mathbf{p}^* \\ -\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}}) \end{pmatrix} \in \nabla F(\bar{\alpha}, \bar{\mathbf{y}})^T \mathbf{v}^* + \widehat{D}^* Q(\Phi(\bar{\alpha}, \bar{\mathbf{y}}))(\mathbf{v}^*). \qquad \text{(RAGE)}$$

Then $\mathbf{p}^* \in D^* S(\bar{\alpha})(\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}}))$.

(ii) *For every* $\mathbf{p}^* \in D^* S(\bar{\alpha})(\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}}))$ *there exists a vector* $\mathbf{v}^* \in (\mathbb{R}^p)^3$ *such that* $(\mathbf{p}^*, \mathbf{v}^*)$ *is a solution of the (limiting) adjoint GE:*

$$\begin{pmatrix} \mathbf{p}^* \\ -\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}}) \end{pmatrix} \in \nabla F(\bar{\alpha}, \bar{\mathbf{y}})^T \mathbf{v}^* + D^* Q(\Phi(\bar{\alpha}, \bar{\mathbf{y}}))(\mathbf{v}^*). \qquad \text{(AGE)}$$

Proof The first assertion follows immediately from [19, Theorem 10.6]. The second one is implied by [9, Theorem 4.1], whose assumptions are fulfilled by virtue of Lemma 8. □

Note that due to Lipschitz continuity of S, AGE attains at least one solution $\mathbf{p}^*$ and whenever Q is normally regular at $\Phi((\bar{\alpha}, \bar{\mathbf{y}}))$, i.e. $\widehat{N}_{\text{Gr}\,Q}(\Phi(\bar{\alpha}, \bar{\mathbf{y}})) = N_{\text{Gr}\,Q}(\Phi(\bar{\alpha}, \bar{\mathbf{y}}))$, RAGE and AGE coincide. On the other hand, in the nonregular case RAGE may be difficult to solve or not solvable at all. Therefore the computation of the desired subgradient $\zeta \in \bar{\partial}\mathscr{J}(\bar{\alpha})$ is usually done via the AGE, while accepting the fact that at nonregular points the computed vector may lie outside of $\bar{\partial}\mathscr{J}(\bar{\alpha})$. In such cases the employed optimization algorithm might collapse and ζ has to be replaced by a correct subgradient.

In light of the previous paragraph we will focus on the solution of the AGE (for details see [11]). In particular, in the sequel we will express the most difficult part of AGE, i.e. the coderivative $D^* Q(\Phi(\bar{\alpha}, \bar{\mathbf{y}}))$ in terms of the problem data.

6.1 Computation of D^*Q

First of all, note that the components of Q are *decoupled* (this fact is a consequence of the assumed model of given friction), hence its coderivative can be computed componentwise:

$$\forall \mathbf{q}^* \in (\mathbb{R}^p)^3 : \quad D^* Q(\bar{\alpha}, \bar{\mathbf{y}}, \bar{\mathbf{q}})(\mathbf{q}^*) = \begin{pmatrix} D^* Q_1(\bar{\alpha}, \bar{\mathbf{y}}_1, \bar{\mathbf{q}}_1)(\mathbf{q}_1^*) \\ 0 \\ D^* N_{\mathbb{R}^p_+}(\bar{\mathbf{y}}_3, \bar{\mathbf{q}}_3)(\mathbf{q}_3^*) \end{pmatrix},$$

at any reference point $(\bar{\alpha}, \bar{\mathbf{y}}, \bar{\mathbf{q}}) \in \text{Gr}\, Q$.

The third component is standard and the exact formula for it may be found e.g. in [17, Lemma 2.2].

In order to deal with the first component, let us write the multifunction $Q_1 : \mathbb{R}^p \times \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ as a composition of an outer multifunction Z_1 and an inner single-valued, smooth mapping Ψ:

$$Q_1(\alpha, \mathbf{u}) = \begin{pmatrix} \omega_1(\alpha)\mathscr{F}(|\mathbf{u}_1|)\partial|\mathbf{u}_1| \\ \omega_2(\alpha)\mathscr{F}(|\mathbf{u}_2|)\partial|\mathbf{u}_2| \\ \vdots \\ \omega_p(\alpha)\mathscr{F}(|\mathbf{u}_p|)\partial|\mathbf{u}_p| \end{pmatrix} = (Z_1 \circ \Psi)(\alpha, \mathbf{u}), \tag{52}$$

