Akademie věd České republiky
Ústav teorie informace a automatizace

Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation

RESEARCH REPORT

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Construction of Mass Migration Process

No. 2316 December 2011

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ABSTRACT. A general model of conservative particle systems on $\mathbb{Z}^d$ is treated in this report. We call it the Mass Migration Process. We bring out a construction of an appropriate Markov process, we set conditions on existence, attractiveness and we present many particular examples.

1. INTRODUCTION

In this paper, we are interested in a rather general model of conservative particle systems with interactions which generalizes or includes known particle systems: particle systems with zero range/exclusion/Misanthrope interactions studied in the 80's by Spitzer, Liggett, Andjel, Coccoza and others ([1, 11, 2]), the migration processes recently studied by Godreche, Luck et al. ([7, 8]). All these particle systems have dynamics consisting in separate jumps of particles among sites (nodes): if an exponential clock (corresponding to a couple of sites - departure and arrival) rings then one particle from the departure site attempts to jump to the arrival site. Whether the particle really jumps and how long is time between two rings, it depends generally on the current number of particles on departure site and on the current number of particles on arrival site. Particulars of the dynamics are given by an individual type of the model.

The generalization, we are interested in, consists in allowing jumps of more than one particle in one moment. It means that if one of the exponential clocks rings then a multiple jump of k-many particles may occur.

In such a way generalized zero range dynamics on $\mathbb{Z}$ was previously mentioned by Seppalainen [12], the process called *stick process* had a constant rate of jumps (independent of the number of particles at sites and also independent of the number of jumping particles) and was studied in the context with Ulam's problem.

In the framework of finite volume systems, the *zero range processes* (ZRP) with multiple jumps were studied by Evans et al. [3, 4, 5], Greenblatt and Lebowitz [9]. In [4, 5], authors use a rather different approach when they consider a continuous mass instead of discrete particles moving among sites. These systems are called *mass transport models*. In the work [9] of Greenblatt and Lebowitz, the generalized zero range dynamics is considered in the same sense as we do. However the set of sites is always assumed to be finite: it is interval $\{1, ..., L\} \subset \mathbb{Z}$ in [3, 4], a general finite graph in [5] and cube $\{1, ..., L\}^d \subset \mathbb{Z}^d$ in [9].

This report is aimed on the construction and the description of some basic properties of the model which we will call the *mass migration process* (MMP). It includes as a special case the *generalized zero range process* (GZRP) and it generalizes the migration process (specially Misanthrope process) in the same way.

The concept of this report is following: in Section 2, the model is defined and in Section 3, a construction of an appropriate Markov process with a semigroup relevant to a formal infinitesimal generator is given. Finally, Section 4 collects some useful properties of a coupling process, the sufficient and necessary conditions on attractiveness and some examples.

2. Model description

Let us consider a countable set X of sites where particles move, typically $\mathbb{Z}^d$. Each site is occupied by an arbitrary number $\alpha \in \mathbb{N}$ of particles. Particles move between sites with respect to a *mass migration dynamics*, i.e. k particles from the total amount α at leaving site x jump to a target site y occupied by β particles with rate

$$g^k_{\alpha,\beta}$$

where $g^k_{\alpha,\beta}$ are nonnegative for each α, $\beta \in \mathbb{N}$, $0 < k \leq \alpha$, $g^0_{\alpha,\beta} = 0$ and $g^i_{\alpha,\beta} = 0$ for every $i > \alpha$. The target site y is chosen with respect to probabilities $p(x, y)$, $x \in \mathrm{X}$.

The *misanthrope process* (introduced in [2]) is then a special case with

$$g^k_{\alpha,\beta} = \mathrm{I}_{[k=1]} g(\alpha, \beta) \tag{1}$$

for a nonnegative function $g(\cdot, \cdot)$ on $\mathbb{N}\times\mathbb{N}$ increasing (decreasing) in the first (second) coordinate, it means that only the jumps with one particle moving are possible. In work of Godreche et al., arbitrary nonnegative $g(\cdot, \cdot)$ on $\mathbb{N} \times \mathbb{N}$ is considered in (1) and the process is then called the *dynamic urn model* or the *migration process*. Another special cases are *zero range processes* (generalized or classical) which we obtain when the dependence on β is dropped:

$$g^k_{\alpha,\beta} = g^k_\alpha \qquad \text{(GZRP)}$$
$$g^k_{\alpha,\beta} = \mathrm{I}_{[k=1]} g(\alpha) \qquad \text{(ZRP)}.$$

Whole particle system in a single moment is described as a configuration $\eta = (\eta(x) : x \in \mathrm{X})$ where $\eta(x)$ stands for the number of particles at x. To stress a particular time t, we write η_t.

The dynamics of the particle system can be described by a collection of Poisson processes – one Poisson process (with time dependent rate) for every couple of sites (x, y). In time t, the rate of the (x, y)-th Poisson process is equal to $p(x, y) \sum_{k=1}^{\alpha} g^k_{\alpha,\beta}$ where $\alpha = \eta_{t_-}(x)$ and $\beta = \eta_{t_-}(y)$. We are used to say that (x, y)-th exponential clock rings when an event of (x, y)-th Poisson process occurs. For arbitrary $k > 0$, if (x, y)-th exponential clock rings then k-particles from the total amount α of particles at x leave site x and move to site y with β particles with probability

$$\frac{g^k_{\alpha,\beta}}{\sum_{l=1}^{\alpha} g^l_{\alpha,\beta}},$$

which means exactly that the rate of this jump is

$$p(x, y) g^k_{\alpha,\beta}. \tag{2}$$

In another words, if we denoted a changed configuration after a multiple jump by

$$\eta^{kxy}(z) = \begin{cases} \eta(x) - k & \text{if } z = x \\ \eta(y) + k & \text{if } z = y \\ \eta(z) & \text{otherwise} \end{cases}$$

then (2) refers to a transition rate between particle configurations η and η^{kxy}.

3. Construction and model assumptions

The above description of the mass migration process (MMP) defines under some conditions a Markov process $(\eta_t)_{t\geq 0}$ with a state space $\mathfrak{X} \subset \mathbb{N}^{\mathrm{X}}$ given by transition rates (2). In this section we specify sufficient conditions for existence of a rigorous mathematical object and prove it. The proof is based on methods of Andjel [1] and Liggett, Spitzer [10].

3.1. **The idea.** For finite set X of sites, the rates of transitions between particle configurations $\eta, \zeta \in \mathbb{N}^{\mathrm{X}}$ are

$$q(\eta,\zeta) = \sum_{x,y\in \mathrm{X}} \sum_{k=1}^{\eta(x)} p(x,y)\, g^k_{\eta(x),\eta(y)} \left(\mathrm{I}_{[\zeta=\eta^{kxy}]} - \mathrm{I}_{[\zeta=\eta]} \right) \tag{3}$$

and form a generator matrix $Q = \big(q(\eta,\zeta) : \eta,\zeta \in \mathbb{N}^{\mathrm{X}}\big)$ since the state space $\mathfrak{X} = \mathbb{N}^{\mathrm{X}}$ is countable. From the standard theory of countable state space Markov processes, the *mass migration process with generator matrix* Q is well defined as a Markov process $(\eta_t)_{t\geq 0}$ with transition probability matrices $P(t)$. Here

$$P(t),\ t\geq 0, \tag{4}$$

- form a semigroup,
- are a minimal solution of differential equation $P'(t) = QP(t)$, $P(0) = I$,
- for $\eta,\zeta \in \mathbb{N}^{\mathrm{X}}$, $P(t)(\eta,\zeta) = \mathbb{P}^\eta(\eta_t = \zeta)$,
- for a function f on $\mathbb{N}^{\mathrm{X}}$ and $\eta \in \mathbb{N}^{\mathrm{X}}$, $\big(P(t)\cdot f\big)(\eta) = \mathbb{E}^\eta f(\eta_t)$.

For X countable infinite, we can write down (as analogy to the generator matrix) an infinitesimal generator:

$$\mathcal{L}f(\eta) = \sum_{x,y\in \mathrm{X}} \sum_{k>0} p(x,y)\ g^k_{\eta(x),\eta(y)} \left(f\left(\eta^{kxy}\right) - f(\eta) \right) \tag{5}$$

where $\eta \in \mathbb{N}^{\mathrm{X}}$, f is a function on $\mathbb{N}^{\mathrm{X}}$. But the set $\mathbb{N}^{\mathrm{X}}$ is now uncountable and moreover, we can not apply the Hille-Yosida Theorem (which provides usually the existence of a Markov semigroup for uncountable state space Markov processes given by a generator) since in our case its assumptions fail.

We use the following approximation introduced in [10]. Let $\mathrm{X}_1 \subset \mathrm{X}_2 \subset \dots$ X be finite sets such that $\bigcup \mathrm{X}_n = \mathrm{X}$ and

$$p_n(x,y) = \begin{cases} p(x,x) + \sum_{z\notin \mathrm{X}_n} p(x,z) & x=y,\ x\in \mathrm{X}_n \\ 1 & x=y,\ x\notin \mathrm{X}_n \\ p(x,y) & x\neq y,\ x,y\in \mathrm{X}_n \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Using p_n in (5), we obtain

$$\mathcal{L}_n f(\eta) = \sum_{x\neq y\in \mathrm{X}_n} \sum_{k>0} p(x,y)\ g^k_{\eta(x),\eta(y)} \left(f\left(\eta^{kxy}\right) - f(\eta) \right). \tag{7}$$

Since there is no possible movement of particles outside of X_n, generator $\mathcal{L}_n$ depends on η only through $(\eta(x) : x \in \mathrm{X}_n) \in \mathbb{N}^{\mathrm{X}_n}$. So the dynamics of the process given by $\mathcal{L}_n$ is identical to one of the Markov process with generator matrix $Q_n = \big(q(\eta,\zeta) : \eta,\zeta \in \mathbb{N}^{\mathrm{X}_n}\big)$ and corresponding transition probability matrices $P_n(t)$, with properties as in (4).

Convention: *We will use hereafter a natural embedding* $\mathbb{N}^{\mathrm{X}_n}$ *into* $\mathbb{N}^{\mathrm{X}}$ *by identifying configuration* $\eta \in \mathbb{N}^{\mathrm{X}_n}$ *with configuration* $\eta \times 0^{\mathrm{X}\setminus \mathrm{X}_n} \in \mathbb{N}^{\mathrm{X}}$ *and do not distinguish between processes given by* $\mathcal{L}_n$ *and by* Q_n.

Roughly speaking, since we have $\mathrm{X}_n \to \mathrm{X}$ and $p_n(x,y) \to p(x,y)$, our existence proof will consist in precise defining a limit of semigroups $P_n(t)$, $n\to\infty$.

3.2. **Model assumptions.** The MMP is certainly well defined when starting from a configuration η with a finite total number of particles $\#\eta$. Then there is the same finite number of particles in the system at arbitrary time. In general, we care about $\eta_t(x)$ to be finite for every $x \in \mathrm{X}$ and every $t > 0$. So we need to avoid starting from configurations which could cause explosions. From this reason we can't let the process live on the whole set $\mathbb{N}^{\mathrm{X}}$ of all possible particle configurations and we have to drop configurations with a big mass near infinity. We define

$$\mathfrak{X}_0 = \{\eta \in \mathbb{N}^{\mathrm{X}} : \|\eta\| < \infty\} \tag{8}$$

where $\|\eta\| = \sum_{x\in \mathrm{X}} \eta(x)a_x$. Here a is a positive function on X such that

$$\sum_y p(x,y)a_y + \sum_y a_y p(y,x) \le M a_x, \text{ for every } x \in \mathrm{X} \tag{9}$$

and for some constant M.

Let us assume that $p(x,y)$ is irreducible transition probability on X and

$$\sup_y \sum_x p(x,y) = m_p < \infty. \tag{10}$$

Then it is possible to choose any $M > 1 + m_p$ and set $a_x = \sum_n (\frac{1+m_p}{M})^n \tilde{p}^n(x,x_0)$ where x_0 is a fixed reference site, $\tilde{p}(x,y)$ are symmetric transition probabilities on X defined by $\tilde{p}(x,y) = \frac{1}{1+m_p}\big(p(x,y) + p(y,x)\big)$ for every $x \neq y$. We denote by $\tilde{P}$ the matrix $\big(\tilde{p}(x,y) : x, y \in \mathrm{X}\big)$ and by $\tilde{P}^n = \big(\tilde{p}^n(x,y) : x, y \in \mathrm{X}\big)$ the matrix $\tilde{P}$ to the power of n. It is obvious then (9) holds. If moreover $M > (1+m_p)m_p$ then $\sum_x a_x < \infty$.

We have just reduced the state space to (8) and put assumption (10) on $p(x,y)$. Furthermore we shall need some assumptions on rate function $g^k_{\alpha,\beta}$. A natural condition

$$\sum_{k=1}^{\alpha} k\, g^k_{\alpha,\beta} \le C(\alpha+\beta) \qquad \text{for every } \alpha > 0,\ \beta \ge 0, \text{ for some } C, \tag{11}$$

is not strong enough. A sufficient condition, under which technical inequalities in paragraph 3.3 shall be proved, is

$$\sum_{k=1}^{\alpha\vee\gamma} k\,|g^k_{\alpha,\beta} - g^k_{\gamma,\delta}| \ \le\ C(|\alpha-\gamma| + |\beta-\delta|) \qquad \text{for every } \alpha,\beta,\gamma,\delta \ge 0, \tag{12}$$

for some $C > 0$. The condition (12) can be equivalently reformulate as

$$\sup_{\alpha,\beta\ge 0} \sum_{k=1}^{\alpha+1} k\,|g^k_{\alpha+1,\beta} - g^k_{\alpha,\beta}| + \sum_{k=1}^{\alpha} k\,|g^k_{\alpha,\beta+1} - g^k_{\alpha,\beta}| \ \le\ C$$

and obviously, the condition (12) implies (11).

Let us define the set of Lipshitz functions on $\mathfrak{X}_0$

$$\mathbf{L} = \{f : \mathfrak{X}_0 \to \mathbb{R} \ \text{ such that } |f(\eta) - f(\zeta)| \le L_f\|\eta-\zeta\| \text{ for some constant } L_f\}.$$

Now we can state the existence theorem which assigns a corresponding Markov semigroup to the infinitesimal generator $\mathcal{L}$ given by (5).

Theorem 3.1. *Let us assume that irreducible transition probabilities $p(x,y)$ satisfy (10) and rates $g^k_{\alpha,\beta} \ge 0$ for $k,\alpha,\beta \ge 0$, $g^k_{\alpha,\beta} = 0$ for $k > \alpha$ or $k = 0$, satisfy (12).*

Then there exist a Markov semigroup of operators $S(t)$, $t \ge 0$, defined on Lipshitz functions on $\mathfrak{X}_0$, and a constant $c > 0$ such that for every $f \in \mathbf{L}$, $t \ge 0$ the following items hold:

(1) *$S(t)f \in \mathbf{L}$ and $L_{S(t)f} \le L_f e^{ct}$*

(2) *$S(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}S(s)f(\eta)\mathrm{d}s \quad$ for every $\eta \in \mathfrak{X}_0$ where $\mathcal{L}$ is the infinitesimal generator given by (5)*

(3) $S(t)f(\eta) = \mathbb{E}^\eta f(\eta_t)$ *for every* η *with a finite number of particles where* $(\eta_t)_{t\ge 0}$ *on the right-hand side is a Markov process with rates (3) started from* η *at time* 0.