where

$$\Psi = (\Psi_1, \ldots, \Psi_p) : \mathbb{R}^p \times \mathbb{R}^p \to \big((0, \infty) \times \mathbb{R}\big)^p, \qquad \Psi_j(\alpha, \mathbf{u}) := \big(\omega_j(\alpha), u_j\big),$$

and

$$Z_1 : \big((0, \infty) \times \mathbb{R}\big)^p \rightrightarrows \mathbb{R}^p, \quad \mathbf{y} \mapsto \big(Z(\mathbf{y}_1), \ldots, Z(\mathbf{y}_p)\big),$$

with

$$Z : (0, \infty) \times \mathbb{R} \rightrightarrows \mathbb{R}, \quad (x_1, x_2) \mapsto x_1\mathscr{F}(|x_2|)\partial|x_2|.$$

Now the chain rule from [19, Theorem 10.40] allows us to compute the coderivative of the composite multifunction (52) as follows:

Theorem 14 *Let* $(\bar{\alpha}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in Gr\, Q_1$ *be such that the following condition holds:*

$$Ker\, \nabla\Psi(\bar{\alpha}, \bar{\mathbf{u}})^T \cap D^* Z_1(\Psi(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(\mathbf{0}) = \{\mathbf{0}\}. \tag{53}$$

Then:

$$\forall \mathbf{q}^* \in \mathbb{R}^p : \quad D^* Q_1(\bar{\alpha}, \bar{\mathbf{u}}, \bar{\mathbf{q}})(\mathbf{q}^*) \subset \nabla\Psi(\bar{\alpha}, \bar{\mathbf{u}})^T D^* Z_1(\Psi(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(\mathbf{q}^*)$$
$$= \nabla\Psi(\bar{\alpha}, \bar{\mathbf{u}})^T \begin{pmatrix} D^* Z(\Psi_1(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}}_1)(\mathbf{q}_1^*) \\ D^* Z(\Psi_2(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}}_2)(\mathbf{q}_2^*) \\ \vdots \\ D^* Z(\Psi_p(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}}_p)(\mathbf{q}_p^*) \end{pmatrix}. \tag{54}$$

Observe that the assertion of Proposition 14 requires the validity of the qualification condition (53). We are going to show that (53) is satisfied at all points $(\bar{\alpha}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in \mathrm{Gr}\, Q_1$ and hence the assertion of Proposition 14 holds automatically.

Remark 2 The right inclusion above becomes equality for points $(\bar{\alpha}, \bar{\mathbf{u}}, \bar{\mathbf{q}})$, such that the multifunction Z_1 is normally regular at $(\Psi(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}})$ or $\nabla\Psi(\bar{\alpha}, \bar{\mathbf{u}})$ is surjective. In other cases, however, the formula on the right-hand side may provide a vector outside of $D^* Q_1$.

6.2 Computation of $D^* Z$

In the sequel we will compute the coderivative of Z at a given point $(\bar{x}_1, \bar{x}_2, \bar{z}) \in \mathrm{Gr}\, Z$. The obtained results will then be used to validate condition (53), while at the same time they play a central role in the assertion of Proposition 14 itself.

Let us distinguish several situations according to the position of the reference point $(\bar{x}_1, \bar{x}_2, \bar{z})$ on the graph of Z.

Proposition 3 *Let $z^* \in \mathbb{R}$ be arbitrary and $(\bar{x}_1, \bar{x}_2, \bar{z}) \in Gr\, Z$ such that $\bar{x}_2 > 0$. Then:*

$$D^* Z(\bar{x}_1, \bar{x}_2, \bar{z})(z^*) = \left\{z^* \mathscr{F}(\bar{x}_2)\right\} \times D^* \mathscr{F}(\bar{x}_2)(\bar{x}_1 z^*). \tag{55}$$

Proof Due to the assumption on $\bar{x}_2$ there exists a neighbourhood $\mathscr{O}$ of $(\bar{x}_1, \bar{x}_2)$ so that:

$$Z(x_1, x_2) = x_1 \mathscr{F}(x_2) \qquad \forall (x_1, x_2) \in \mathscr{O}.$$

Note that Z is single-valued and (locally) Lipschitz continuous in $\mathscr{O}$. The computation of the regular normal cone to Gr Z at points of $\mathscr{O}$ is straightforward and yields:

$$\widehat{N}_{\mathrm{Gr}\, Z}(x_1, x_2, z) = \left\{ \left(x_1^*, x_2^*, z^*\right) \mid x_1^* = -z^* \mathscr{F}(x_2),\ \left(x_2^*, x_1 z^*\right) \in \widehat{N}_{\mathrm{Gr}\, \mathscr{F}}(x_2, \mathscr{F}(x_2)) \right\}. \tag{56}$$