A proof of the theorem is a straight consequence of the facts we bring out in paragraph 3.5 below. Paragraphs 3.3, 3.4 include required inequalities and computations.

Having a Markov semigroup $S(t)$ on $\mathbf{L}$, a standard way using Kolmogorov extension theorem derives from this Markov semigroup the corresponding Markov process defined by probabilities P^η on trajectories where projections $P^\eta(\pi_t \in \cdot)$ concentrates on $\mathfrak{X}_0$ and satisfy $\int f(\xi)P^\eta(\pi_t \in d\xi) = S(t)f(\eta)$. Note that a special care is necessary when proving right continuity and left limits existence of the process' trajectories.

3.3. **Key inequalities.** Let X be finite through this paragraph. The key issue is to find a coupling process such that

$$\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma} |k-l|\, G^{k,l}_{\alpha,\beta,\gamma,\delta} \;\le\; C\big(|\alpha-\gamma| + |\beta-\delta|\big) \tag{13}$$

where coupling rate $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ is related to a jump of k particles in the first coordinate from the total amount α and l particles in the second coordinate from the total amount γ where at target sites sit β particles in the first coordinate and δ in the second one.

We employ the coupling defined by Gobron, Saada [6], see (18). Then:

$$\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma} |k-l| G^{k,l}_{\alpha,\beta,\gamma,\delta} = \sum_{k>l}(k-l)G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k\le l}(k-l)G^{k,l}_{\alpha,\beta,\gamma,\delta} =$$

$$= \sum_{l=0}^{\gamma} l \left(\sum_{k=0}^{l} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k=l+1}^{\alpha} G^{k,l}_{\alpha,\beta,\gamma,\delta} \right) - \sum_{k=0}^{\alpha} k \left(\sum_{l=k}^{\gamma} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l=0}^{k-1} G^{k,l}_{\alpha,\beta,\gamma,\delta} \right) =$$

$$= \sum_{l=0}^{\gamma} lR_l - \sum_{k=0}^{\alpha} kR^*_k$$

where we have denoted

$$\begin{aligned} R_l &= \sum_{k=0}^{l} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{k=l+1}^{\alpha} G^{k,l}_{\alpha,\beta,\gamma,\delta}, \quad l = 0,\dots,\gamma, \\ R^*_k &= \sum_{l=k}^{\gamma} G^{k,l}_{\alpha,\beta,\gamma,\delta} - \sum_{l=0}^{k-1} G^{k,l}_{\alpha,\beta,\gamma,\delta}, \quad k = 0,\dots,\alpha, \end{aligned}$$

using convention $\sum_a^b \cdots = 0$ if $a > b$. So $R_l = g^l_{\gamma,\delta}$ for $l > \alpha\wedge\gamma$ and $R^*_k = -g^k_{\alpha,\beta}$ for $k > \alpha\wedge\gamma$ and we can write down:

$$\sum_{k=0}^{\alpha}\sum_{l=0}^{\gamma} |k-l| G^{k,l}_{\alpha,\beta,\gamma,\delta} = \sum_{i=1}^{\alpha\wedge\gamma} iR_i - iR^*_i \;+ \sum_{i>(\alpha\wedge\gamma)}^{\gamma} i\, g^i_{\gamma,\delta} \;+ \sum_{i>(\alpha\wedge\gamma)}^{\alpha} i\, g^i_{\alpha,\beta} \tag{14}$$

Using telescopic argument [6, Lemma 3.6] (see (19) below) for partial sums of $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ and notation $\mathbf{\Sigma}^i_{\alpha,\beta} = \sum_{k>i} g^k_{\alpha,\beta}$ we obtain

$$\begin{aligned} R_i &= g^i_{\gamma,\delta} - 2\Big[g^i_{\gamma,\delta} \wedge \Big(\mathbf{\Sigma}^i_{\alpha,\beta} - \mathbf{\Sigma}^i_{\alpha,\beta}\wedge\mathbf{\Sigma}^i_{\gamma,\delta}\Big)\Big] \\ &= \begin{cases} g^i_{\gamma,\delta} & \text{if } \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \\ g^i_{\gamma,\delta} - 2\big(\mathbf{\Sigma}^i_{\alpha,\beta} - \mathbf{\Sigma}^i_{\gamma,\delta}\big) & \text{if } \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} < \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \\ -g^i_{\gamma,\delta} & \text{if } \qquad \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \ge \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \end{cases} \end{aligned}$$

$$\begin{aligned} R_i^* &= 2\Big[g^i_{\alpha,\beta} \wedge \Big(\mathbf{\Sigma}^{i-1}_{\gamma,\delta} - \mathbf{\Sigma}^i_{\alpha,\beta} \wedge \mathbf{\Sigma}^{i-1}_{\gamma,\delta}\Big)\Big] - g^i_{\alpha,\beta} \\ &= \begin{cases} -g^i_{\alpha,\beta} & \text{if } \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \geq \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \\ 2\big(\mathbf{\Sigma}^{i-1}_{\gamma,\delta} - \mathbf{\Sigma}^i_{\alpha,\beta}\big) - g^i_{\alpha,\beta} & \text{if } \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} < \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \\ g^i_{\alpha,\beta} & \text{if } \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \geq \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \end{cases} \end{aligned}$$

for $i = 1, \dots, \alpha \wedge \gamma$. We can compute

$R_i - R_i^* =$

$$= \begin{cases} g^i_{\alpha,\beta} - g^i_{\gamma,\delta} & \text{if } \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \geq \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \\ g^i_{\gamma,\delta} - g^i_{\alpha,\beta} & \text{if } \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \leq \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} \ \& \ \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \leq \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \\ g^i_{\alpha,\beta} - g^i_{\gamma,\delta} & \text{if } \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} < \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \\ g^i_{\gamma,\delta} - g^i_{\alpha,\beta} + & \text{if } \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} < \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \ \& \\ \quad + 2(\mathbf{\Sigma}^i_{\gamma,\delta} - \mathbf{\Sigma}^i_{\alpha,\beta}) & \qquad \& \ \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \leq \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} \\ g^i_{\alpha,\beta} - g^i_{\gamma,\delta} + & \text{if } \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} \leq \sum_{j=i+1}^{\gamma} g^j_{\gamma,\delta} \ \& \\ \quad + 2(\mathbf{\Sigma}^i_{\alpha,\beta} - \mathbf{\Sigma}^i_{\gamma,\delta}) & \qquad \& \ \sum_{j=i+1}^{\alpha} g^j_{\alpha,\beta} < \sum_{j=i}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=i}^{\alpha} g^j_{\alpha,\beta} \end{cases}$$

and we immediately obtain that $|R_i - R_i^*| \leq |g^i_{\alpha,\beta} - g^i_{\gamma,\delta}|$. Therefore, from (14) and using assumption (12) we obtain the desired inequality

$$\sum_{k=0}^{\alpha} \sum_{l=0}^{\gamma} |k-l|\, G^{k,l}_{\alpha,\beta,\gamma,\delta} \ \leq\ \sum_{i=1}^{\alpha\vee\gamma} i\, |g^i_{\alpha,\beta} - g^i_{\gamma,\delta}| \ \leq\ C|\alpha-\gamma| + |\beta-\delta|.$$

We employ this result to prove item (*ii*) of the following lemma.

Lemma 3.2. *Let $\mathcal{L}_n$ be given by (7). Then*
(i) $\mathcal{L}_n\Big(\|\eta\|\Big) \leq C\,(M+m_p)\,\|\eta\|$ *for every* $\eta \in \mathfrak{X}_0$, $n \geq 1$,

(ii) $\bar{\mathcal{L}}_n\Big(\|\eta-\zeta\|\Big) \leq C\,(M+m_p+1)\,\|\eta-\zeta\|$ *for every* $\eta, \zeta \in \mathfrak{X}_0$, $n \geq 1$,

where $\bar{\mathcal{L}}_n$ denotes the generator of the coupling process derived from rates $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ and probabilities $p_n(x,y)$,
(iii) $\mathcal{L}_n\Big(|||\eta|||\Big) \leq C\,(M+3)\,|||\eta|||$ *for every* $\eta \in \mathfrak{X}_0$, $n \geq 1$,
where $|||\eta||| = \sum_{x,y} p(x,y)(a_x+a_y) \sum_k k\, g^k_{\eta(x),\eta(y)}$.
Note that constants C, M, m_p were introduced in section 3.2.

Proof. $\mathcal{L}_n\Big(\|\eta\|\Big) =$

$$= \sum_{x\neq y \in \mathrm{X}_n} \sum_k p(x,y) g^k_{\eta(x),\eta(y)} (k a_y - k a_x) \leq \sum_{x \neq y} p(x,y) a_y \sum_k k\, g^k_{\eta(x),\eta(y)} \overset{\text{by (11)}}{\leq}$$

$$\leq C \sum_{x\neq y} p(x,y) a_y \,(\eta(x)+\eta(y)) \leq C \sum_x M a_x \eta(x) + C \sum_y m_p a_y \eta(y)$$

$$\bar{\mathcal{L}}_n\Big(\|\eta-\zeta\|\Big) = \sum_{x,y} \sum_{k,l} p_n(x,y) G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)} \Big(\|\eta^{kxy} - \zeta^{lxy}\| - \|\eta-\zeta\|\Big) \leq$$

$$\leq \sum_{x\neq y \in \mathrm{X}_n} \sum_{k,l} p(x,y) G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)} (a_x+a_y)|k-l| \overset{\text{by (13)}}{\leq}$$

$$\leq C \sum_{x\neq y\in \mathrm{X}_n} p(x,y)(a_x+a_y)\left(|\eta(x)-\zeta(x)|+|\eta(y)-\zeta(y)|\right) \leq$$
$$\leq C\sum_x |\eta(x)-\zeta(x)| \sum_y (p(x,y)+p(y,x))(a_y+a_x)$$

$$\mathcal{L}_n\Big(|||\eta|||\Big) = \sum_{x\neq y\in \mathrm{X}_n}\sum_k p(x,y) g^k_{\eta(x),\eta(y)}$$
$$\sum_{u,v} p(u,v)(a_u+a_v)\sum_l l\left(g^l_{\eta(u)-k\mathrm{I}_{[u=x]},\eta(v)+k\mathrm{I}_{[v=y]}} - g^l_{\eta(u),\eta(v)}\right) \overset{\text{by (12)}}{\leq}$$
$$\leq \sum_{x,y} p(x,y)\sum_k k g^k_{\eta(x),\eta(y)}\Big((M+1)a_x k + (M+1)a_y k + (a_x+a_y)2k\Big)$$

□

3.4. **Technicalities.** Based on inequalities (i) a (ii) of Lemma 3.2, the following technical lemmas ([1, Lemma 2.1–2.4]) can be proved for the MMP. Note that through this paragraph set X is still finite.

Lemma 3.3. X *finite.*

$$\mathbb{E}^\eta\Big(\eta_t(y)\Big) \leq e^{Cm_p t}\sum_{x\in\mathrm{X}}\eta(x)\sum_{l=0}^\infty \frac{(Ct)^l}{l!}p^l(x,y)$$

Proof. If we fix $\eta\in\mathbb{N}^{\mathrm{X}}$ then Markov process η_t starting from η has finite state space $\mathrm{S}_{\#\eta}=\{\xi\in\mathbb{N}^{\mathrm{X}} : \#\xi=\#\eta\}$ and if $P(t)=(\mathbf{p}_t(\eta,\zeta))$ stand for its transition probabilities we can write

$$\mathbb{E}^\eta\Big(\eta_t(y)\Big) = \sum_{\zeta\in\mathbb{N}^{\mathrm{X}}}\zeta(y)\mathbf{p}_t(\eta,\zeta) = \sum_{\zeta:\#\zeta=\#\eta}\zeta(y)\sum_{j=0}^\infty\frac{t^j}{j!}q^j(\eta,\zeta)$$

where $Q^j=(q^j(\eta,\zeta))$ is matrix $Q=(q(\eta,\zeta):\eta,\zeta\in\mathrm{S}_{\#\eta})$ powered by j and rates $q(\eta,\zeta)$ are given by (3). Since

$$\sum_{\zeta\in\mathbb{N}^{\mathrm{X}}}\zeta(y)q(\eta,\zeta) \leq \sum_{x\in\mathrm{X}}\sum_{k=1}^{\eta(x)} p(x,y)g^k_{\eta(x),\eta(y)}k \leq C\sum_x \eta(x)p(x,y) \;+\; Cm_p\eta(y)$$

and $q^j(\eta,\zeta)=\sum_\xi q^{j-1}(\eta,\xi)q(\xi,\zeta)$ we can show by induction that

$$\sum_{\zeta\in\mathbb{N}^{\mathrm{X}}}\zeta(y)q^j(\eta,\zeta) \leq C^j\sum_{l=0}^j\binom{j}{l}m_p^{j-l}\sum_z\eta(z)p^l(z,y)$$

where $p^0(x,y)=\mathrm{I}_{[x=y]}$. We conclude that

$$\mathbb{E}^\eta\Big(\eta_t(y)\Big) \leq \sum_{j=0}^\infty\frac{t^j}{j!}C^j\sum_{l=0}^j\binom{j}{l}m_p^{j-l}\sum_x\eta(x)p^l(x,y)$$

and when summing in the opposite direction we obtain the statement of Lemma.

□

Remark 3.4. *Let us note that matrix* $Z_t := \exp(tQ)$ *is well defined also for* $Q=(q(\eta,\zeta))_{\eta,\zeta\in\mathbb{N}^{\mathrm{X}}}$ *by formula* $Z_t(\eta,\zeta)=\sum_{j=0}^\infty\frac{t^j}{j!}q^j(\eta,\zeta)$ *since sum*

$$\sum_{j=0}^\infty\frac{t^j}{j!}q^j(\eta,\zeta) \leq e^{C(m_p+1)t}\#\eta,$$

where $q(\eta,\zeta)$ is given by (3), is convergent for every η uniformly in ζ. Writing formal derivation Z_t' as a limit of $(Z_{t+h}-Z_t)/h$ when $h\to 0$ we can see that Z_t solve differential equation $Z_t' = QZ_t$, $Z_0 = I$. Since from Markov process theory we know that $P(t)$ is the minimal solution of the differential equation we can conclude that $P(t)\le Z_t = \exp(tQ)$.

Similarly we can proceed with the coupling process. For X finite, the process is well defined by coupling rates $G^{k,l}_{\alpha,\beta,\gamma,\delta}\ge 0$. The matrix $\bar{Q}$ has then elements

$\bar{q}((\eta,\zeta),(\hat\eta,\hat\zeta)) = \sum_{x\neq y}\sum_{k,l} p(x,y) G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)}(\mathrm{I}_{[\hat\eta=\eta^{kxy},\hat\zeta=\zeta^{lxy}]} - \mathrm{I}_{[\hat\eta=\eta,\hat\zeta=\zeta]})$
which are bounded by $\sum_{x\neq y}\sum_k p(x,y) g^k_{\eta(x),\eta(y)}$. So $\exp(t\bar{Q})$ is well defined.

Let us denote by $S(t)$ an operator on functions $f:\mathbb{N}^{\mathrm{X}}\to\mathbb{R}$ derived from transition probability matrix $P(t)$ such that $S(t)f$ is function equal to matrix product $P(t)\cdot f$. Note that f is here written as a vector $(f(\eta):\eta\in\mathbb{N}^{\mathrm{X}})$ since X is finite.