Thus

$$N_{\mathrm{Gr}\, Z}(\bar{x}_1, \bar{x}_2, \bar{z}) = \left\{ (x_1^*, x_2^*, z^*) \mid x_1^* = -z^* \mathscr{F}(\bar{x}_2), (x_2^*, \bar{x}_1 z^*) \in N_{\mathrm{Gr}\, \mathscr{F}}(\bar{x}_2, \mathscr{F}(\bar{x}_2)) \right\},$$

and the assertion follows immediately from the definition of the coderivative. □

Proposition 4 *Let $z^* \in \mathbb{R}$ be arbitrary and $(\bar{x}_1, \bar{x}_2, \bar{z}) \in Gr\, Z$ such that $\bar{x}_2 < 0$. Then:*

$$D^* Z(\bar{x}_1, \bar{x}_2, \bar{z})(z^*) = \left\{-z^* \mathscr{F}(-\bar{x}_2)\right\} \times \left(- D^* \mathscr{F}(-\bar{x}_2)(-\bar{x}_1 z^*)\right). \tag{57}$$

Proof In this case there exists a neighbourhood $\tilde{\mathscr{O}}$ of $(\bar{x}_1, \bar{x}_2)$ such that:

$$Z(x_1, x_2) = -x_1 \mathscr{F}(-x_2) \qquad \forall (x_1, x_2) \in \tilde{\mathscr{O}}.$$

The rest is done in a similar fashion. □

Remark 3 The previous two cases have the mechanical interpretation of *sliding*, i.e. represent those contact points, where the displacement in the tangential direction is nonzero.

Proposition 5 *Let $z^* \in \mathbb{R}$ be arbitrary and $(\bar{x}_1, 0, \bar{z}) \in Gr\, Z$ such that $|\bar{z}| < \bar{x}_1 \mathscr{F}(0)$. Then:*

$$D^* Z(\bar{x}_1, 0, \bar{z})(z^*) = \begin{cases} \{0\} \times \mathbb{R}, & \text{if } z^* = 0 \\ \emptyset, & \text{otherwise.} \end{cases} \tag{58}$$

Proof As readily seen, there exists a neighourhood $\mathscr{U}$ of $(\bar{x}_1, 0, \bar{z})$ such that:

$$\mathscr{U} \cap \mathrm{Gr}\, Z = \mathscr{U} \cap \left(\mathbb{R} \times \{0\} \times \mathbb{R}\right),$$

whence we immediately get:

$$\widehat{N}_{\mathrm{Gr}\, Z}(x_1, 0, z) = \{0\} \times \mathbb{R} \times \{0\} \qquad \forall (x_1, 0, z) \in \mathscr{U} \cap \mathrm{Gr}\, Z. \tag{59}$$

□

The setting of the previous proposition corresponds to contact points, where *strong sticking* is present, i.e. the tangential component of the stress vector is below the threshold value to trigger motion in the tangential direction. If this critical value is attained at a contact point, but there is still no tangential motion, we speak of *weak sticking*, which is investigated below.

Proposition 6 *Let $z^* \in \mathbb{R}$ and $\bar{x}_1 > 0$ be arbitrary. Then:*

$$\left.\begin{aligned} &D^*Z(\bar{x}_1, 0, \bar{x}_1\mathscr{F}(0))(z^*) \subset \{z^*\mathscr{F}(0)\} \times D^*\mathscr{F}(0)(\bar{x}_1 z^*), \ \textit{if}\ z^* > 0,\\ &D^*Z(\bar{x}_1, 0, \bar{x}_1\mathscr{F}(0))(z^*) = \{z^*\mathscr{F}(0)\} \times (-\infty,\ \bar{x}_1 z^* D^+\mathscr{F}(0)], \ \textit{if}\ z^* < 0,\\ &D^*Z(\bar{x}_1, 0, \bar{x}_1\mathscr{F}(0))(z^*) = \{0\} \times \mathbb{R}, \ \textit{if}\ z^* = 0. \end{aligned}\right\} \tag{60}$$

where the symbol $D^+\mathscr{F}(0) := \limsup_{\eta\to 0_+} \frac{\mathscr{F}(\eta)-\mathscr{F}(0)}{\eta}$ stands for the upper Dini derivative of $\mathscr{F}$ at 0.