We have

$$S(t)f(\eta) = \mathbb{E}^\eta(f(\eta_t)) = \sum_{\zeta\in\mathbb{N}^{\mathrm{X}}} f(\zeta)\mathrm{p}_t(\eta,\zeta) \le \sum_{j=0}^{\infty}\frac{t^j}{j!}\sum_{\zeta\in\mathbb{N}^{\mathrm{X}}} f(\zeta) q^j(\eta,\zeta) = \sum_{j=0}^{\infty}\frac{t^j}{j!} Q^j f(\eta). \tag{15}$$

Lemma 3.5. X *finite. If $f\in\mathbf{L}$ then $S(t)f\in\mathbf{L}$ and $L_{\mathcal{S}(t)f}\le L_f e^{C(M+m_p+1)t}$*

Proof. For $f\in\mathbf{L}$ we define $g(\eta,\zeta) = f(\eta)-f(\zeta)$ and $h(\eta,\zeta) = \|\eta-\zeta\|$ for every η,ζ. If we denote $\bar{S}(t)$ the semigroup of operators for coupling process given by rates $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ we have

$$|S(t)f(\eta) - S(t)f(\zeta)| = |\bar{S}(t)g(\eta,\zeta)| \le L_f\bar{S}(t)h(\eta,\zeta) \overset{\text{by (15)}}{\le} L_f\sum_{j=0}^{\infty}\frac{t^j}{j!}\bar{Q}^j h(\eta,\zeta).$$

Using Lemma 3.2 (ii) we obtain

$$\bar{Q}h(\eta,\zeta) = \bar{\mathcal{L}}\|\eta-\zeta\| \le C(M+m_p+1)\|\eta-\zeta\|.$$

Therefore $\bar{Q}^j h(\eta,\zeta)\le C^j(M+m_p+1)^j\|\eta-\zeta\|$. □

Lemma 3.6. X *finite. If $f\in\mathbf{L}$ & $|f(\eta)|\le c_0\|\eta\|$ then $|S(t)f(\eta)|\le c_0 e^{C(M+m_p)t}\|\eta\|$.*

Proof. $|S(t)f(\eta)|\le c_0\sum_\zeta \mathrm{p}_t(\eta,\zeta)\|\zeta\|\le c_0\sum_{j=0}^{\infty}\frac{t^j}{j!}Q^j\|\cdot\|(\eta)$ where by Lemma 3.2(i) $Q\|\cdot\|(\eta) = \mathcal{L}(\|\eta\|)\le C(M+m_p)\|\eta\|$. □

In the following lemma, we consider two different probabilities $p(x,y)$ and $\tilde{p}(x,y)$ on X. Generator $\tilde{\mathcal{L}}$ is then derived from $\tilde{p}$ instead of p.

Lemma 3.7. X *finite, $f\in\mathbf{L}$.*

$$|(\mathcal{L}-\tilde{\mathcal{L}})f(\eta)|\le CL_f\sum_{x,y}|p(x,y)-\tilde{p}(x,y)|(\eta(x)+\eta(y))(a_x+a_y)$$

Proof. $|(\mathcal{L}-\tilde{\mathcal{L}})f(\eta)|\le \sum_{x\neq y}\sum_k g^k_{\eta(x),\eta(y)}|p(x,y)-\tilde{p}(x,y)|L_f\|\eta^{kxy}-\eta\|\le$
$\le L_f\sum_{x\neq y}|p(x,y)-\tilde{p}(x,y)|\sum_k k g^k_{\eta(x),\eta(y)}(a_x+a_y)$

□

Corollary 3.8.

$$S(t)f(\eta)-\tilde{S}(t)f(\eta)\le CL_f e^{Cm_pt}\int_0^t e^{C(M+1)(t-s)}$$
$$\sum_{x\neq y}\sum_z \eta(z)\Big(\sum_{l=0}^{\infty}\frac{(Cs)^l}{l!}(p^l(z,x)+p^l(z,y))\Big)(a_x+a_y)|p(x,y)-\tilde{p}(x,y)|ds$$

Proof. The result follows from the fact

$$S(t)f(\eta) - \tilde{S}(t)f(\eta) = \int_0^t S(s)(\mathcal{L} - \tilde{\mathcal{L}})\tilde{S}(t-s)f(\eta)ds$$

and using Lemmas 3.3, 3.5, 3.7. □

3.5. **Approximation of infinite volume.** Through this paragraph X is infinite. X_n, p_n, $\mathcal{L}_n$ and $P_n(t)$ are as in paragraph 3.1.

For each $n \in \mathbb{N}$ let $S_n(t), t \geq 0$, denote the semigroup of operators on functions $f : \mathbb{N}^{\mathrm{X}} \to \mathbb{R}$, $f \in \mathbf{L}$, derived from probability matrices $P_n(t)$: $S_n(t)f(\eta) = \big(P_n(t) \cdot f\restriction_{\mathbb{N}^{\mathrm{X}_n}}\big)(\eta\restriction_{\mathbb{N}^{\mathrm{X}_n}})$ and corresponding with $\mathcal{L}_n$ by

$$S_n(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}_n S_n(s)f(\eta)ds. \tag{16}$$

In the previous we obtained some properties of finite volume processes with generators $\mathcal{L}_n$ and semigroups $S_n(t)$, $t \geq 0$ which now ensure a good sense of limiting procedure when n is approaching infinity.

Lemma 3.9. *$S_n(t)f(\eta)$ converges when $n \to \infty$ for every $f \in \mathbf{L}$, $\eta \in \mathfrak{X}_0$ and $t \geq 0$.*

Proof. From Corollary 3.9 we obtain

$$|S_n(t)f(\eta) - S_m(t)f(\eta)| \leq CL_f e^{Cm_p t} \int_0^t \text{integrand } ds \qquad \text{where}$$

$$\text{integrand} \leq 2e^{C(M+2)(t-s)} \sum_{x,z} \eta(z) \sum_{l=0}^{\infty} \frac{(Cs)^l}{l!} p_n^l(z,x) a_x(M+1) \leq$$

$$\leq 2(M+1)e^{C(M+2)t}\|\eta\|$$

because $\sum_x a_x p_n^l(z,x) \leq a_z(M+2)^l$. So we are allowed to use the dominate convergence theorem. Since also

$$|S_n(t)f(\eta) - S_m(t)f(\eta)| \leq CL_f e^{Cm_p t} \int_0^t 2(M+1)e^{C(M+2)t} ds\, |p_n(x,y) - p_m(x,y)|$$

and $|p_n(x,y) - p_m(x,y)|$ tends to 0 when $n, m \to \infty$ the proof is done. □

Definition 3.10. *For every $f \in \mathbf{L}$, $\eta \in \mathfrak{X}_0$ and $t \geq 0$ we define*

$$S(t)f(\eta) = \lim_{n \to \infty} S_n(t)f(\eta). \tag{17}$$

The following lemma shows that such defined $(S(t) : t \geq 0)$ has semigroup properties and that corresponds to infinitesimal generator $\mathcal{L}$ given by (5). We bring it without the proof since the properties of our process stated and proved in Lemmas above are sufficient to use exactly the same arguments as in the paper [10].

Lemma 3.11. [10, Lemma 2.16] *Let $f \in \mathbf{L}$, $t, t_1, t_2 \geq 0$*

(i) $S(t_1 + t_2) = S(t_1)S(t_2)$, $S(0) = I$

(ii) $S(t)f(\eta) = f(\eta) + \int_0^t \mathcal{L}S(s)f(\eta)\mathrm{d}s$ *for every* $\eta \in \mathfrak{X}_0$

(iii) $|S(t)f(\eta) - f(\eta)| \leq \|\eta\| L_f (e^{C(M+m_p+1)t} - 1)$

(iv) $S(s)f(\eta)$ *is continuous in* s *for every* $\eta \in \mathfrak{X}_0$

(v) $\mathcal{L}S(s)f(\eta)$ *is continuous in* s *for every* $\eta \in \mathfrak{X}_0$

(vi) $\lim_{t \searrow 0} \frac{S(t)f(\eta) - f(\eta)}{t} = \mathcal{L}f(\eta)$

(vii) $\mathcal{L}S(t)f(\eta) = S(t)\mathcal{L}f(\eta)$

4. Attractiveness, Examples

4.1. **Coupling process.** Let us consider coupling rates define by Gobron and Saada [6]. For each $\alpha,\beta,\gamma,\delta\geq 0$

$$\begin{aligned}
G^{k,l}_{\alpha,\beta,\gamma,\delta} &= \Big(g^k_{\alpha,\beta}-g^k_{\alpha,\beta}\wedge\Big(\mathbf{\Sigma}^l_{\gamma,\delta}-\mathbf{\Sigma}^l_{\gamma,\delta}\wedge\mathbf{\Sigma}^k_{\alpha,\beta}\Big)\Big)\wedge\\
&\qquad\Big(g^l_{\gamma,\delta}-g^l_{\gamma,\delta}\wedge\Big(\mathbf{\Sigma}^k_{\alpha,\beta}-\mathbf{\Sigma}^l_{\gamma,\delta}\wedge\mathbf{\Sigma}^k_{\alpha,\beta}\Big)\Big),\quad k,l\geq 1,\\
G^{k,0}_{\alpha,\gamma} &= g^k_{\alpha,\beta}-g^k_{\alpha,\beta}\wedge\Big(\mathbf{\Sigma}^0_{\gamma,\delta}-\mathbf{\Sigma}^0_{\gamma,\delta}\wedge\mathbf{\Sigma}^k_{\alpha,\beta}\Big),\quad k\geq 1, \qquad (18)\\
G^{0,l}_{\alpha,\gamma} &= g^l_{\gamma,\delta}-g^l_{\gamma,\delta}\wedge\Big(\mathbf{\Sigma}^0_{\alpha,\beta}-\mathbf{\Sigma}^0_{\alpha,\beta}\wedge\mathbf{\Sigma}^l_{\gamma,\delta}\Big),\quad l\geq 1,
\end{aligned}$$

where

$$\mathbf{\Sigma}^k_{\alpha,\beta}=\sum_{k'>k}g^{k'}_{\alpha,\beta}.$$

We also use the following expressions [6, Lemma 3.6]: for arbitrary $\alpha,\beta,\gamma,\delta\geq 0$, each $1\leq k\leq\alpha$, $1\leq l\leq\gamma$

$$\begin{aligned}
G^{k,l}_{\alpha,\beta,\gamma,\delta} &= g^k_{\alpha,\beta}\wedge\Big(\mathbf{\Sigma}^{l-1}_{\gamma,\delta}-\mathbf{\Sigma}^k_{\alpha,\beta}\wedge\mathbf{\Sigma}^{l-1}_{\gamma,\delta}\Big)-g^k_{\alpha,\beta}\wedge\Big(\mathbf{\Sigma}^l_{\gamma,\delta}-\mathbf{\Sigma}^k_{\alpha,\beta}\wedge\mathbf{\Sigma}^l_{\gamma,\delta}\Big)\\
\text{or} & \qquad (19)\\
G^{k,l}_{\alpha,\beta,\gamma,\delta} &= g^l_{\gamma,\delta}\wedge\Big(\mathbf{\Sigma}^{k-1}_{\alpha,\beta}-\mathbf{\Sigma}^{k-1}_{\alpha,\beta}\wedge\mathbf{\Sigma}^l_{\gamma,\delta}\Big)-g^l_{\gamma,\delta}\wedge\Big(\mathbf{\Sigma}^k_{\alpha,\beta}-\mathbf{\Sigma}^k_{\alpha,\beta}\wedge\mathbf{\Sigma}^l_{\gamma,\delta}\Big).
\end{aligned}$$

We can obtain an explicit formula from which it is obvious that the coupling is well defined: for arbitrary $\alpha,\beta,\gamma,\delta\geq 0$ and each $1\leq k\leq\alpha$

$$G^{k,l}_{\alpha,\beta,\gamma,\delta}=\begin{cases}
0 & \text{for } l\in L_1\\
g^k_{\alpha,\beta}+\mathbf{\Sigma}^k_{\alpha,\beta}-\mathbf{\Sigma}^l_{\gamma,\delta} & \text{for } l\in L_2^*\\
g^l_{\gamma,\delta} & \text{for } l\in L_3\\
g^k_{\alpha,\beta} & \text{for } l\in L_4^*\\
g^l_{\gamma,\delta}-\mathbf{\Sigma}^k_{\alpha,\beta}+\mathbf{\Sigma}^l_{\gamma,\delta} & \text{for } l\in L_5^*\\
0 & \text{for } l\in L_6
\end{cases} \qquad (20)$$

where

$$\begin{aligned}
L_1 = &\quad \{0\leq l<\gamma:\ \mathbf{\Sigma}^l_{\gamma,\delta}\geq g^k_{\alpha,\beta}+\mathbf{\Sigma}^k_{\alpha,\beta}\}\\
l_2 = &\quad \min\ \{0,...,\gamma\}\setminus L_1\\
L_2^* = &\quad \{0\leq l<\gamma:\ \mathbf{\Sigma}^l_{\gamma,\delta}\geq\mathbf{\Sigma}^k_{\alpha,\beta}\}\cap\{l_2\}\\
L_3 = &\quad \{0<l<\gamma:\ \mathbf{\Sigma}^l_{\gamma,\delta}\geq\mathbf{\Sigma}^k_{\alpha,\beta}\}\setminus(L_1\cup L_2)\\
L_4^* = &\quad \{l_2\}\setminus L_2^*\\
L_5^* = &\quad \{0<l\leq\gamma:\ g^l_{\gamma,\delta}+\mathbf{\Sigma}^l_{\gamma,\delta}\geq\mathbf{\Sigma}^k_{\alpha,\beta}\}\setminus(L_1\cup L_2^*\cup L_3\cup L_4^*)\\
L_6 = &\quad \{0<l\leq\gamma\}\setminus(L_1\cup L_2^*\cup L_3\cup L_4^*\cup L_5^*).
\end{aligned}$$

Let us note that
– the star by sets L_2^*,L_4^*,L_5^* symbolizes that these sets are at most singleton;
– L_4 is nonempty if and only if L_2 is empty and also if and only if $L_2\cup L_3$ is empty;
– L_4 is nonempty if and only if L_5 is empty.

Furthermore, the following characterization holds

$l \in L_1$ iff $0 \le l < \gamma$ & $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k}^{\alpha} g^j_{\alpha,\beta}$

$l \in L_2^*$ iff $0 \le l < \gamma$ & $\left[(l=0) \vee \sum_{j=l}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k}^{\alpha} g^j_{\alpha,\beta}\right]$ &
& $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=k}^{\alpha} g^j_{\alpha,\beta}$ & $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$

$l \in L_3$ iff $0 < l < \gamma$ & $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=k}^{\alpha} g^j_{\alpha,\beta}$ & $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$

$l \in L_4^*$ iff $0 \le l \le \gamma$ & $\left[(l=0) \vee \sum_{j=l}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k}^{\alpha} g^j_{\alpha,\beta}\right]$ &
& $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$

$l \in L_5^*$ iff $0 < l \le \gamma$ & $\sum_{j=l}^{\gamma} g^j_{\gamma,\delta} \ge \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$ & $\sum_{j=l+1}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$

$l \in L_6$ iff $0 < l \le \gamma$ & $\sum_{j=l}^{\gamma} g^j_{\gamma,\delta} < \sum_{j=k+1}^{\alpha} g^j_{\alpha,\beta}$.