Proof The analysis in this case becomes more involved, since the point $\bar{\mathbf{a}} := (\bar{x}_1, 0, \bar{x}_1\mathscr{F}(0))$ may be approached by sequences corresponding to different mechanical regimes:

$$N_{\mathrm{Gr}\,Z}(\bar{\mathbf{a}}) = \operatorname*{Lim\,sup}_{(x_1,x_2,z)\xrightarrow{\mathrm{Gr}\,Z}\bar{\mathbf{a}}} \widehat{N}_{\mathrm{Gr}\,Z}(x_1, x_2, z) = \mathscr{N}_1 \cup \mathscr{N}_2 \cup \mathscr{N}_3,$$

where

$$\mathscr{N}_1 := \operatorname*{Lim\,sup}_{\substack{(x_1,x_2,z)\xrightarrow{\mathrm{Gr}\,Z}\bar{\mathbf{a}}\\ x_2>0}} \widehat{N}_{\mathrm{Gr}\,Z}(x_1, x_2, z), \qquad \mathscr{N}_2 := \operatorname*{Lim\,sup}_{\substack{(x_1,0,z)\xrightarrow{\mathrm{Gr}\,Z}\bar{\mathbf{a}}\\ z<x_1\mathscr{F}(0)}} \widehat{N}_{\mathrm{Gr}\,Z}(x_1, 0, z),$$

and

$$\mathscr{N}_3 := \operatorname*{Lim\,sup}_{x_1\to\bar{x}_1} \widehat{N}_{\mathrm{Gr}\,Z}(x_1, 0, x_1\mathscr{F}(0)).$$

Observe that the regular normal cones generating in $\mathscr{N}_1$ and $\mathscr{N}_2$ have already been computed in (56) and in (59), respectively. From these relations it is clear that

$$\mathscr{N}_1 \subset \left\{ \left(x_1^*, x_2^*, z^*\right) \mid x_1^* = -z^*\mathscr{F}(0),\ \left(x_2^*, \bar{x}_1 z^*\right) \in N_{\mathrm{Gr}\,\mathscr{F}}(0, \mathscr{F}(0)) \right\}, \tag{61}$$

and

$$\mathscr{N}_2 = \{0\} \times \mathbb{R} \times \{0\}.$$

The treatment of $\mathscr{N}_3$ is, however, more delicate. As a first step, let us compute the contingent cone to $\mathrm{Gr}\,Z$ at $\mathbf{a} := (x_1, 0, x_1\mathscr{F}(0))$, for $x_1 > 0$ fixed. Note that $\mathrm{Gr}\,Z$ locally around the reference point $\mathbf{a}$ coincides with the union $G_1 \cup G_2$, where

$$\begin{aligned} G_1 &= \left\{ \left(x_1', x_2', z'\right) \mid |x_1' - x_1| < \varepsilon,\ x_2' = 0,\ x_1'\mathscr{F}(0) - \varepsilon < z' \le x_1'\mathscr{F}(0) \right\},\\ G_2 &= \left\{ \left(x_1', x_2', z'\right) \mid |x_1' - x_1| < \varepsilon,\ 0 \le x_2' < \varepsilon,\ z' = x_1'\mathscr{F}\left(x_2'\right) \right\}, \end{aligned}$$

for a sufficiently small $\varepsilon > 0$. Moreover, the following holds:

$$T_{\mathrm{Gr}\,Z}(\mathbf{a}) = T_{G_1}(\mathbf{a}) \cup T_{G_2}(\mathbf{a}). \tag{62}$$

By the definition of the contingent cone:

$$T_{G_1}(\mathbf{a}) = \{ (h,k,l) \mid \exists h_i \to h\ \exists k_i \to k\ \exists l_i \to l\ \exists \lambda_i \to 0_+ : \\ \lambda_i k_i = 0,\ x_1 \mathscr{F}(0) + \lambda_i l_i \le (x_1 + \lambda_i h_i)\mathscr{F}(0) \} = \{ (h,0,l) \mid l \le h\mathscr{F}(0) \}.$$