4.2. **Attractiveness.** Let us start this section with defining some basic terms. We consider the following partial order on state space $\mathfrak{X} \subset \mathbb{N}^{\mathrm{X}}$. For $\eta, \zeta \in \mathfrak{X}$

$$\eta \le \zeta \text{ if and only if } \eta(x) \le \zeta(x) \text{ for every } x \in \mathrm{X}.$$

We say that a function f on X is *monotone* if $f(\eta) \le f(\zeta)$ whenever $\eta \le \zeta$.

We call a particle system *attractive* if for every bounded, monotone continuous function f and every time $t > 0$, function $S(t)f$ is again bounded, monotone continuous function.

A particle system (η_t, ζ_t) on $\mathfrak{X} \times \mathfrak{X}$ is called the *coupling process* (derived from MMP) if both its marginals η_t, ζ_t are a copies of the same MMP.

We usually define a coupling process (η_t, ζ_t) by setting rates of coupled jumps: for a time t and sites $x, y \in \mathrm{X}$, a jump of k particles in the first coordinate and together l particles in the second coordinate from x to y occurs with a rate $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ where $\alpha = \eta_{t_-}(x)$, $\beta = \eta_{t_-}(y)$, $\gamma = \zeta_{t_-}(x)$, $\delta = \zeta_{t_-}(y)$, $0 \le k \le \alpha$ and $0 \le l \le \gamma$. Another jumps are not allowed. Note that jumps in both coordinates in the same time may occur just simultaneously from a common site x to a common site y (chosen with probability $p(x,y)$). The coupling process is then given by a following generator:

$$\bar{\mathcal{L}}\bar{f}(\eta,\zeta) = \sum_{x\in\mathrm{X}}\sum_{y\in\mathrm{X}} p(x,y) \sum_{k\ge 0}\sum_{l\ge 0} G^{k,l}_{\eta(x),\eta(y),\zeta(x),\zeta(y)} \left(f\left(\eta^{kxy},\zeta^{lxy}\right) - f\left(\eta,\zeta\right)\right). \tag{21}$$

Since by the definition of a coupling process, the following equality

$$\bar{\mathcal{L}}\bar{f}(\eta,\zeta) = \mathcal{L}f(\eta) \text{ for each } \zeta$$

must hold for every η and every $\bar{f}$ on $\mathfrak{X} \times \mathfrak{X}$ which does not depend on the second coordinate, i.e. $\bar{f}(\eta,\zeta) = f(\eta)$, then the rates have to satisfied:

$$\sum_{l=0}^{\gamma} G^{k,l}_{\alpha,\beta,\gamma,\delta} = g^k_{\alpha,\beta} \text{ for each } \alpha,\beta,\gamma,\delta \ge 0 \text{ and } 0 \le k \le \alpha. \tag{22}$$

The same for the second coordinate symmetrically.

If we know that there exists a coupling process which even preserves ordering of its marginals, i.e.

$$\eta_0 \le \zeta_0 \text{ implies } \eta_t \le \zeta_t,\ P^{(\eta_0,\zeta_0)} \text{ almost surely, for every } t > 0,$$

(so called *increasing coupling*) then the original process is attractive.

For the classical ZRP, a construction of an increasing coupling process can be found in [1] under the condition that rate function $g(\cdot)$ is nondecreasing. The Misanthrope process was originally (in [2]) considered just attractive, it means when rate function $g(\cdot,\cdot)$ is nondecreasing in the first coordinate and nonincreasing in the second one (here the *Misanthrope* name is originated).

A thorough characterization of coupling rates, and also equivalent conditions on the original jump rates to have attractiveness, for a big class of conservative processes, MMP among them, can be found in [6]. As a consequence of the general results presented in [6], we can obtain the following lemma concerning coupling rates of GZRP and MMP.

Lemma 4.1. *Let us consider (a) the mass migration process given by generator (5) with rate function $g^k_{\alpha,\beta}$ and, specially, (b) the generalized zero range process with rate g^k_α.*

(a) *The MMP is attractive if and only if for every $\alpha,\beta,\gamma,\delta\geq 0$, $\alpha\leq\gamma$, $\beta\leq\delta$,*

$$\forall l\geq 0 \qquad \sum_{k'>\delta-\beta+l}^{\alpha} g^{k'}_{\alpha,\beta} \;\leq\; \sum_{l'>l}^{\gamma} g^{l'}_{\gamma,\delta}$$
$$\&$$
$$\forall k\geq 0 \qquad \sum_{k'>k}^{\alpha} g^{k'}_{\alpha,\beta} \;\geq\; \sum_{l'>\gamma-\alpha+k}^{\gamma} g^{l'}_{\gamma,\delta}.$$

(b) *The GZRP is attractive if and only if for every $\alpha\geq 0$, $k\geq 0$,*

$$\sum_{k'>k+1}^{\alpha+1} g^{k'}_{\alpha+1} \leq \sum_{k'>k}^{\alpha} g^{k'}_{\alpha} \;\leq\; \sum_{k'>k}^{\alpha+1} g^{k'}_{\alpha+1}.$$

If the conditions in (a) hold, there exists an increasing coupling process of MMP's with rate $G^{k,l}_{\alpha,\beta,\gamma,\delta}$ given by formula (20). Analogously, if the condition in (b) holds, there exists an increasing coupling process of GZRP's with rate $G^{k,l}_{\alpha,\gamma}$ given by formula (20) where we omit the dependence on β,δ.

Remark 4.2. *(a') The MMP is attractive if and only if $\forall\alpha\geq 0$, $0\leq\beta\leq\delta$, $k\geq 0$,*

$$\sum_{k'>k+1}^{\alpha+1} g^{k'}_{\alpha+1,\delta} \;\leq\; \sum_{k'>k}^{\alpha} g^{k'}_{\alpha,\beta} \quad \& \quad \sum_{k'>k+(\delta-\beta)}^{\alpha} g^{k'}_{\alpha,\beta} \;\leq\; \sum_{k'>k}^{\alpha+1} g^{k'}_{\alpha+1,\delta} \quad \&$$
$$\sum_{k'>k+1}^{\alpha} g^{k'}_{\alpha,\beta} \leq \sum_{k'>k}^{\alpha} g^{k'}_{\alpha,\beta+1} \;\leq\; \sum_{k'>k}^{\alpha} g^{k'}_{\alpha,\beta}.$$

4.3. **Examples.** In this paragraph, we bring some particular examples of MMPs, we focus on existence condition (12) and the attractiveness conditions, possibly coupling rates. The examples can be sorted into three basic types: examples on generalized zero range processes when rate $g^k_{\alpha,\beta}=g^k_\alpha$ does not depend on β, examples on generalized target processes when rate $g^k_{\alpha,\beta}=g^k_{*\beta}\mathrm{I}_{[k\leq\alpha]}$ does not depend on α except the condition $k\leq\alpha$ and examples on mass transport processes when rate is depending on both α,β.

Example 4.3. *Special case of GZRP with $g^k_\alpha=h(k)\,\mathrm{I}_{[k\leq\alpha]}$ for $k\geq 1$.*

(i) A sufficient condition on existence is: $\sum_{k=1}^{\alpha} kh(k)\leq C\alpha$ for every α, for some C.

(ii) The process is attractive if and only if

$h(k)$ is nonincreasing for $k\geq 1$, nonnegative.

Then there exists an increasing coupling process with rate function $G^{k,l}_{\alpha,\gamma}$:
for each $\alpha,\gamma\geq 0$ and each $1\leq k\leq\alpha$

$$G^{k,0}_{\alpha,\gamma} = H^{k-1,0}_{\alpha,\gamma} - H^{k,0}_{\alpha,\gamma}$$
$$G^{k,l}_{\alpha,\gamma} = \left(H^{k-1,l}_{\alpha,\gamma} - H^{k,l}_{\alpha,\gamma}\right)\wedge\left(H^{k,l-1}_{\alpha,\gamma} - H^{k,l}_{\alpha,\gamma}\right) \text{ for } 0\leq l<\gamma$$

where $H^{i,j}_{\alpha,\gamma} = \left[\sum_{k'>i}^{\alpha} h(k') - \sum_{l'>j}^{\gamma} h(l')\right]^+$ using convention $\sum_a^b \cdots = 0$ if $a > b$.

Example 4.4. *Special case of GZRP with* $g^k_\alpha = r(\alpha)\,\mathrm{I}_{[k\le\alpha]}$ *for* $k \ge 1$ *where* $r(\cdot)$ *is a nonnegative function.*

(i) A sufficient condition on existence is: there exists C such that

$$\forall\alpha \ge 0: \qquad \alpha^2\,|r(\alpha+1) - r(\alpha)| + \alpha r(\alpha) \le C.$$

Note that if specially r is a constant function, this condition failed.

(ii) The process is attractive if and only if

$$r(\alpha) \text{ is nonincreasing } \& \ \alpha r(\alpha) \text{ nondecreasing for } \alpha \ge 1.$$

Then there exists an increasing coupling process with rate function $G^{k,l}_{\alpha,\gamma}$:
for each $\alpha < \gamma$ and each $1 \le k \le \alpha$

$$G^{k,l}_{\alpha,\gamma} = \begin{cases} 0 & \text{for } l = 0, ..., l_1 \\ r(\alpha)(\alpha-k+1) - r(\gamma)(\gamma-l) & \text{for } l = l_1+1 \\ r(\gamma) & \text{for } l = l_1+2, ..., l_3 \\ r(\gamma)(\gamma-l+1) - r(\alpha)(\alpha-k) & \text{for } l = l_3+1 \\ 0 & \text{for } l = l_3+2, ..., \gamma \end{cases}$$

where

$$\begin{aligned} l_1 &= \max\{l = 0, ..., \gamma-2 : \ r(\gamma)(\gamma-l) \ge r(\alpha)(\alpha-k+1)\} \\ l_3 &= \max\{l = 0, ..., \gamma : \ r(\gamma)(\gamma-l) \ge r(\alpha)(\alpha-k)\}. \end{aligned}$$

Note that l_3 may be equal to l_1+1 and then the third case does not occur.

For $\alpha \ge \gamma$ and $1 \le k \le \alpha$

$$G^{k,l}_{\alpha,\gamma} = \begin{cases} 0 & \text{for } l = 0, ..., l^*-1 \\ r(\alpha) - c^* & \text{for } l = l^* \\ c^* & \text{for } l = l^*+1 \\ 0 & \text{for } l = l^*+2, ..., \gamma \end{cases}$$

where

$$\begin{aligned} l^* &= \min\{l = 0, ..., \gamma : \ r(\gamma)(\gamma-l) < r(\alpha)(\alpha-k+1)\} \\ c^* &= (r(\gamma)(\gamma-l^*) - r(\alpha)(\alpha-k))^+ . \end{aligned}$$

Note that l^* may be equal to 0 and then the first case does not occur.

Example 4.5. *Special case of GZRP with*
$g^k_\alpha = \rho(\alpha)\rho(\alpha-1)\cdot\ldots\cdot\rho(\alpha-k+1)\,\mathrm{I}_{[k\le\alpha]}$ *for* $k \ge 1$.

(i) A sufficient condition on existence is: there exists C such that
for all $\alpha \ge 0,\ 1 \le k \le \alpha$:

$$\rho(\alpha)\rho(\alpha-1)\cdot\ldots\cdot\rho(1) \le \tfrac{C}{\alpha}$$

and

$$|\rho(\alpha+1)\rho(\alpha)\cdot\ldots\cdot\rho(\alpha-k+2) - \rho(\alpha)\rho(\alpha-1)\cdot\ldots\cdot\rho(\alpha-k+1)| \le \tfrac{C}{k\alpha}.$$

(ii) The process is attractive if and only if

$$0 < \rho(\alpha) \le 1 \ \& \ \sum_{i=1}^{k} \rho(k)\cdot\ldots\cdot\rho(i) \le \tfrac{\rho(\alpha+1)}{1-\rho(\alpha+1)} \text{ for all } \alpha \ge 0,\ 1 \le k \le \alpha.$$

Note that a sufficient condition is ρ positive, bounded by 1 and nondecreasing.

Example 4.6. *Special case of GZRP with* $g_\alpha^k = \frac{\pi(\alpha-k)}{\pi(\alpha)}$ *for* $k \geq 1$ *where* $\pi(\cdot)$ *is a positive function on* $\mathbb{N}^+$*. It is useful to consider an extension of* π *on* $\mathbb{Z}$ *by* $\pi(0) = 1$, $\pi(i) = 0$ *for* $i \in \mathbb{Z}^-$.

(i) A sufficient condition on existence is: there exists C such that

$$\sum_{k=1}^{\alpha+1} k|\frac{\pi(\alpha+1-k)}{\pi(\alpha+1)} - \frac{\pi(\alpha-k)}{\pi(\alpha)}| \leq C \quad \text{for every } \alpha \geq 0.$$

(ii) The process is attractive if and only if

$$\pi(\alpha) \leq \pi(\alpha+1) \leq \pi(\alpha)\frac{\sum_{i=0}^{j}\pi(i)}{\sum_{i=0}^{j-1}\pi(i)} \quad \text{for all } 1 \leq j \leq \alpha.$$

Example 4.7. *Special case of MMP called the target process with* $g_{\alpha,\beta}^k = \mathrm{I}_{[k\leq\alpha]} g_{*\beta}^k$.

(i) A sufficient condition on existence is: there exists C such that

$$\sum_{k=1}^{\alpha} k\,|g_{*\beta+1}^k - g_{*\beta}^k| \ + \ \alpha g_{*\beta}^\alpha \ \leq C \qquad \text{for every } \beta, \alpha \geq 0.$$

Note that if $g_{*\beta}^k$ is independent of k, this condition failed.

(ii) The process is attractive if and only if for all $0 \leq \beta \leq \delta$, $k \geq 0$

$$\begin{array}{rcccl} & g_{*\beta+1}^k & \leq & g_{*\beta}^k & \leq \ g_{*\beta+1}^k + g_{*\beta+1}^{k-1} \\ \text{and} & g_{*\delta}^{k+1} & \leq & g_{*\beta}^k & \leq \ g_{*\delta}^{k+1} + g_{*\delta}^k + \cdots + g_{*\delta}^{k-(\delta-\beta)}. \end{array}$$

Example 4.8. *Special case of MMP with* $g_{\alpha,\beta}^k = \mathrm{I}_{[k\leq\alpha]} b(\alpha,\beta)$ *where* b *is a nonnegative function.*

(i) A sufficient condition on existence is: there exists C such that

$$\forall \alpha, \beta \geq 0: \quad \alpha^2\,|b(\alpha+1,\beta) - b(\alpha,\beta)| + \alpha^2\,|b(\alpha,\beta+1) - b(\alpha,\beta)| + \alpha b(\alpha,\beta) \leq C.$$

(ii) The process is attractive if and only if for all $0 \leq \alpha \leq \gamma,\ 0 \leq \beta \leq \delta$

$$b(\alpha,\beta) \geq b(\gamma,\delta) \text{ and } b(\alpha,\beta)(\alpha+\beta) \leq b(\gamma,\delta)(\gamma+\delta) + \delta\left(b\left(\alpha,\beta\right) - b\left(\gamma,\delta\right)\right).$$

Note that sufficient conditions on attractiveness are
$b(\alpha,\beta) \geq b(\gamma,\delta)$ and $b(\alpha,\beta)(\alpha+\beta) \leq b(\gamma,\delta)(\gamma+\delta)$ for all $0 \leq \alpha \leq \gamma,\ 0 \leq \beta \leq \delta$.

Acknowledge

This work was supported by the Grant Agency of the Czech Republic under Grant No. 201/09/1931.

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