Analogously:

$$\begin{aligned} T_{G_2}(\mathbf{a}) &= \{ (h,k,l) \mid \exists h_i \to h\ \exists k_i \to k\ \exists l_i \to l\ \exists \lambda_i \to 0_+ : \\ &\qquad 0 \le \lambda_i k_i,\ x_1 \mathscr{F}(0) + \lambda_i l_i = (x_1 + \lambda_i h_i)\mathscr{F}(\lambda_i k_i) \} \\ &= \left\{ (h,k,l) \,\middle|\, \exists h_i \to h\ \exists k_i \to k\ \exists l_i \to l\ \exists \lambda_i \to 0_+ : \right. \\ &\qquad \left. 0 \le k_i,\ l_i = h_i \mathscr{F}(\lambda_i k_i) + x_1 k_i \frac{\mathscr{F}(\lambda_i k_i) - \mathscr{F}(0)}{\lambda_i k_i} \right\} \\ &= \{ (h,k,l) \mid 0 \le k,\ l = h\mathscr{F}(0) + x_1 \xi k,\ \xi \in \Xi \}, \end{aligned}$$

where $\Xi := \operatorname{Lim\,sup}_{\eta \to 0_+} \frac{\mathscr{F}(\eta) - \mathscr{F}(0)}{\eta}$. Since $\mathscr{F}$ is assumed to be Lipschitz continuous, the inclusion $\Xi \subset [D^- \mathscr{F}(0), D^+ \mathscr{F}(0)] \subset \mathbb{R}$ holds, whereas Ξ contains at least the endpoints of the interval. Now it is sufficient to compute the (negative) polars to these cones to obtain:

$$\widehat{N}_{G_1}(\mathbf{a}) = \left(T_{G_1}(\mathbf{a})\right)^0 = \{ (x_1^*, x_2^*, z^*) \mid x_1^* = -z^* \mathscr{F}(0),\ z^* \ge 0 \} \tag{63}$$

and similarly:

$$\widehat{N}_{G_2}(\mathbf{a}) = \{ (x_1^*, x_2^*, z^*) \mid x_1^* = -z^* \mathscr{F}(0),\ x_2^* \le -x_1 z^* \xi \quad \forall \xi \in \Xi \}. \tag{64}$$

Finally, combining (62), (63) and (64) yields:

$$\begin{aligned} \widehat{N}_{\mathrm{Gr}\,Z}(\mathbf{a}) &= \widehat{N}_{G_1}(\mathbf{a}) \cap \widehat{N}_{G_2}(\mathbf{a}) \\ &= \{ (x_1^*, x_2^*, z^*) \mid x_1^* = -z^* \mathscr{F}(0),\ x_2^* \le -x_1 z^* D^+ \mathscr{F}(0),\ z^* \ge 0 \}. \end{aligned}$$

From this it is obvious that $\mathscr{N}_3 = \widehat{N}_{\mathrm{Gr}\,Z}(\bar{\mathbf{a}})$.

In this way we have now an upper estimate of $N_{\mathrm{Gr}\,Z}(\bar{\mathbf{a}})$ and the result follows easily by the definition of the coderivative. Indeed, for instance, the first formula in (60) follows from (61) and the fact that for $z^* > 0$ and $i = 2, 3$ there does not exist any (x_1^*, x_2^*) such that $(x_1^*, x_2^*, -z^*) \in \mathscr{N}_i$. The statement has been established. □

Remark 4 Note that if $\mathscr{F}$ is continuously differentiable, then the inclusion in (60) becomes equality in the form $D^* Z(\bar{x}_1, 0, \bar{x}_1 \mathscr{F}(0))(z^*) = \{z^* \mathscr{F}(0)\} \times \{\bar{x}_1 z^* \mathscr{F}'_+(0)\}$, with $\mathscr{F}'_+(0)$ being the right-hand derivative of $\mathscr{F}$ at 0.

A straightforward modification of the proof of Proposition 6 implies the following result, concerning the point $\bar{\mathbf{a}} := (\bar{x}_1, 0, -\mathscr{F}(0))$.

Proposition 7 *Let $z^* \in \mathbb{R}$ and $\bar{x}_1 > 0$ be arbitrary. Then:*

$$\left.\begin{aligned} &D^*Z(\bar{x}_1, 0, -\bar{x}_1\mathscr{F}(0))(z^*) = \{-z^*\mathscr{F}(0)\} \times [\bar{x}_1 z^* D^+\mathscr{F}(0),\ +\infty),\ \textit{if}\ z^* > 0,\\ &D^*Z(\bar{x}_1, 0, -\bar{x}_1\mathscr{F}(0))(z^*) \subset \{-z^*\mathscr{F}(0)\} \times \big(-D^*\mathscr{F}(0)(-\bar{x}_1 z^*)\big),\ \textit{if}\ z^* < 0,\\ &D^*Z(\bar{x}_1, 0, -\bar{x}_1\mathscr{F}(0))(z^*) = \{0\} \times \mathbb{R},\ \textit{if}\ z^* = 0. \end{aligned}\right\} \quad (65)$$

We are now in a position to verify the qualification condition (53).

Corollary 2 *Let $(\bar{\alpha}, \bar{\mathbf{u}}, \bar{\mathbf{q}}) \in Gr\, Q_1$ be arbitrary. Then (53) holds.*

Proof By (55), (57), (58), (60) and (65) we see that $D^*Z(\bar{x}_1, \bar{x}_2, \bar{z})(0) \subset \{0\} \times \mathbb{R}$ for any $(\bar{x}_1, \bar{x}_2, \bar{z}) \in \text{Gr}\, Z$, implying:

$$D^*Z_1(\Psi(\bar{\alpha}, \bar{\mathbf{u}}), \bar{\mathbf{q}})(0) \subset \big(\{\mathbf{0}\} \times \mathbb{R}\big)^p.$$

Choosing now $\mathbf{w} \in \left(\mathbb{R}^2\right)^p$ such that $\mathbf{w}_i = (0, c_i)^T$ for all $i = 1, \ldots, p$, then:

$$\mathbf{0} = \nabla\Psi(\bar{\alpha}, \bar{\mathbf{u}})^T\mathbf{w} = \sum_{i=1}^{p} \nabla\Psi_i(\bar{\alpha}, \bar{\mathbf{u}})^T\mathbf{w}_i = \sum_{i=1}^{p} \begin{pmatrix} \nabla\omega_i(\bar{\alpha})^T & \mathbf{0} \\ \mathbf{0} & \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 \\ c_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{c} \end{pmatrix}.$$

□

In this way we have proved that the upper estimate (54), needed in AGE, is valid.

Although sensitivity analysis has been designed primarily for numerical realization along the lines of [1], the numerical treatment of ($\mathbb{P}$) is not studied in this paper. Nevertheless, the obtained results enable us to establish necessary optimality conditions, that may serve e.g. as a stopping criterion in the prepared numerical algorithm or for testing optimality of a design computed in some other way.

Theorem 15 *Let $(\bar{\alpha}, \bar{\mathbf{y}})$ be a local solution to ($\mathbb{P}$) (in particular $\bar{\mathbf{y}} = S(\bar{\alpha})$). Then:*

(1) $\mathbf{0} \in \nabla_\alpha J(\bar{\alpha}, \bar{\mathbf{y}}) + D^*S(\bar{\alpha})(\nabla_y J(\bar{\alpha}, \bar{\mathbf{y}})) + N_{\mathscr{U}_{ad}}(\bar{\alpha})$;
(2) $\exists\ \mathbf{v}^* \in (\mathbb{R}^p)^3$:

$$\mathbf{0} \in \nabla J(\bar{\alpha}, \bar{\mathbf{y}}) + \nabla F(\bar{\alpha}, \bar{\mathbf{y}})^T\mathbf{v}^* + D^*Q(\bar{\alpha}, \bar{\mathbf{y}}, -F(\bar{\alpha}, \bar{\mathbf{y}}))(\mathbf{v}^*) + N_{\mathscr{U}_{ad}\times(\mathbb{R}^p)^3}(\bar{\alpha}, \bar{\mathbf{y}}).$$

Proof The optimality condition in (1) amounts directly to the respective condition in [15, Corollary 5.35]. This relation together with Theorem 13 (ii) it yields (2). □

7 Conclusion

In this paper we studied shape optimization for 2D contact problems with given friction and a coefficient of friction, which depends on the solution. In particular, we have shown existence of an optimal domain (optimal with respect to a given cost functional) in a class of admissible ones, whose contact boundaries can be described by functions which are together with their first derivatives Lipschitz equicontinuous. A suitable discretization was then introduced and besides proving existence of discrete optimal domains, we established convergence results as well. Since the proposed discretization is not suitable for direct computer implementation, another level of approximation was introduced, resulting in an algebraic shape optimization problem. Sensitivity analysis of the algebraic model was then conducted on the basis of the generalized differential calculus of B. Mordukhovich. In this way, the paper paved a way for an efficient numerical treatment of the considered model in a number of applications.

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